JOURNAL OF FUNCTIONAL ANALYSIS 18, 27–59 (1975)

# Potential and Scattering Theory on Wildly Perturbed Domains*

JEFFREY RAUCH AND MICHAEL TAYLOR

*Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48104*

*Communicated by Ralph Phillips*

Received February 1, 1974

We study the potential, scattering, and spectral theory associated with boundary value problems for the Laplacian on domains which are perturbed in very irregular fashions. Of particular interest are problems in which a "thin" set is deleted and the behavior of the Laplace operator changes very little, and problems where many tiny domains are deleted. In the latter case the "clouds" of tiny obstacles may tend to disappear, to solidify, or to produce an intermediate effect, depending on the relative numbers and sizes of the tiny domains. These phenomena vary according to the specific boundary value problem and in many cases their behavior is contrary to crude intuitive guesses.

## INTRODUCTION

In many situations one studies the behavior of elliptic boundary value problems defined on domains which in some sense are approximated by simpler domains, and it is of interest to know when the different domains yield solutions which are close. The usual techniques for such problems are power series expansions, e.g., the Hadamard variational formula, or methods related to the direct method of the calculus of variations (see [8, 4]). For these methods to work the domains must be approximated in some smooth sense. We study situations where physically the approximation is reasonable but it is not smooth. To give the flavor of the results we describe two problems.

*The Fireman's Pole Problem.* Let $\Omega$ be a bounded open set in $\mathbb{R}^3$ with some small degree of regularity. Let $\Omega_n$ be $\Omega$ with a cylinder of of radius $1/n$ removed. We think of $\Omega$ as the interior of a firehouse and

* Research supported by NSF Grant GP 34260.

$\Omega_n$ as the firehouse with one fireman's pole. The physical interest is that the presence of the pole does not affect, to any large extent, the propagation of sound, or heat, in the firehouse.

This is proven by considering the Laplacian, $\Delta$ on $\Omega$ and $\Omega_n$ (which we write $\Delta_n$) with domain defined by Dirichlet on Neumann boundary conditions and proving that as $n \to \infty$ the resolvents of $\Delta_n$ converge to the resolvent of $\Delta$ (e.g., Lemma 1.1). We show that this is strong enough to justify the statements made above about sound and heat propagation. A similar problem can be formulated for exterior domains with a fireman's pole, in which case it is also interesting to study the convergence as $n \to \infty$ of the wave and scattering operators. This we do in Section 5.

*The Crushed Ice Problem.* With $\Omega$ as above, let $\Omega_n$ be $\Omega$ with $n$ closed balls, of radius $r_n$, removed. We suppose $r_n \to 0$ as $n \to \infty$, and we suppose the balls are evenly spaced in some subregion $\Omega' \subset \Omega$. The question is how fast must $r_n$ decrease in order to render the balls negligible and when this condition fails, what happens. A physical problem which this would model is the flow of heat in $\Omega_n$, where the balls are little coolers maintained at temperature zero. Crude guesses might be that the rate of cooling grows unless the volume $(4/3)\pi n r_n^3 \to 0$, or that the total surface area $4\pi n r_n^2$ is the critical parameter. This is not correct; it is $nr_n$ that determines the behavior. If $nr_n \to 0$ then the balls are negligible in the limit in the same sense as for the fireman's pole (Theor. 4.2). If $nr_n \to \infty$ then the behavior of heat in $\Omega_n$ converges to the behavior for the region $\Omega - \Omega'$, so the cloud of balls appears to be solid (Theor. 4.4). Such phenomena are also common everyday experience where the atomic nature of matter is not easily observable. The borderline case where $nr_n$ is bounded seems to be quite delicate. If the balls are placed so that in the limit of number of balls $n_U$ in any open set $U \subseteq \Omega$ is given by $1/r_n \int_U \rho(x)\, dx$ then $\Delta_n$ converges to $\Delta - 2\pi\rho$ for "most" placements. The method of proof is probabilistic and, unfortunately, does not yield conditions of convergence for any specific placement. A stimulating lecture of M. Kac contributed to our interest in this problem and the methods of Section 6 were inspired by his paper [6]. As in the case of the fireman's pole problem questions related to scattering for these obstacles are discussed in Section 5.

Our basic method of proof is indirect and was inspired by a lecture of Tom Beale at the 1973 CMBS conference on scattering theory at Buffalo.

## 1. The Dirichlet Problem on Converging Domains

In order to treat domains $\Omega \subset \mathbb{R}^m$ which lack strong regularity properties we will treat the Laplacian wirh Dirichlet boundary conditions on $\partial\Omega$ by using the theory of quadratic forms. Suppose $\Omega$ is an open set in $\mathbb{R}^m$. Then $\Delta$ is the self-adjoint operator on $L^2(\Omega)$ defined by the quadratic form,

$$a(u, v) = -\int \operatorname{grad} u \cdot \operatorname{grad} v \qquad \mathscr{D}(a) = C_0^\infty(\Omega).$$

The general theory asserts that $\mathscr{D}((-\Delta)^{1/2})$ is the closure of $C_0^\infty(\Omega)$ in the norm $\| u \|_{L^2(\Omega)} + (a(u, u))^{1/2}$. This is the Sobolev space $\mathring{H}_1(\Omega)$. In addition $u \in \mathscr{D}(\Delta)$ and $\Delta u = f$ is equivalent to $u \in \mathring{H}_1(\Omega)$ and

$$a(u, \phi) = (f, \phi)_{L^2(\Omega)} \qquad \forall \phi \in C_0^\infty(\Omega).$$

This defines a nonpositive self-adjoint operator on $\Omega$. Thus $\sigma(\Delta)$, its spectrum, lies on the negative real axis.

We define here one basic notion of convergence of domains $\Omega_n$ to $\Omega$, and we study convergence of $\Delta_n$, the Laplacian on $\Omega_n$ with Dirichlet boundary conditions, to $\Delta$.

Definition. $\Omega_n$ converges metrically to $\Omega$ if for any compact set $K \subset \Omega$ (respectively, $K \subset \bar{\Omega}^c$) we have $K \subset \Omega_n$ (respectively, $K \subset \Omega_n{}^c$) if $n$ is sufficiently large.

A special case of metrical convergence is $\Omega_n$ increasing to $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$.

The theorems we shall prove require, for our proof, a mild regularity assumption on $\Omega$, which is the following.

(I) $\mathring{H}_1(\Omega)$ is exactly the set of distributions $u \in H_1(\mathbb{R}^m)$ with supp $u \subset \bar{\Omega}$.

Many sufficient conditions for (I) to hold are known. For example, if $\Omega^c$ has the restricted cone property [1, p. 11] then (I) is satisfied. To prove this merely regularize by a kernel with support in the appropriate cone.

In the interesting case of $\Omega_n \subset \Omega$, this assumption will be unnecessary.

We define the cut off $P_n: L^2(\Omega) \to L^2(\Omega_n)$ by setting $P_n u = u$ on $\Omega \cap \Omega_n$, $P_n u = 0$ on the complement of $\Omega \cap \Omega_n$. As a general convention, we regard elements of $L^2(\Omega)$ and $L^2(\Omega_n)$ as elements of $L^2(\mathbb{R}^m)$, set equal to 0 outside $\Omega$ and $\Omega_n$, respectively.

LEMMA 1.1. *If $\Omega_n$ converges to $\Omega$ metrically and $\Omega$ satisfies* (I), *then* $\|(1-\Delta_n)^{-1}P_n u - (1-\Delta)^{-1}u\|_{H^1(\mathbb{R}^m)} \to 0$ *as* $n \to \infty$ *for all* $u \in L^2(\Omega)$.

*Proof.* Fix $u \in L^2(\Omega)$ and let $v_n = (1-\Delta_n)^{-1}P_n u$ and $v = (1-\Delta)^{-1}u$. $\|v_n\|_{L^2(\Omega_n)} \leqslant \|(1-\Delta_n)^{-1}\| \, \|P_n u\|_{L^2(\Omega_n)} \leqslant \|u\|_{L^2(\Omega)}$ for all $n$ and $\|v\|_{L^2(\Omega)} \leqslant \|u\|_{L^2(\Omega)}$ also. In addition, $\|v_n\|^2_{H_1(\mathbb{R}^m)} = ((1-\Delta_n)v_n, v_n)_{\Omega_n} = (P_n u, v_n) \leqslant \|u\|^2_{L^2(\Omega)}$ with a similar estimate for $v$. Therefore $\{v_n\}$ is precompact in the weak topology of $H_1(\mathbb{R}^m)$. If $w$ is a weak limit point, suppose $v_{n_j} \to w$, then for any $\phi \in C_0^\infty(\Omega)$ we have $\phi \in C_0^\infty(\Omega_n)$ for $n$ large and $a(v_{n_j}, \phi) = (P_{n_j}u, \phi)$. Passing to the limit $j \to \infty$ we get $a(w, \phi) = (u, \phi)$ so $(1-\Delta)w = u$ in $\Omega$. It is also easy to see that supp $w \in \bar{\Omega}$ so by (I) $w \in \mathring{H}_1(\Omega)$ and we have shown that $w = v$. Therefore $v_n$ converges weakly to $v$ in $H_1(\mathbb{R}^m)$. To complete the proof notice that $\|v_n\|^2_{H_1(\mathbb{R}^m)} = (P_n u, v_n) \to (Pu, v) = \|v\|^2_{H_1(\mathbb{R}^m)}$ since $P_n u \to u$ strongly in $L^2(\mathbb{R}^m)$ and $v_n \to v$ weakly in $L^2(\mathbb{R}^m)$. ∎

Using Lemma 1.1 we can investigate the convergence of more general operators $F(\Delta_n)$.

THEOREM 1.2. *Let $\Omega$, $\Omega_n$ be as in Lemma* 1.1. *If $F$ is a bounded Borel function on the negative real axis which is continuous in a neighborhood of $\sigma(\Delta)$, then for all $u \in L^2(\Omega)$, $F(\Delta_n)P_n u \to F(\Delta)u$ in $L^2(\mathbb{R}^m)$.*

*Proof.* The proof we use is not the most direct but it generalizes smoothly to the problems of Sections 5 and 6. It suffices to prove the theorem for real-valued $F$. Let $\Gamma$ be the Banach space of continuous real-valued functions on $(-\infty, 0]$ which vanish at $-\infty$ and let $\mathcal{A}$ be the set of $\Gamma \in \Gamma$ for which the theorem is true. $\mathcal{A}$ is a subalgebra of $\Gamma$ since for $F$, $G \in \mathcal{A}$ we have $F(\Delta_n)G(\Delta_n)P_n u - F(\Delta)G(\Delta)u = [F(\Delta_n)P_n G(\Delta)u - F(\Delta)G(\Delta)u] + F(\Delta_n)[G(\Delta_n)P_n u - P_n G(\Delta)u]$. The first term goes to zero because $F \in \mathcal{A}$ and the second because $G \in \mathcal{A}$. $\mathcal{A}$ is clearly closed in $\Gamma$ and Lemma 1.1 asserts that $f(x) = (1-x)^{-1}$ is in $\Gamma$. Since $f$ separates points of $(-\infty, 0]$, the Stone–Weierstrass theorem implies that $\mathcal{A} = \Gamma$.

If $F$ is a bounded continuous function on $(-\infty, 0]$ it suffices to show that $F(\Delta_n)P_n u \to F(\Delta)u$ for all $u$ in a dense subset of $L_2(\Omega)$, in particular for all $v$ of the form $\exp(\Delta)u$. Now $F(\Delta_n)P_n \exp(\Delta)u = F(\Delta_n)\exp(\Delta_n)P_n u + F(\Delta_n)[P_n \exp(\Delta)u - \exp(\Delta_n)P_n u]$. By the above result the first term converges to $F(\Delta)\exp(\Delta)u$ and the second to zero since $\exp(\Delta_n)P_n u \to \exp(\Delta)u$.

Finally suppose $F$ is a bounded Borel function on $(-\infty, 0]$, continuous on a neighborhood $U$ of $\sigma(\Delta)$. Choose two positive continuous functions $\psi_1, \psi_2$ on $(-\infty, 0]$ with $\psi_1 + \psi_2 \equiv 1$, $\operatorname{supp} \psi_1 \subset U$ and $\psi_1 \equiv 1$ on a neighborhood of $\sigma(\Delta)$. Then $F(\Delta_n) P_n u - (\psi_1 F)(\Delta_n) P_n u + (\psi_2 F)(\Delta_n) P_n u$. $\psi_1 F$ is bounded and continuous so $(\psi_1 F)(\Delta_n) P_n \to (\psi_1 F)(\Delta) u = F(\Delta) u$. On the other hand

$$\|(\psi_2 F)(\Delta_n) P_n u\|_{L^2(\Omega_n)} \leqslant \sup |F| \, \|\psi_2(\Delta_n) P_n u\|_{L^2(\Omega_n)} \to 0$$

since $\psi_2(\Delta_n) P_n u \to \psi_2(\Delta) u = 0$. ∎

As applications of this theorem we first consider mixed problems for the wave and heat equations. The solution of

$$w_t = \Delta_n w \text{ in } \Omega_n, \qquad w = 0 \text{ on } \partial\Omega_n$$

with initial value $w(0) \in L^2(\Omega_n)$ is $e^{t\Delta_n} w(0)$. Therefore the fact that $e^{t\Delta_n} P_n w(0) \to e^{t\Delta} w(0)$ shows that the solutions to the corresponding mixed problems converge as $n \to \infty$. Similar arguments yield convergence of solutions to Schrödinger's equation; $w_t = i\Delta_n w$ in $\Omega_n$, $w = 0$ on $\partial\Omega_n$, which models the motion of a quantum mechanical particle confined, by an infinite potential barrier, to stay in $\Omega_n$.

For the wave equation,

$$w_{tt} = \Delta_n w \text{ in } \Omega_n, \qquad w = 0 \text{ on } \partial\Omega_n, \qquad (1.1)_n$$

with $w(0) = u$, $w_t(0) = v$ prescribed, the solution is

$$w(t) = [(-\Delta_n)^{-1/2} \sin t(-\Delta_n)^{1/2}] v + [\cos t(-\Delta_n)^{1/2}] u, \qquad (1.2)_n$$

so the solutions to the wave equation converge in $L^2$, given initial data in $L^2$.

However, for the wave equation there is a natural norm associated with the physical energy, a norm defined on pairs of functions $\binom{u}{v}$ by the formula

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{E(\Omega)}^2 = \| \operatorname{grad} u \|_{L^2(\Omega)}^2 + \| v \|_{L^2(\Omega)}^2,$$

and the appropriate Hilbert space, $E(\Omega)$, is the closure of $C_0^\infty(\Omega) \oplus L_2(\Omega)$ in this norm. Note that if $\Omega \subset \Omega'$, $E(\Omega) \subset E(\Omega')$, in particular $E(\Omega) \subset E(\mathbb{R}^m)$ for all $\Omega$. Another useful fact is that $E(\Omega) = \mathring{H}_1(\Omega) \oplus L_2(\Omega)$ for bounded domains. The solution to $(1.1)_n$ can be regarded as an operator $U_n^t$ on $E(\Omega_n)$ taking initial data $\binom{u}{v}$ into the solution $\binom{u(t)}{u_t(t)}$. This operator is easily seen to be unitary, reflecting energy conservation for solutions to $(1.1)_n$.

What we would like to have is convergence of solutions to $(1.1)_n$ in energy norm, as $n \to \infty$, given $\binom{u}{v} \in E(\Omega)$. One problem here is that if we simply cut off on $\Omega_n$, $\binom{P_n u}{P_n v}$ may fail to belong to $E(\Omega_n)$. We will content ourselves with proving that $U_n{}^t\binom{u}{v} \to \binom{u}{v}$ in energy norm provided $u$, $v$ are supported in all the regions $\Omega_n$. The next lemma is the key tool.

LEMMA 1.3. *Let $\Omega, \Omega_n, \Delta, \Delta_n, F$ be as in Theorem 1.2. If $u \in C_0{}^\infty(\Omega)$ and $u \in C_0{}^\infty(\Omega_n)$ for all $n$, then for all $\tau \in \mathbb{R}$*

$$(1 - \Delta_n)^\tau F(\Delta_n)u \to (1 - \Delta)^\tau F(\Delta)u \qquad \textit{in} \quad H_1(\mathbb{R}^m).$$

*Proof.* First since $(1 - \Delta_n)^\tau F(\Delta_n)u = (1 - \Delta_n)^{\tau-k} F(\Delta_n)(1 - \Delta)^k u$ for $k \geqslant \tau$, it suffices to prove that $F(\Delta_n)u \to F(\Delta)u$ in $H_1(\mathbb{R}^m)$. We already know the convergence is true in $L^2(\mathbb{R}^m)$ and

$$\begin{aligned} \| \operatorname{grad} F(\Delta_n)u \|_{L^2(\mathbb{R}^m)}^2 &= (\Delta_n F(\Delta_n)\, u, F(\Delta_n)u) = (F(\Delta_n)\, \Delta u, F(\Delta_n)u) \\ &\leqslant \sup |F|\, \| \Delta u \|_{L^2(\mathbb{R}^m)} \| u \|_{L^2(\mathbb{R}^m)}. \end{aligned}$$

To show that $F(\Delta_n)u$ converges weakly to $F(\Delta)u$ in $H_1(\mathbb{R}^m)$ we need only show that for any weakly convergent subsequence $v_j = F(\Delta_{n_j}) \to v$ we must have $v = F(\Delta)u$. Clearly $\operatorname{supp} v \subset \bar{\Omega}$ so $v \in \mathring{H}_1(\Omega)$, in addition for any $\psi \in C_0{}^\infty(\Omega)$ we have for $j$ large $(v_j, \psi)_\Omega = (u, F(\Delta_{n_j})\psi)_{\Omega_{n_j}}$.

By Theorem 1.2 the right-hand side converges to $(u, F(\Delta)\psi) = (F(\Delta)u, \psi)$, proving that $(v, \psi) = (F(\Delta)u, \psi)$ so $v = F(\Delta)u$.

To complete the proof we show that $\| F(\Delta_n)u\|_{H_1(\mathbb{R}^m)} \to \| F(\Delta)u \|_{H_1(\mathbb{R}^m)}$. This follows since $\| \operatorname{grad} F(\Delta_n)u \|^2 = (F(\Delta_n)\, \Delta u, F(\Delta_n)u) \to (F(\Delta)\Delta u, F(\Delta)u) = \| \operatorname{grad} F(\Delta)u \|_{L^2(\mathbb{R}^m)}^2$. ∎

THEOREM 1.4. *Suppose $\Omega_n$, $\Omega$ are as in Theorem 1.2 and $\binom{u}{v} \in E(\Omega)$. If $\binom{u}{v} \subset E(\Omega_n)$ for $n$ large then*

$$\lim_{n\to\infty} \left\| U_n{}^t \begin{pmatrix} u \\ v \end{pmatrix} - U^t \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{E(\mathbb{R}^m)} = 0. \tag{1.3}$$

*The convergence is uniform on compact time intervals.*

*Sketch of proof.* By uniform boundedness it suffices to prove the theorem for $\binom{u}{v} \in C_0{}^\infty(\Omega) \oplus C_0{}^\infty(\Omega)$. For fixed $t$ this is a simple consequence of Theorem 1.2, Lemma 1.3, and $(1.2)_n$. The uniformity follows since if the quantity in (1.2) is $< \epsilon$ then it is $< \epsilon$ for nearby $t$. ∎

As a second sort of application we investigate the spectral properties of the operators $\Delta$ and $\Delta_n$. Let $I \subset \mathbb{R}$ be a bounded open interval whose endpoints do not belong to $\sigma(\Delta)$, and let $\Pi$ and $\Pi_n$ be the spectral projections of $\Delta$ and $\Delta_n$ on this interval. Then $\operatorname{rank} \Pi = \dim(\operatorname{range} \Pi)$ is the number of eigenvalues of $\Delta$ in $I$, counted with multiplicities.

THEOREM 1.5. *With $\Omega$, $\Omega_n$ as in Theorem 1.2, and bounded we have* $\operatorname{rank} \Pi_n = \operatorname{rank} \Pi$, *for $n$ large.*

*Proof.* The proof consists of three steps: for $n$ large,

(i) $\operatorname{rank} \Pi_n P_n \geqslant \operatorname{rank} \Pi$,

(ii) $\operatorname{rank} \Pi_n \leqslant \operatorname{rank} \Pi$,

(iii) $\operatorname{range} \Pi_n = \operatorname{range} \Pi_n P_n$.

To prove (i), let $u_1, \ldots, u_k$ be an orthonormal basis of the range of $\Pi$. By Theorem 1.2 we have $\| \Pi_n P_n u_i - u_i \|_{L^2(\mathbb{R}^m)} < 1/2$ for $n$ large. It follows that $\{\Pi_n P_n u_i\}_{i=1}^k$ is a linearly independent set, for $n$ large, so (i) is established.

To prove (ii), consider both range $\Pi_n$ and range $\Pi$ as subsets of $L^2(\mathbb{R}^m)$. If (ii) fails for arbitrarily large $n$ we may assume, passing to a subsequence if necessary, that $\dim \operatorname{range} \Pi_n > \dim \operatorname{range} \Pi$ for all $n$. Choose $v_n \in \operatorname{range} \Pi_n$ with $\| v_n \|_{L^2(\mathbb{R}^m)} = 1$ and $v_n \perp \operatorname{range} \Pi$. Now $\operatorname{range} \Pi_n \subset \mathscr{D}(\Delta_n) \subset \mathring{H}_1(\Omega_n)$ and $|(\Delta_n v_n, v_n)| \leqslant M \| v_n \|_{L^2(\Omega_n)}$ with $M = \sup_{x \in I} | x |$. Hence the $v_n$ form a bounded set in $H_1(\mathbb{R}^m)$. Thus we may apply Rellich's theorem to get a convergent subsequence $v_n \to v$ in $L^2(\mathbb{R}^m)$, with $v \in \mathring{H}_1(\Omega)$, $\| v \|_{L^2(\Omega)} = 1$, and $v \perp \operatorname{range} \Pi$. We show that $v \in \operatorname{range} \Pi$, giving a contradiction.

Theorem 1.2 implies that $\Pi_n P_n v \to \Pi v$ in $L^2(\mathbb{R}^m)$. In addition we have $\| \Pi_n P_n v_n - \Pi_n P_n v \|_{L^2(\mathbb{R}^m)} \leqslant \| v_n - v \|_{L^2(\mathbb{R}^m)} \to 0$, so $\Pi_n P_n v_n \to \Pi v$ in $L^2(\mathbb{R}^m)$. However since $v_n \in \operatorname{range} \Pi_n \subset L^2(\Omega_n)$, we have $v_n = \Pi_n v_n = \Pi_n P_n v_n + \Pi_n(1 - P_n) v_n$, so $v_n \to \Pi v$. Hence $v = \Pi v \in \operatorname{range} \Pi$, and this contradiction establishes (ii).

To prove (iii), we need the following facts. First, all eigenfunctions of $\Delta_n$ are real analytic, so range $\Pi_n$ consists of real analytic functions on $\Omega_n$. Second, the decomposition $\Omega_n = \bigcup_{j=1}^N \Omega_n{}^j$ (possibly $N = \infty$) of $\Omega_n$ into its connected components yields a natural decomposition of $L^2(\Omega_n)$ which is preserved by $\Delta_n$. If $\Delta_{n,i}$ is the Laplacian on $\Omega_n{}^i$ with Dirichlet boundary conditions we have $\Delta_n = \bigoplus_i \Delta_{n,i}$, and $\sigma(\Delta_n) = \bigcup_i \sigma(\Delta_{n,i})$. The third fact is that for the eigenvalues $\lambda$ of $\Delta_{n,i}$ we have the isoperimetric inequality $-\lambda \geqslant C_m(\operatorname{vol} \Omega_{n,i})^{-2/m}$, so if vol $\Omega_{n,i}$ is sufficiently small then $\Delta_{n,i}$ has no spectrum in a fixed bounded interval $I$. Now, on with the proof.

If (iii) were false, there would exist a nonzero $v \in \text{range}\, \Pi_n$ with $v \perp \text{range}\, \Pi_n P_n$. Thus for all $u \in L^2(\Omega)$, we have

$$0 = (\Pi_n P_n u, v)_{L^2(\Omega_n)} = (P_n u, \Pi_n v) = (P_n u, v).$$

Thus for all $u$ with support in $\Omega \cap \Omega_n$ we have $(u, v)_{L^2(\Omega_n)} = 0$, so $v = 0$ on $\Omega \cap \Omega_n$. Since $v \in \text{range}\, \Pi_n$, it is real analytic on $\Omega_n$, so $v = 0$ on any component of $\Omega_n$ which intersects $\Omega$. Chose $\epsilon > 0$ so that if $\text{vol}\, \Omega_{n,i} \leqslant \epsilon$ then all eigenvalues of $\Delta_{n,i}$ lie outside $I$. It follows that if $v \in \text{range}\, \Pi_n$ and $v = 0$ on all components of $\Omega_n$ of volume greater than $\epsilon$, then $v = 0$. Since $\Omega_n$ converges metrically to $\Omega$ it follows that if $n$ is sufficiently large then $\Omega$ must intersect every component of $\Omega_n$ whose volume is greater than $\epsilon$. Thus if $n$ is large we conclude that $v \equiv 0$. This establishes (iii). ▮

*Remark.* If one wishes to extend these results to equations with variable coefficients which are merely smooth, the well-known unique continuation principle for second-order elliptic operators will allow one to carry through an argument as given above.

## 2. Convergence to a Domain Less a Small Part

In this section we discuss the same sort of problems as in Section 1 except that $\Omega_n$ converges to $\Omega - K$ where $K$ is a closed subset of $\Omega$. An example is the Fireman's pole problem, where $\Omega_n$ is $\Omega$ less a thin cylinder, and $K$ is a line. The criterion for smallness of $K$ will be its Newtonian capacity, as defined in [3]. If $K$ has capacity zero, it is called a *polar set*.

LEMMA 2.1. *$K$ is a polar set if and only if the only element of $H_{-1}(\mathbb{R}^m)$ with support contained in $K$ is the zero element. (Assume $m \geqslant 3$.)*

*Proof.* This lemma is probably familiar to many mathematicians, and a proof is essentially given in [3, p. 88]; we will fill in the details from that argument.

It $K$ has positive capacity, so does some compact subset $K_0$. Hence there is a measure $\mu$ supported by $K_0$ whose potential

$$u(x) = \int_{K_0} \frac{d\mu(y)}{|x - y|^{m-2}}$$

is bounded. Then $\| \operatorname{grad} u \|^2_{L^2(\mathbb{R}^m)} = (\Delta u, u)_{\mathbb{R}^m} = (\mu, u)_{\mathbb{R}^m} < \infty$ so $u \in H_1^{\mathrm{loc}}$, and $\mu = \Delta u \in H_{-1}$.

On the other hand, if $K$ is a polar set, Carleson shows that any compact subset $K_0$ of $K$ is a removable set for the class of harmonic function with finite Dirichlet integral. If $\mu \in H_{-1}$ and supp $\mu \subset K_0$ then $f = \Delta^{-1}\mu \in H_1$ and is harmonic off $K_0$. But if $K_0$ is a set of removable singularities for $f$, then $\Delta f = 0$, so $\mu = 0$. ▮

The reader who is unfamiliar with potential theory can take the conclusion of the previous lemma as a definition of a polar set. In such a case, the following lemma is helpful.

LEMMA. *Suppose there exist functions $\psi_k \in C_0^1(\mathbb{R}^m)$ such that $\psi_k = 1$ on a neighborhood of $K$, $\psi_k \to 0$ in $L^2(\mathbb{R}^m)$, and $\{\psi_k\}$ is bounded in $H_1(\mathbb{R}^m)$. Then $K$ is a polar set.*

*Proof.* Suppose $\mu \in H_{-1}$, supp $\mu \subset K_0$, a compact subset of $K$. We need $\langle \mu, \phi \rangle = 0$ for all $\phi \in C_0^\infty(\mathbb{R}^m)$. But $\langle \mu, \phi \rangle = \langle \psi_k \mu, \phi \rangle = \langle \mu, \psi_k \phi \rangle = \langle \phi\mu, \psi_k \rangle \to 0$ since $\psi_k \to 0$ weakly in $H_1(\mathbb{R}^m)$. ▮

EXAMPLE 1. A finite set of points is a polar set in $\mathbb{R}^m$ for $m \geqslant 2$.

EXAMPLE 2. A simple smooth arc is a polar set in $\mathbb{R}^m$ for $m \geqslant 3$.

*Proof.* It is easy to see that being a closed polar set is a local property that is invariant under diffeomorphisms. Hence we need only consider a closed interval $I = \{(x_1, 0, \ldots, 0): -1 \leqslant x_1 \leqslant 1\}$ in $\mathbb{R}^m$. Let $f \in C_0^\infty(-2, 2)$ be identically 1 on $[-\frac{3}{2}, \frac{3}{2}]$. Let

$$\psi_k(x) = f(x_1) f(k(x_2^2 + \cdots + x_m^2)).$$

One easily verifies that the conditions of the previous lemma are satisfied.

EXAMPLE 3. A compact smooth codimension 2 surface is polar in $\mathbb{R}^m$. (Proof same as above.)

EXAMPLE 4. A codimension 1 surface is not polar. In fact, surface measure on such a surface yields an element of $H_{-1}$ by the trace theorem.

EXAMPLE 5. A polar set must have Lebesgue measure zero. Otherwise it has a compact subset of positive measure and the characteristic function of that set gives the desired element of $H_{-1}$.

The preceding examples are all well known in potential theory. The reason for our interest in polar sets is contained in the following proposition.

PROPOSITION 2.2. *Let $K$ be a closed polar subset of the bounded domain $\Omega$. Let $\Delta$ be the Laplacian on $\Omega$ and $\Delta_1$ the Laplacian on $\Omega - K$, with Dirichlet boundary conditions. Then $(\lambda - \Delta)^{-1} = (\lambda - \Delta_1)^{-1}$ on $L^2(\Omega) = L^2(\Omega - K)$, for $\lambda \in \rho(\Delta) = \rho(\Delta_1)$.*

*Proof.* Given $u \in L^2(\Omega)$, let $v_0 = (\lambda - \Delta)^{-1}u$ and $v_1 = (\lambda - \Delta_1)^{-1}u$ Let $w = v_0 - v_1$. Then $w \in \mathring{H}_1(\Omega)$ and $\operatorname{supp}(\lambda - \Delta)w \subset K$. Since $K$ is polar, this says $(\lambda - \Delta)w = 0$. Hence $w = 0$ if $\lambda \in \rho(\Delta)$. The other assertions of the proposition are very easy to prove. ∎

We can now state the main result of this section.

THEOREM 2.3. *Suppose that $\Omega_n$ converges metrically to $\Omega\backslash K$ and that either $\Omega_n \subset \Omega\backslash K$ or that $\Omega$ satisfies assumption (I) of Section 1. If $K$ is a polar set, then for any Borel function $F$ bounded on the negative axis and continuous on a neighborhood of $\sigma(\Delta)$, we have $F(\Delta_n)\, P_n u \to F(\Delta)u$. in $L^2(\mathbb{R}^m)$ for all $u \in L^2(\Omega)$.*

*Proof.* In case $\Omega_n \subset \Omega\backslash K$ the theorem is an immediate consequence of Theorem 1.2, given the previous proposition. In the other case, we must establish analogues of Lemma 1.1. This done, the proof is the same as that of Theorem 1.2. The proof of Lemma 1.1 survives verbatim except for the proof that $(1 - \Delta)w = u$. We get $w \in \mathring{H}_1(\Omega)$ but only that $(w, \phi) - a(w, \phi) = (u, \phi)\ \forall \phi \in C_0^\infty(\Omega\backslash K)$. It follows that as a distribution $\psi = (1 - \Delta)w - u \in H_{-1}(\mathbb{R}^m)$ and $(\operatorname{supp} \psi \cap \Omega) \subset K$. Since $K$ is polar it follows that $\psi = 0$ in $\Omega$. ∎

Theorem 2.3 combined with Example 2 of a polar set yields a solution of the fireman's pole problem, in the case of Dirichlet boundary conditions.

The direct analogue of Theorems 1.3 and 1.4 carry over to the present situation. The details are left to the reader.

## 3. GENERAL COERCIVE BOUNDARY CONDITIONS

In this section we introduce the notions necessary to carry the previous analysis over to the case of more general boundary conditions. We shall define the self-adjoint extensions of the Laplacian by means of quadratic forms.

For complex-valued functions $u$ (vector-valued functions can be handled similarly) let

$$a_\Omega(u, v) = \int_\Omega \sum_{j,k=1}^{m} a_{jk} D_j u \overline{D_k v}\, dx$$

be a quadratic form such that

$$a_\Omega(u, v) = \int_\Omega (\Delta u)\bar{v} \qquad \text{if} \quad u, v \in C_0^\infty(\Omega).$$

We suppose that $a_{jk}$ are constants. If $\Omega_n$ is a domain in $\mathbb{R}^m$, $H_1(\Omega_n)$ is the closure of the set of smooth functions, $u$, on $\Omega_n$ with finite $H_1$ norm

$$\| u \|_{H_1(\Omega_n)}^2 = \| u \|_{L_2(\Omega_n)}^2 + \sum_{j=1}^{m} \| D_j u \|_{L^2(\Omega_n)}^2 .$$

Let $B_n$ be a closed linear subspace of $H_1(\Omega_n)$ such that $\mathring{H}_1(\Omega_n) \subset B_n \subset H_1(\Omega_n)$. We make the following assumptions on the quadratic form $a_{\Omega_n}(\cdot, \cdot)$:

(i) $a_{\Omega_n}(u, u) \leqslant 0$ if $u \in H_1(\Omega_n)$,

(ii) $-a_{\Omega_n}(u, u) \geqslant C_n \| u \|_{H_1(\Omega_n)}^2 - C_n' \| u \|_{L^2(\Omega_n)}^2$, if $u \in B_n$.

The second of these is a coerciveness condition. It implies that the norm

$$\| u \|_{B_n}^2 = -a_{\Omega_n}(u, u) + \| u \|_{L^2(\Omega_n)}^2 \tag{3.1}$$

is equivalent to the $H_1(\Omega_n)$ norm. From now on we shall take (3.1) as the norm on $B_n$.

We define the negative self-adjoint operator $\Delta_n$ as follows: $u \in \mathscr{D}(\Delta_n)$ if and only if $u \in B_n$ and there is a $g \in L^2(\Omega_n)$ with $a_{\Omega_n}(u, f) = (g, f)_{L^2(\Omega_n)}$ for all $f \in B_n$. In this case define $\Delta_n u = g$. See [8] for the details of this construction.

If $\Omega_n$ is smooth, $B_n$ may be defined by some boundary condition, and then further boundary conditions, known as natural boundary conditions may arise to specify $\mathscr{D}(\Delta_n)$. For example, if each $B_n = H_1(\Omega_n)$, $\mathscr{D}(\Delta_n)$ is said to satisfy Neumann-type boundary conditions. We shall take this as a definition even if $\Omega_n$ is not smooth. Two examples of boundary value problems for the Laplacian acting on functions with values in $\mathbb{R}^3$ that arise in this fashion are, repectively, $E x \nu = 0$, $\operatorname{div} E = 0$ on $\partial\Omega_n$, and $B \cdot \nu = 0$, $\nu x \operatorname{curl} B = 0$ on $\partial\Omega_n$, where $\nu$ is the normal to $\partial\Omega_n$. These are of interest because free electric and magnetic fields traveling in a region bounded by a perfect conductor satisfy the wave equations $(1/c^2)(\partial^2/\partial t^2)E - \Delta E = 0$, $(1/c^2)(\partial^2/\partial t^2)B - \Delta B = 0$, and the above boundary conditions.

We are interested in studying the convergence of functions of $\Delta_n$ as $n \to \infty$. We suppose $\Delta$ is defined on $\Omega$ in a manner as above, with $a_\Omega(\cdot, \cdot)$ a quadratic form on $B \subset H_1(\Omega)$ satisfying hypothesis (i) and (ii). In place of metrical convergence of $\Omega_n$ to $\Omega$ we will require that $\text{meas}(\Omega\backslash\Omega_n) \to 0$ and the following important property.

(II) $\Omega_n \subset \Omega$ and there exist continuous extension maps $E_n$: $B_n \to B$ of uniformly bounded norm, that is, $E_n u \,|_{\Omega_n} = u$ and $\| E_n u \|_B \leqslant M \| u \|_{B_n}$ with $M$ independent of $n$. Furthermore, if $\Omega$ is unbounded then $\Omega - \Omega_n$ is contained in a bounded set independent of $n$.

If $\Omega_n \not\subset \Omega$ then interesting new phenomena can occur. Beale [2] gives some examples involving the Neumann problem. We must also insist on some mild regularity for $\Omega$.

(III) $\partial\Omega$ is compact and Rellich's theorem holds in $\Omega$. That is, if $\{u_n\} \subset H_1(\Omega)$ with $\| u_n \|_{H_1(\Omega)} \leqslant M$ then for any bounded subset $\beta \subset \Omega$ there is a subsequence $u_{n_k}$ convergent in $L^2(\beta)$.

A general sufficient condition for (III) is that $\Omega$ have the restricted cone property. Our last assumption involves the boundary conditions as well as the regions.

(IV) If $u \in B$, $\exists u_j \to u$ in $B$ such that $u_j \,|_{\Omega_j} \in B_j$.

In the case of Neumann boundary conditions $\Omega_n \subset \Omega$ implies that $B \,|_{\Omega_j} \subset B_j$ so (IV) is automatic. For Dirichlet conditions metrical convergence of $\Omega_n$ to $\Omega$ implies (IV) .The main result is the following.

THEOREM 3.1. *Let the regions $\Omega_n$, $\Omega$ and self-adjoint extensions $\Delta_n$, $\Delta$ of the Laplacian satisfy* II, III, *and* IV *above and suppose that F is a bounded Borel function on the negative real axis which is continuous on a neighborhood of $\sigma(\Delta)$. If* $\text{meas}(\Omega\backslash\Omega_n) \to 0$ *then $F(\Delta_n) P_n u \to F(\Delta)u$ in $L^2(\Omega)$ for all $u \in L^2(\Omega)$.*

*Proof.* We must show that $(1 - \Delta_n)^{-1}P_n u \to (1 - \Delta)^{-1}u$ in $L^2(\Omega)$ for all $u \in L^2(\Omega)$. The reasoning of Theorem 1.2 can then be applied. If $v_n = (1 - \Delta_n)^{-1}P_n u$ then $v_n \in B_n$ and

$$\| v_n \|_{B_n}^2 = \| v_n \|_{L^2(\Omega_n)}^2 - a_{\Omega_n}(v_n, v_n) = (P_n u, v_n)_{L^2(\Omega_n)} \leqslant C.$$

Thus $w_n = E_n v_n$ is a bounded sequence in $B$.

We first verify that $w_n$ converges to $(1 - \Delta)^{-1}u$ weakly in $B$. Let $w$ be a weak limit point of $\{w_n\}$. Relabeling if necessary we suppose $w_n \rightharpoonup w$. For $\phi \in B$ choose $\phi_n \to \phi$, $\phi_n|_{\Omega_n} \in B_n$. Thus

$$a_\Omega(w, \phi) = \lim a_\Omega(w_n, \phi_n) = \lim \left\{ a_{\Omega_n}(v_n, \phi_n|_{\Omega_n}) + \int_{\Omega - \Omega_n} a_{ij} D_i w_n \overline{D_j \phi_n} \right\}.$$

Now $a_{\Omega_n}(v_n, \phi_n|_{\Omega_n}) = (v_n - P_n u, \phi_n)_{L^2(\Omega_n)}$ and this converges to $(w, \phi)_{L^2(\Omega)} - (u, \phi)_{L^2(\Omega)}$. On the other hand

$$\left| \int_{\Omega \setminus \Omega_n} \sum a_{ij} D_i w_n \overline{D_j \phi_n} \right| \leqslant c \| w_n \|_{H_1(\Omega)} \| \phi_n \|_{H_1(\Omega \setminus \Omega_n)}$$

which converges to zero since $\operatorname{meas}(\Omega \setminus \Omega_n) \to 0$. It follows that $a_\Omega(w, \phi) = (w, \phi)_{L^2(\Omega)} - (u, \phi)_{L^2(\Omega)}$ for all $\varphi \in B$ so $w = (1 - \Delta)^{-1}u$.

If $\Omega$ is bounded, Rellich's theorem implies that $w_n \to w$ in $L^2(\Omega)$. What we need is $v_n \to w$, but

$$\| v_n - w_n \|^2_{L^2(\Omega)} = \int_{\Omega - \Omega_n} | w_n |^2 \, dx$$

$$\leqslant 2 \int_{\Omega - \Omega_n} | w |^2 \, dx + 2 \int_{\Omega - \Omega_n} | w - w_n |^2 \, dx \to 0,$$

which completes the proof for bounded $\Omega$.

If $\Omega$ is unbounded we must do some additional work. It suffices to consider $u \in C_0^\infty(\Omega)$ as this is a dense subset of $L^2(\Omega)$. Since (III) holds we may choose $\rho > 0$ so that $\Omega_n$ contains $\Gamma = \{x : | x | > \rho/2\}$ for all $n$ and $\operatorname{supp} u \subset \mathbb{R}^m \setminus \Gamma$. Then $(1 - \Delta) w_n = 0$ on $\Gamma$ and $w_n \to w$ weakly in $L^2(\Gamma)$. From the interior estimates of elliptic theory, it follows that on compact subsets of $\Gamma$, $w_n$ converges to $w$ uniformly with all its derivatives.

To apply Rellich's theorem in $\Omega$ to conclude that $w_n \to w$ in $L^2(\Omega)$ what is needed is that for any $\epsilon > 0$. There is an $R > \rho$ with $\| w_n \|_{L^2(|x| > R)} < \epsilon$ for all $n$. This can be proven using the fundamental solution, $G(x)$, of the operator $1 - \Delta$. $G(x)$ is defined by $\hat{G}(\xi) = (2\pi)^{-m/2}(1 + | \xi |^2)^{-1}$ so that $(1 - \Delta)G = \delta$. Furthermore $\widehat{x^\alpha G} = (-i)^{|\alpha|} D_\xi^\alpha (1 + | \xi |^2)^{-1}$ is integrable if $| \alpha | > m - 2$, so

$$x^\alpha G = C_\alpha \int e^{ix \cdot \xi} D_\xi^\alpha (1 + | \xi |^2)^{-1} \, d\xi.$$

In addition the contour of integration can be shifted to $\xi + ia(x/| x |)$ for any $a \in [0, 1)$. In this way we can show that as $| x | \to \infty$ $G =$

$0(e^{-a|x|})$ for all $a \in [0, 1)$. Similarly we can show that $|\nabla G| = 0(e^{-a|x|})$ for $a \in [0, 1)$. The desired smallness of $w_n$ for large $x$ follows from the representation

$$w_n(X) = \int_{|Y|=\rho} \left[ w_n(Y) \frac{\partial G}{\partial r}(X - Y) - G(X - Y) \frac{\partial w_n}{\partial r}(Y) \right] dY.$$

The proof is completed as for bounded domains by showing that $\| w_n \|_{L^2(\Omega \setminus \Omega_n)} \to 0$. ∎

EXAMPLE 1. Let $A_r = \{x \in \mathbb{R}^m: r < |x| < 2r\}$, $B_r = \{x: |x| \leqslant r\}$. If $u \in H_1(A_r)$, we extend $u$ to $e_r u \in H_1(A_r \cup B_r)$ by setting $e_r u$ to be the unique harmonic function $v$ inside $B_r$ which agrees with $u$ on $\partial B_r$. Note that $u|_{\partial B_r} \in H_{\frac{1}{2}}[\partial B_r)$. We claim $\| e_r \|$ is bounded independent of $r$, $0 < r \leqslant 1$, if $m \geqslant 2$.

This one deduces by scaling the following two inequalities.

$$\| v \|^2_{L^2(B_1)} \leqslant C \| u \|^2_{L^2(A_1)} + C \| \nabla u \|^2_{L^2(A_1)}, \tag{3.2}$$

$$\| \nabla v \|^2_{L^2(B_1)} \leqslant C' \| \nabla u \|^2_{L^2(A_1)}, \qquad \text{if} \quad m \geqslant 2. \tag{3.3}$$

In fact the first scales to $\| v \|^2_{L^2(B_r)} \leqslant C \| u \|^2_{L^2(A_r)} + Cr^2 \| \nabla u \|^2_{L^2(A_r)}$ and the second scales to $\| \nabla v \|^2_{L^2(B_r)} \leqslant C' \| \nabla u \|^2_{L^2(A_r)}$.

The first inequality, (3.2), is an immediate consequence of the trace theorem and standard elliptic theory, which also provide the inequality $\| \nabla v \|^2_{L^2(B_1)} \leqslant C \| u \|^2_{H_1(A_1)}$. If (3.3) were false, there would exist $u_n \in H_1(A_1)$ such that $\| \nabla u_n \|_{L^2(A_1)} \leqslant 1/n$ and $\| \nabla v_n \|_{L^2(B_1)} \geqslant 1$. Taking $\alpha_n = 1/\text{vol}\, A_1 \int_{A_1} u_n$, we can show that $\| u_n - \alpha_n \|_{H_1(A_1)} \leqslant c/n$ where $c$ is independent of $n$. The extension of $u_n - \alpha_n$ is given by $v_n - \alpha_n$. Now $\| \nabla v_n \|_{L^2(B_1)} = \| \nabla(v_n - \alpha_n) \|_{L^2(B_1)} \leqslant \| v_n - \alpha_n \|_{H_1(B_1)} \leqslant C \| u_n - \alpha_n \|_{H_1(A_1)} \leqslant (cC/n)$, a contradiction.

Suppose now that $\Omega$ is a region and $x_1, \ldots, x_n$ are points in $\Omega$ with $\text{dist}(x_j, \partial\Omega) \geqslant 2r_n$ and $|x_i - x_j| \geqslant 4r_n$ if $i \neq j$. Let $\Omega_n = \Omega \setminus \{x: |x - x_i| \leqslant r_n$ for some $i\}$. Then II is satisfied. Hypothesis IV is not generally valid, except for the Neumann problem.

Thus we see that for the Neumann problem where $\Omega_n$ consists of $\Omega$ with $n$ well spaced balls removed the obstacles disappear in the limit provided that the total volume $n{r_n}^m \to 0$. In the next section we will see that for general boundary value problems the obstacles fade away if $nr_n^{m-2} \to 0$, with no condition on spacing. If $n{r_n}^m \to 0$ but $nr_n^{m-2}$ does not go to zero, the Dirichlet problem behaves much differently from the Neumann problem, as we will see in the next section and in Section 6.

EXAMPLE 2 (fireman's pole). Let $\Omega = \{(x, y, z) \in \mathbb{R}^3: x^2 + y^2 < 1, |z| < 1\}$ and let $K$ be the $z$ axis, $K_n = \{x, y, z) \in \Omega: x^2 + y^2 \leqslant 1/n\}$, $\Omega_n = \Omega \backslash K_n$. This is the fireman's pole problem in a cylinder. By considering diffeomorphic images, more general fireman's pole problems can be handled. We now show that (II), (IV) are satisfied. As in the previous example we can define an extension operator $e_n$: $H_1(\Omega_n) \to H_1(\Omega)$ by setting $e_n u = v$ in $K_n$ where $\Delta v = 0$ on $\operatorname{int}(K_n)$, $v = u$ on the lateral boundary of $K_n$ and $(\partial v/\partial n) = 0$ if $|z| = 1$. The Neumann condition on $v$ is taken in the generalized sense of quadratic forms. A scaling argument as in Example 1 shows that $\| e_n \|$ is bounded independent of $r$, so (II) is satisfied. Furthermore, $\Omega$ has the restricted cone property so (III) is satisfied.

We want to verify that hypothesis IV is satisfied under very mild additional conditions on the boundary value problems specifying the spaces $B_n$, $B$. What we assume is the following.

(iii) Let $u \in B$. If $\phi \in C^\infty(\mathbb{R}^m)$, then $\phi u \in B$. Furthermore, if $u$ vanishes in a neighborhood of $K_n$, then $u \in B_n$.

Let $\tilde{B}_1{}^*$ be the set of $u \in B$ such that there is a sequence $\{u_j\} \subset B$ with $u_j \to u$ in $B$ and $u_j|_{\Omega_j} \subset B_j$. We must show that $\tilde{B} = B$. Now $\tilde{B}$ is a closed subspace of $B$ so it suffices to show that any element $w \in \mathring{H}_{-1}(\Omega) \equiv H_1(\Omega)'$ which annihilates $\tilde{B}$ must annihilate $B$. By (iii) any member of $C_0^\infty(\Omega \backslash K)$ is in $B_n$ for $n$ large so $\tilde{B} \supset C_0^\infty(\Omega \backslash K)$ so we must have $\operatorname{supp} w \subset \partial\Omega \cup K$. Since $\operatorname{cap} K = 0$ it follows that $\operatorname{supp} w \subset \partial\Omega$ and a second application of (iii) shows that $w$ annihilates all element $u$ of $B$ which vanish in a neighborhood of the end points of $K$. If the set of all such $u$ is called $\tilde{\tilde{B}}$ we must show that $\tilde{\tilde{B}}$ is weakly (and therefore strongly) dense in $B$.

Define smooth functions $\psi_\nu$ to be 0 for $x$ within $1/2\nu$ of these end points, 1 for $x$ a distance greater than $1/\nu$ from the end points, such that $\sup |\psi_\nu| \leqslant 1$ and $\sup |\nabla\psi_\nu| \leqslant C\nu$. Given $u \in B$, let $u_\nu = \psi_\nu u$. Then $u_\nu \in \tilde{\tilde{B}}$ and $u_\nu \to u$ in $L^2(\Omega)$. If we show that $\| u_\nu \|_{H_1(\Omega)}$ is uniformly bounded, it will follow that $u_\nu \to u$ weakly in $B$. Now

$$\| u_\nu \|_{H_1(\Omega)}^2 \leqslant \| \psi_\nu u \|_{L^2(\Omega)}^2 + \| \psi_\nu \nabla u \|_{L^2(\Omega)}^2 + \| u \nabla\psi_\nu \|_{L^2(\Omega)}^2 ,$$

and every term but the last is clearly uniformly bounded. To bound the last term, use the following inequality.

$$\int_{|x|<r} |u|^2 \leqslant C \| u \|_{H_1(\mathbb{R}^3)}^2 r^2 \qquad \text{if} \quad u \in H_1(\mathbb{R}^3), \qquad 0 \leqslant r \leqslant 1. \tag{3.4}$$

This, in turn, is an immediate consequence of the inequality $\| \, |x|^{-1} u \|_{L^2(\mathbb{R}^3)} \leqslant 2 \| \nabla u \|_{L^2(\mathbb{R}^3)}$ which is proven in Courant–Hilbert

[4, p. 446–447]. Note that this argument would fail if we tried to apply it directly to cutoff functions $\psi_\nu$ vanishing in a neighborhood of $K$.

EXAMPLE 3. Let $\Omega$ be the square $\{(x, y): 0 < x < \pi, 0 < y < \pi\}$ and let $\Omega_n = \{(x, y)$: either $0 \leqslant x < (2/3)\pi$ and $0 < y < \pi/n$ or $\pi((2/3) + (1/n)) < x < \pi$, or $2\pi/n < y < \pi\}$, for $n \geqslant 4$. If $\Delta$, $\Delta_n$ are the Laplacians in these regions with Neumann boundary conditions, then every eigenvalue of $\Delta$ is an integer, while $-(3/2)^2$ is an eigenvalue of $\Delta_n$ for all $n \geqslant 4$. Thus we do not have convergence of $(\lambda - \Delta_n)^{-1}$ to $(\lambda - \Delta)^{-1}$. In this example, hypothesis (II) does not hold.

## 4. DOMAINS WITH MANY TINY OBSTACLES

In this section we look at the Laplacian on regions $\Omega_n = \Omega - K_n$ where each $K_n$ is a union of a large number of separate bodies, each of which is small; e.g., $K_n$ could consist of $n$ spheres of radius $r_n$. We look for conditions where the $K_n$ disappear in the limit, given Dirichlet or other boundary conditions on $\partial\Omega_n$. In the case of Dirichlet boundary conditions on $\partial\Omega_n$, we find conditions under which the "clouds" $K_n$ seem to become solid.

DEFINITION. A sequence of closed (in $\mathbb{R}^m$) sets $K_n \subset \Omega$ is said to be *fading* if

$$\mu_n \in \mathscr{E}'(K_n), \|\mu_n\|_{H_{-1}} \leqslant M \Rightarrow \mu_n \to 0 \text{ weakly}.$$

One test that yields fading sequences of sets is given by the following.

LEMMA. *Suppose there exists $\psi_n \in H_1(\mathbb{R}^m)$ such that*

(1) $\psi_n = 1$ *on a neighborhood of $K_n$, and*

(2) $\|\psi_n\|_{H_1} \to 0$ *as $n \to \infty$.*

*Then $\{K_n\}$ is fading.*

*Proof.* If $\mu_n \in \mathscr{E}'(K_n)$, $\|\mu_n\|_{H_{-1}} \leqslant M$, $\phi \in C_0^\infty(\mathbb{R}^m)$, then $|\langle\mu_n, \phi\rangle| = |\langle\mu_n, \psi_n\phi\rangle| \leqslant C\|\mu_n\|_{H_1}\|\psi_n\|_{H_{-1}} \to 0$. Hence $\mu_n \to 0$ weakly. ∎

With the help of this lemma, we can give a precise characterization of fading sequences of compact sets. Let cap$(A)$ denote the capacity of the set $A$.

PROPOSITION 4.1. *Let $K_n \subset K$ be compact sets. Then $\{K_n\}$ is fading if and only if* cap$(K_n) \to 0$ *as $n \to \infty$.*

*Proof.* First, suppose cap$(K_n) \to 0$. Let $K_n'$ be compact neighborhoods of $K_n$ with cap$(K_n') \to 0$. Let $\nu_n$ be the equilibrium measure of $K_n'$; $\nu_n$ is a positive measure supported by $K_n'$ with potential $U_n(x) = \Delta^{-1}\nu_n \leqslant 1$, $U_n = 1$ at the regular points of $K_n'$, (in particular on interior $K_n'$), and $\int d\nu_n = \text{cap}(K_n')$. Note that $U \in L^2_{\text{loc}}$ and $(\Delta U_n, U_n) = \int U_n(x)\, d\nu_n(x) = \text{cap}(K_n')$ so $U_n \in H^1_{\text{loc}}$. Also for bounded sets $s$,

$$\int_s u_n(x)\, dx = \int_s \int d\nu_n(y)/|x - y|\, dx \leqslant (\text{cap}\, K_n')(\sup_y \int_s dx/|x - y|),$$

so $U_n \to 0$ in $L^1_{\text{loc}}$; hence $U_n \to 0$ in $H^1_{\text{loc}}(\mathbb{R}^m)$. Thus if we fix $\phi \in C_0^\infty(\mathbb{R}^m)$, with $\phi = 1$ on a neighborhood of $K$, we can take $\psi_n = \phi U_n$, and the previous lemma shows that $\{K_n\}$ is fading. On the other hand, if $\text{cap}(K_n) \geqslant \alpha > 0$ and if $\nu_n$ is the equilibrium measure of $K_n$, the previous calculation shows that $\nu_n \in H_{-1}$ and $\|\nu_n\|_{H_{-1}} \leqslant M$. But since each $\nu_n$ is a positive measure of mass at least $\alpha$, these measures do not go weakly to 0. ▮

EXAMPLE. Let $K_n$ consist of $n$ balls in $\Omega \subset \mathbb{R}^3$, of radius $r_n$. The capacity of one such ball is $r_n$, so by the subadditivity of capacity $\text{cap}(K_n) \leqslant nr_n$. It follows that $\{K_n\}$ is fading if $nr_n \to 0$.

More generally, if $K_n$ consists of $n$ balls in $\Omega \subset \mathbb{R}^m$, of radius $r_n$, we see that $\{K_n\}$ is fading if $nr_n^{m-2} \to 0$, for $m \geqslant 3$. In case $m = 2$, put $\Omega$ inside a disc $B_R$ of radius $R$, $R \gg \text{diam}\, \Omega$. Let $\Delta^{-1}$ be the Green's operator of this disc with Dirichlet conditions on $|x| = R$, and define capacity as before. Then the capacity of a disc of radius $r$, concentric with $B_r$, is exactly $2\pi/|\log r/R|$, and for any disc of radius $r$ in $\Omega$, this is close to its capacity. Thus in the two dimensional case, $\{K_n\}$ is fading provided $n/|\log r_n| \to 0$.

THEOREM 4.2. *Consider $\Delta_n$, the Laplacian on $\Omega_n = \Omega \backslash K_n$. We pose either Dirichlet boundary conditions, or general coercive boundary conditions as described in Section* 3. *In the latter case we assume $\Omega$ and $\Omega_n$ satisfy hypotheses* II *and* III. *If $\{K_n\}$ is fading, then the conclusions of Theorem* 1.2 *hold.*

*Proof.* As usual it suffices to establish the analog of Lemma 1.1. We verify this for the case of the Dirichlet problem since this illustrates the only new idea. The proof of Lemma 1.1 is followed verbatim except where it is shown that $(1 - \Delta)w = u$ in $\Omega$. Instead we reason that in $\Omega$, $(1 - \Delta)\, v_{n_j} = P_{n_j}u + \mu_{n_j}$ where $\mu_{n_j} \in \mathscr{E}'(K_{n_j})$ and $\|\mu_{n_j}\|_{H_{-1}(\mathbb{R}^m)} \leqslant C$, since the $K_n$ are fading it follows that $\mu_{n_j} \to 0$ so passing to the limit we have $(1 - \Delta)w = u$ in $\Omega$. ▮

The same proof yields removable singularities theorem for harmonic functions.

THEOREM 4.3. *Suppose $K_n \subset K \subset \Omega$ with $K$ compact and* $\operatorname{cap}(K_n) \to 0$. *If $f_n \in H_1(\Omega)$ is a sequence of functions harmonic on $\Omega \backslash K_n$ with $\| f_n \|_{H_1(\Omega)}$ uniformly bounded and if $f_n \to f$ is $\mathscr{D}'(\Omega \backslash K)$ then $f$ has a unique harmonic extension to $\Omega$.*

A similar result is true if we replace the bound on the $H_1$ norm of $f_n$ by $\sup_\Omega | f_n | \leqslant M$; a simple proof uses the maximum principle.

We now consider the opposite case. Let $K_n$ be a sequence of compact subsets of $\Omega_0$, an open subset of $\Omega$. The smallest eigenvalue of the Laplacian on $\Omega_0 \backslash K_n$ with Dirichlet conditions on $\partial K_n$ and Neumann conditions on $\partial \Omega_0$ is denoted $\alpha_n$. Then

$$\alpha_n = \inf \frac{\int_{\Omega_0 \backslash K_n} | \nabla u |^2}{\int_{\Omega_0 \backslash K_n} u^2}$$

where the infimum is over all $u \in H_1(\Omega_0 \backslash K_n)$ which vanish on $\partial K_n$ in the usual generalized sense.

DEFINITION. $K_n$ becomes solid in $\Omega_0$ if $\alpha_n \to \infty$.

The reason for this name is the following theorem.

THEOREM 4.4. *Let $\Delta_n$ denote the Laplacian on $\Omega \backslash K_n$ and $\Delta$ the Laplacian on $\Omega \backslash \bar{\Omega}_0$ both with Dirichlet boundary conditions. Suppose that $\Omega \backslash \bar{\Omega}_0$ satisfies assumption* I *of Section* 1 *and that $F$ is a bounded Borel function on the negative real axis which is continuous on a neighborhood of $\sigma(\Delta)$. If $\{K_n\}$ becomes solid in $\Omega_0$ then $F(\Delta_n) P_n u \to F(\Delta) u$ in $L_2(\mathbb{R}^m)$ for all $u \in L^2(\Omega \backslash \bar{\Omega}_0)$.*

*Proof.* Again, the goal is to obtain an analog of Lemma 1.1.

The proof of that lemma must only be modified where $w$ is identified. We must show (1) $w \in \mathring{H}_1(\Omega \backslash \bar{\Omega}_0)$ and (2) $(1 - \Delta) w = u$ on $\Omega \backslash \bar{\Omega}_0$. Of these (2) presents no difficulties; it is (1) which makes use of the assumption that $\{K_n\}$ becomes solid. We have $\| v_n \|_{\mathring{H}_1(\Omega \backslash K_n)} \leqslant M$ so

$$\begin{aligned} \int_{\Omega_0} | w |^2 &\leqslant \liminf \int_{\Omega_0} | v_n |^2 \\ &\leqslant \liminf \frac{1}{\alpha_n} \int_{\Omega_0 \backslash K_n} | \nabla v_n |^2 \leqslant \lim \frac{M}{\alpha_n} = 0. \end{aligned}$$

Thus $w$ must vanish on $\Omega_0$ so using (I) we have $w \in \mathring{H}_1(\Omega \backslash \bar{\Omega}_0)$. ∎

EXAMPLE. Suppose $K_n$ consists of $n$ balls of radius $r_n$, evenly spaced inside a region $\Omega_0 \subset \mathbb{R}^3$. We claim that $K_n$ becomes solid if $n r_n \to \infty$. Contrast this with the fact that $\{K_n\}$ is fading if $n r_n \to 0$.

To verify this claim, cover $\Omega_0$ with balls of radius $\rho_n$, with centers at the center of the little balls making up $K_n$. Take $\rho_n{}^3 = \text{const}/n$, so that no point of $\Omega_0$ is covered more than a fixed number of times, independent of $n$. The claim made above is then an easy consequence of the following.

ASSERTION. If $\mathcal{O} = \{x \in \mathbb{R}^3: r_n \leqslant |x| \leqslant \rho_n\}$, then

$$\frac{\int_{\mathcal{O}} |\nabla u|^2\, dx}{\int_{\mathcal{O}} |u|^2\, dx} \geqslant C\frac{r_n}{\rho_n{}^3}$$

for any smooth $u$ which vanishes on $\{x: |x| = r_n\}$. (Assume $r_n \leqslant 1/2\rho_n$.)

Since the minimum of the above quotient occurs at a function $u$ which is rotationally symmetric one can show that the above assertion is equivalent to the assertion that for $r \leqslant 1/2\rho$,

$$\frac{\int_r^\rho |u'(t)|^2\, t^2\, dt}{\int_r^\rho |u(t)|^2\, t^2\, dt} \geqslant C\frac{r}{\rho^3}$$

for any smooth $u$ on $[r, \rho]$ with $u(r) = 0$. The problem of estimating this quotient can be translated into a problem involving Bessel functions. Fortunately, however, the estimate we want to obtain is an immediate consequence of the following simple inequality.

LEMMA 4.5. *If $\phi \in \rho[r, \rho]$ and $\phi > 0$, then*

$$\frac{\int_r^\rho |f'(t)|^2 \phi(t)\, dt}{\int_r^\rho |f(t)|^2 \phi(t)\, dt} \geqslant \left(\int_r^\rho \phi(t)\, dt\right)^{-1} \left(\int_r^\rho \frac{1}{\phi(t)}\, dt\right)^{-1}$$

*for all smooth $f$ on $[r, \rho]$ with $f(r) = 0$.*

*Proof.* Suppose $\int_r^\rho |f(t)|^2 \phi(t)\, dt = 1$. Then there is a $t_0$ such that $|f(t_0)|^2 \geqslant (1/\Phi)$, where we set $\Phi = \int_r^\rho \phi(t)\, dt$. Hence $\int_r^\rho |f'(t)|\, dt \geqslant 1/(\Phi)^{1/2}$. Applying Cauchy's inequality, we get

$$\int_r^\rho \phi(t)\, |f'(t)|^2\, dt \geqslant \frac{[\int_r^\rho |f'(t)|\, dt]^2}{\int_r^\rho (1/\phi(t))\, dt} \geqslant \frac{1}{\Phi \int_r^\rho (1/\phi(t))\, dt}. \quad \blacksquare$$

The previous example can be generalized to higher dimensions. Thus if $K_n$ consists of $n$ evenly spaced balls of radius $r_n$ in a region $\Omega_0$ in $\mathbb{R}^m$, we see that $K_n$ becomes solid if $nr_n^{m-2} \to \infty$, provided $m \geqslant 3$. In case $m = 2$, the above analysis shows that $K_n$ becomes solid if $n/|\log r_n| \to \infty$.

We end this section with a treatment of a slightly more delicate problem. Here we will suppose $\Omega$ is a bounded region in $\mathbb{R}^3$, $S$ a smooth compact surface, $S \subset \Omega$, and we shall take $K_n$ to consist of $n$ balls, of radius $r_n$, which centers evenly spaced on $S$. We know that, if $nr_n \to 0$, then $K_n$ fades; now we shall see that if $nr_n \to \infty$, $K$ tends to "solidify" to the surface $S$. We consider only the Dirichlet problem.

Let $\Delta_n$ be the Laplacian on $\Omega - K_n$, and let $\Delta$ be the Laplacian on $\Omega \backslash S$, each with Dirichlet boundary conditions, and let $P_n$: $L^2(\Omega) \to L^2(\Omega \backslash K_n)$ be the usual restriction map. In addition, suppose that $\Omega$ satisfies assumption (I) of Section 1. Suppose $nr_n \to \infty$.

THEOREM 4.6. *If $F$ is a bounded Borel functions defined on the negative real axis and continuous on a neighborhood of $\sigma(\Delta)$, then*

$$F(\Delta_n)\, P_n u \to F(\Delta) u \text{ in } L^2(\Omega) \qquad \text{for all} \quad u \in L^2(\Omega).$$

*Proof.* As usual it is the analog of Lemma 1.1 which must be established. The proof that $w \in \mathring{H}_1(\Omega)$, $(1 - \Delta)w = u$ in $\Omega \backslash S$ goes exactly as before. What remains to be shown is that $w \in \mathring{H}_1(\Omega \backslash S)$, that is, that $w\,|_S = 0$.

If $K_n$ consists of $n$ balls $B_{n,j}$ with center at $\xi_{n,j} \in S$, $j = 1,..., n$, let $\Gamma_{n,j}^{\rho_n,H}$ be the circular cylinder with center at $\xi_{n,j}$, axis normal to $S$, radius $\rho_n$, and height $H$. We suppose that $\rho_n{}^2 = \text{const}/n$, so that $\bigcup_{j=1}^n \Gamma_{n,j}^{\rho_n,H}$ covers the set $S_H = \{x \in \Omega\colon \text{dist}(x, S) \leqslant H/4\}$, at least for $H$ small, but covers no point in $\Omega$ more then a fixed number of times, independent of $n$. We will establish the following inequality, where $\mathcal{O} = \Gamma_{n,j}^{\rho_n,H} - B_{n,j}$:

$$\frac{\int_{\mathcal{O}} |\nabla u|^2\, dx}{\int_{\mathcal{O}} |u|^2\, dx} \geqslant \frac{C}{H(H + \rho_n{}^2/r_n)}, \tag{4.1}$$

for all smooth $u$ on $\mathcal{O}$, vanishing on $\partial B_{n,j}$ (assuming $\rho_n \ll H$). Granted (4.1), we see that the limit function $w$ satisfies the inequality

$$\int_{S_H} |w|^2\, dx \leqslant C'H^2.$$

Since the volume of $S_H$ is roughly proportional to $H$, the fact that $w \in \mathring{H}_1(\Omega \backslash S)$ is an immediate consequence of this inequality and the trace theorem.

It remains only to establish (4.1). To do this we will replace $\nabla u$ by $Xu$, where $X$ is a properly chosen vector field of unit length on $\mathcal{O}$. We will then reduce (4.1) to an estimate along the integral curves of $X$, where we can apply Lemma 4.5. The problem is to choose $X$ so that the *same* estimate is obtained along *each* such integral curve. We choose the vector field $X$ to be approximately normal to the level surfaces of what one would guess to be the eigenfunction for which the left hand side of (4.1) assumes its minimum. To do that, we suppose a coordinate system is given so that

$$\mathcal{O} = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 \leqslant \rho_n^2, |z| \leqslant (1/2) H, x^2 + y^2 + z^2 \geqslant r_n^2\}.$$

Consider the subregion

$$\mathscr{V} = \Big\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + \left(z - \frac{3\rho_n}{4}\right)^2 \leqslant \frac{25}{16} \rho_n^2,$$
$$x^2 + y^2 + \left(z + \frac{3\rho_n}{4}\right)^2 \leqslant \frac{25}{16} \rho_n^2,$$
$$\text{and } x^2 + y^2 + z^2 \geqslant r_n^2\Big\}.$$

We define $X$ to be the radial derivative $\partial/\partial r$ on $\mathscr{V}$ and (sign $z$) $\partial/\partial z$ on $\mathcal{O}\backslash\mathscr{V}$. Thus the integral curves of $X$ are broken lines, best described by the following figure.

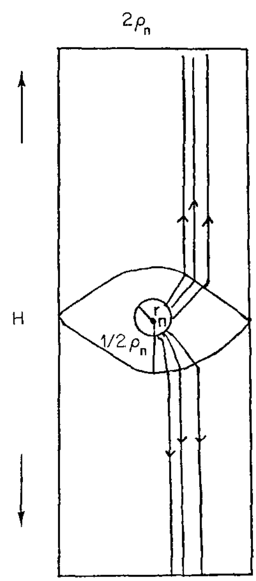


FIGURE 1

Along each integral curve of $X$, the quantity we have to estimate from below is

$$\frac{\int_{r_n}^{H_1} \phi(t)\,|f'(t)|^2\,dt}{\int_{r_n}^{H_1} \phi(t)\,|f(t)|^2\,dt}, \tag{4.2}$$

where

$$\phi(t) = \begin{cases} ct^2 & r_n \leqslant t \leqslant \rho \\ \rho^2 & \rho \leqslant t \leqslant H_1, \end{cases}$$

and $H_1 \approx H$, $C \approx 1$, $(1/2)\,\rho_n \leqslant \rho \leqslant \rho_n$. Here $\phi(t)$ is so chosen that if we parametrize the integral curves of $X$ by points $(x, y)$ in the disc of radius $\rho_n$ (the end points of the integral curves) and then parametrize points in $\mathcal{O}$ by $(x, y)$ and arc length along an integral curve, then $(1/\rho_n{}^2)\,\phi(t)\,dx\,dy\,dt$ is an element of volume in $\mathcal{O}$, comparable to Lebesgue measure.

If (4.2) can be estimated from below (for $\rho \ll H_1$ and $r_n \ll \rho$) by $C/H(H + (\rho^2/r_n))$ then these estimates can be put together to yield (4.1). However, this estimate follows from Lemma 4.5. Our proof is complete. ∎

This result stands in striking contrast to the result obtained for the Neumann problem. In fact if $K_n$ consists of $n$ balls $B_{n,j}$ with centers $\xi_{n,j} \in S$ and radius $r_n \leqslant \frac{1}{4} \min_{j \neq k} \operatorname{dist}(\xi_{n,j}, \xi_{n,k})$, the balls are well spaced and $\operatorname{vol}(K_n) \to 0$ as $n \to \infty$. Hence by the results of Section 3 the obstacles $K_n$ tend to have negligible effect for the Neumann problem, even though they might solidify for the Dirichlet problem, if evenly spaced.

## 5. Scattering Theory

In this section we study scattering by obstacles which fade or become solid. Similar methods could be applied to converging domains as in Section 1, in fact to any sequence of obstacles contained in a fixed compact set for which Theorem 1.2 is valid. Let $\Omega = \mathbb{R}^m \backslash \mathcal{O}$ and $\Omega_n = \Omega \backslash K_n$ when $\mathcal{O}$ and $K_n$ are closed and contained in a fixed compact set. Let $U_n{}^t$ be the unitary groups in $\mathcal{E}(\Omega_n)$ which yields the solution to the wave equations with Dirichlet boundary conditions. We treat two problems simultaneously. From now on, $m > 1$.

(1) $K_n$ is fading. In this case $U^t$ is the group associated with the wave equation in $\Omega$ with Dirichlet boundary conditions.

For this problem we could also treat more general coercive boundary conditions.

(2) $K_n$ becomes solid in $\Omega_0$. Here $U^t$ is the group on $\mathscr{E}(\Omega\backslash\bar{\Omega}_0)$ with Dirichlet condition on $\partial(\Omega\backslash\bar{\Omega}_0)$. We assume that $\Omega\backslash\bar{\Omega}_0$ satisfies (I).

We are interested in similarities of behavior in $U_n{}^t$ and $U^t$ as $t \to \pm\infty$. Let $V^t$ be the solution operator to the free-space wave equation. Lax and Phillips [9, 10] have shown that there exist isometric operators, called wave operators, $W^{\pm}: \mathscr{E}(\mathbb{R}^m) \to \mathscr{E}(\Omega)$ such that $\| U^t W^{\pm}\phi - V^t\phi \|_{\mathscr{E}(\mathbb{R}^m)} \to 0$ as $t \to \pm\infty$ for each $\phi \in \mathscr{E}(\mathbb{R}^m)$. Similarly wave operators $W_n{}^{\pm}: \mathscr{E}(\mathbb{R}^m) \to \mathscr{E}(\Omega_n)$ are defined. Using local energy decay of solutions $U^t\phi$, Lax and Phillips further show that $W_n{}^{\pm}$ is surjective, hence unitary. Thus we have

$$\| U^t\psi - V^t(W^{\pm})^{-1}\psi \|_{\mathscr{E}(\Omega)} \to 0 \qquad \text{as} \quad t \to \pm\infty \tag{5.1}$$

for each $\psi \in \mathscr{E}(\Omega)$. Using this, one easily derives the following useful refinement of energy decay.

LEMMA 5.1. *For any $\epsilon > 0, \phi \in \mathscr{E}(\Omega)$, there is a cone $| x | < t - T_0$ in $(t, x)$ space and a $T$ such that*

$$\| U^t\phi \|_{\mathscr{E}(\{|x|<t-T_0\})} < \epsilon \qquad \textit{if} \quad t \geqslant T.$$

*Proof.* By (5.1), it is sufficient to remark that the lemma is valid for the free space solution operator $V^t$. ∎

We can now state and prove our first result.

THEOREM 5.2. *If $\phi \in \mathscr{E}(\Omega)$ and $\phi \in \mathscr{E}(\Omega_n)$ for all large $n$, then*

$$\| U_n{}^t\phi - U^t\phi \|_{\mathscr{E}(\mathbb{R}^m)} \to 0 \qquad \textit{as} \quad n \to \infty.$$

*The convergence is uniform in $t$.*

*Proof.* Using the ideas of Theorem 1.3, one can deduce uniform convergence on compact time intervals. What is new here is the uniformity for all $t$. We consider the case $t \geqslant 0$. It suffices to prove that for any $\epsilon > 0$ there is a $T > 0$, and $n_0$ such that

$$\| U_n{}^t\phi - U^t\phi \|^2_{\mathscr{E}(\mathbb{R}^m)} < \epsilon \qquad \text{for} \quad t \geqslant T, \quad n \geqslant n_0 .$$

Given $\epsilon > 0$, choose $r > 0$ and $T$ so that the obstacles $\mathcal{O}$ and $K_n$ are all contained in the ball $B = \{x: |x| < r\}$, and

$$\| U^t\phi \|^2_{\mathscr{E}(\{x: |x| < r+(t-T)\})} < \epsilon \qquad \text{for} \quad t \geqslant T.$$

Choose $n_0$ so that, for $n \geqslant n_0$, $\| U^T\phi - U_n{}^T\phi \|^2_{\mathscr{E}(\mathbb{R}^m)} < \epsilon$; in particular the energy norm over $\mathbb{R}^m - B$ is less than $\epsilon$.

Using finite signal speed and the fact that $\mathbb{R}^m - B$ is free of obstacles, we see that for $t \geqslant T$ the values of $U_n{}^t\phi$ and $U^t\phi$ in the region $|x| > r + (t - T)$ are determined by their values at time $T$ in $\mathbb{R}^m - B$. Therefore, $\| U_n{}^t\phi - U^t\phi \|^2_{\mathscr{E}(|x|>r+(t-T))} < \epsilon$ for $t \geqslant T$ and $n \geqslant n_0$. The proof is completed by showing that

$$\| U_n{}^t\phi \|^2_{\mathscr{E}(|x|<r+(t-T))} < \epsilon + 2 \| \phi \|_{\mathscr{E}(\mathbb{R}^m)} \epsilon^{1/2}$$

for the same values of $t$, $n$, since this implies that

$$\| U_n{}^t\phi - U^t\phi \|^2_{\mathscr{E}(\mathbb{R}^m)} < 5\epsilon + 4 \| \phi \|_{\mathscr{E}(\mathbb{R}^m)} \epsilon^{1/2} \qquad \text{for} \quad t \geqslant T, \quad n \geqslant n_0 .$$

Now

$$\begin{aligned} \| \phi \|^2_{\mathscr{E}(\mathbb{R}^m)} &= \| U_n{}^t\phi \|^2_{\mathscr{E}(\mathbb{R}^m)} \\ &= \| U_n{}^t\phi \|^2_{\mathscr{E}(|x|<r+(t-T))} + \| U_n{}^t\phi \|^2_{\mathscr{E}(|x|>r+(t-T))} \\ &\geqslant \| U_n{}^t\phi \|^2_{\mathscr{E}(|x|<r+(t-T))} + \| U^t\phi \|^2_{\mathscr{E}(|x|>r+(t-T))} - 2 \| \phi \|_{\mathscr{E}(\mathbb{R}^m)} \epsilon^{1/2} \\ &\geqslant \| U_n{}^t\phi \|^2_{\mathscr{E}(|x|<r+(t-T))} + \| U^t\phi \|^2_{\mathscr{E}(\mathbb{R}^m)} - \epsilon - 2 \| \phi \|_{\mathscr{E}(\mathbb{R}^m)} \epsilon^{1/2}. \end{aligned}$$

Here we have used the inequalities

$$\| a \|^2 - \| b \|^2 \leqslant (\| a \| + \| b \|) \| a - b \|$$

and

$$\| a - b \|^2 \leqslant 2 \| a \|^2 + 2 \| b \|^2.$$

The fact that $\| U^t\phi \|^2_{\mathscr{E}(\mathbb{R}^m)} = \| \phi \|^2_{\mathscr{E}(\mathbb{R}^m)}$ completes the proof. ∎

Next we prove convergence of the wave operators $W_n{}^{\pm}$ and scattering operators $S_n = (W_n{}^+)^{-1} W_n{}^-$ for the groups $U_n{}^t$. Though this result also deals with behavior of $U_n{}^t$ for $t$ large, notice that it is somewhat different from the point of view of Theorem 5.2. Here we are given a free solution $V^t\Phi$ for $\Phi \in \mathscr{E}(\mathbb{R}^m)$ and seek for instance, $\Psi_n = W_n{}^+\Phi$ with $V^t\Phi \sim U_n{}^t\Psi_n$ for $t \to +\infty$. Theorem 5.2 does not directly give information about the convergence of $W_n{}^{\pm}$ to $W^{\pm}$, but the proofs are similar in spirit. $S = (W^+)^{-1}W^-$ is the scattering operator for $U^t$.

THEOREM 5.3. *For any $\phi \in \mathscr{E}(\mathbb{R}^m)$ we have, as $n \to \infty$,*

$$\| W_n^{\pm}\phi - W^{\pm}\phi \|_{\mathscr{E}(\mathbb{R}^m)} \to 0$$

*and*

$$\| S_n\phi - S\phi \|_{\mathscr{E}(\mathbb{R}^m)} \to 0.$$

*Proof.* We treat $W^+$ first; $W^-$ follows similarly. Suppose the obstacles $\mathcal{O}$, $K_n$ are contained in the ball, $B_r(0)$, of radius $r$ and center 0.

The proof is easier if $m$ is odd, for then we can appeal to Huyghens' principle. In this case, fix $\phi \in C_0^{\infty}(\mathbb{R}^m)$ and use the formula $W_n^+\phi = \lim_{t\to\infty} U_n^{-t}V^t\phi$. Pick $T > 0$ so large that $V^T\phi$ is outgoing and has support disjoint from $B_r(0)$. Thus $U_n^{-t}V^tV^T\phi = V^T\phi$ for all $t > 0$, so $W_n^+\phi = U_n^{-T}V^T\phi \to U^{-T}V^T\phi = W^+\phi$.

If $m$ is even, Huyghens' principle fails, and we must modify our argument.

Given $\phi \in \mathscr{E}(\mathbb{R}^m)$ and $\epsilon > 0$, choose $T > 0$ so that

$$\| V^t\phi \|^2_{\mathscr{E}(|x|<r+(t-T))} < \epsilon \qquad \text{for} \quad t \geqslant T.$$

Smoothly cut off $V^T\phi$ to produce $\psi$ so that $\psi = V^T\phi$ outside $B_r(0)$ and $\| \psi \|^2_{\mathscr{E}(B_r(0))} < c\epsilon$. By finite speed we have $U_n{}^t\psi = V^t\psi = V^{t+T}\phi$ for $| x | > r + (t - T)$ and also $\| U_n{}^t\psi \|^2_{\mathscr{E}(\mathbb{R}^m)} = \| \psi \|^2_{\mathscr{E}(\mathbb{R}^m)} \leqslant \| \phi \|^2_{\mathscr{E}(\mathbb{R}^m)} + c\epsilon$. Thus since $\| V^t\psi \|^2_{\mathscr{E}(|x|>r+t)} > \| \phi \|^2_{\mathscr{E}(\mathbb{R}^m)} - \epsilon$ we have $\| U_n{}^t\psi \|^2_{\mathscr{E}(|x|<r+t)} < c'\epsilon$ for all $n$ and $t \geqslant 0$. Therefore

$$\| U_n^{t+T}U^{-T}\psi - V^{t+T}\phi \|^2_{\mathscr{E}(\mathbb{R}^m)} < c\epsilon$$

for all $n$ and $t \geqslant 0$ since the two terms agree for $| x | > r + t$ and have energy $\mathcal{O}(\epsilon)$ in $| x | < r + t$. It follows that $\| W_n^+\phi - U_n^{-T}\psi \|^2_{\mathscr{E}(\mathbb{R}^m)} < c\epsilon$ and similarly $\| W^+\phi - U^{-T}\psi \|^2_{\mathscr{E}(\mathbb{R}^m)} < c\epsilon$. But, since $U_n^{-T}\psi \to U^{-T}\psi$ as $n \to \infty$, we are done.

Finally, we prove that $S_n$ converges strongly to $S$. The operator $W_n^-$ is a unitary map of $\mathscr{E}(\mathbb{R}^m)$ into $\mathscr{E}(\Omega_n)$. It follows that $(\eta, W_n^-\psi)_{\mathscr{E}(\Omega_n)} = ((W_n^-)^{-1}\eta, \psi)_{\mathscr{E}(\mathbb{R}^m)}$ for all $\psi \in \mathscr{E}(\mathbb{R}^m)$, $\eta \in \mathscr{E}(\Omega_n)$. Thus if $\phi, \psi \in \mathscr{E}(\mathbb{R}^m)$ we have

$$\begin{aligned}(S_n\phi, \psi)_{\mathscr{E}(\mathbb{R}^m)} &= ((W_n^-)^{-1}W_n^+\phi, \psi)_{\mathscr{E}(\mathbb{R}^m)} \\ &= (W_n^+\phi, W_n^-\psi)_{\mathscr{E}(\mathbb{R}^m)} \\ &\to (W^+\phi, W^-\psi)_{\mathscr{E}(\mathbb{R}^m)} = (S\phi, \psi)_{\mathscr{E}(\mathbb{R}^m)} .\end{aligned}$$

This shows that $S_n$ converges weakly to $S$ however a weakly convergent sequence of isometries is strongly convergent. ∎

We can also prove convergence of the scattering matrices, in the strong operator topology, and the transmission coefficients. Details are omitted.

## 6. A Diffusion Theory Approach

In this section we deduce the convergence properties of functions of operators $\Delta_n$ like those studied before from the convergence of the solution operators $e^{t\Delta_n}$ to the heat equations. Two powerful tools available in the study of the heat equation are the Wiener integral representation of the solution, and the probabilistic approach to potential theory. We use these to obtain a refinement of some of the results of Section 4. We begin with some functional analysis.

Proposition 6.1. *Let $A_n$, $A$ be positive (perhaps unbounded) self-adjoint operators on a Hilbert space $H$. Suppose $e^{-tA_n} \to e^{-tA}$ strongly for some $t > 0$. If $F$ is any bounded Borel function on the positive real axis which is continuous in a neighborhood of $\sigma(A)$ then $F(A_n) \to F(A)$ strongly.*

*Proof.* The proof of this result is virtually identical to the proof of Theorem 1.2 with $e^{-tx}$ taking the place of $(1 - x)^{-1}$. ∎

To take one example of semigroups which arise as above, let $\Omega_n = \Omega \backslash K_n$ where $\Omega$ is an open region in $\mathbb{R}^m$, $K_n$ a compact subset, and let $\Delta_n$ be the Laplacian on $\Omega_n$ with Dirichlet boundary conditions. Then $e^{t\Delta_n}$ is a semigroup of positive operators, but for each $n$ it is only defined in a subspace $L^2(\Omega \backslash K_n)$ of $L^2(\Omega)$. However, we can get a semigroup to which we can apply the above lemma by a trick.

If $P_n: L^2(\Omega) \to L^2(\Omega \backslash K_n)$ is the restriction map, then let $Q_n{}^t = e^{t\Delta_n}P_n + (1 - P_n)$. Now $Q_n{}^t$ is a strongly continuous semigroup of positive operators on $L^2(\Omega)$, with $Q_n{}^t \to I$ strongly as $t \to 0$. If $\text{meas}(K_n) \to 0$ as $n \to \infty$ and if $e^{t\Delta_n}P_n \to e^{t\Delta}$ strongly as $n \to \infty$, then $1 - P_n \to 0$ strongly, so $Q_n{}^t$ converges strongly to $e^{t\Delta}$.

Let us recall a couple of facts about $e^{t\Delta_n}$. It is given by a kernel $p_n(x, y, t)$; $e^{t\Delta_n}u(x) = \int_{\Omega_n} p_n(x, y, t)\, u(y)\, dy$. If we extend $p_n(x, y, t)$ to $\Omega \times \Omega \times \mathbb{R}^+$ by setting it equal to 0 if $x$ or $y$ belongs to $K_n$, then $p_n(x, y, t)$ is the kernel of $e^{t\Delta_n}P_n$ on $L^2(\Omega)$. Furthermore, we have the domination

$$p_n(x, y, t) \leqslant p_0(x, y, t)$$

where $p_0(x, y, t) = (1/(2\pi t)^{m/2})\, e^{-(x-y)^2/2t}$ is the free space fundamental solution of the heat equation. From this, the following is a simple consequence of the Lebesgue dominated convergence theorem.

PROPOSITION 6.2. *Suppose* $p_n(x, y, t) \to p(x, y, t)$ *a.e. on* $\Omega \times \Omega$, *for some* $t > 0$. *Then* $e^{t\Delta_n} P_n \to e^{t\Delta}$ *strongly on* $L^2(\Omega)$.

We shall make use of the representation of the kernel $p(x, y, t)$ of $e^{t\Delta}$, where $\Delta$ is the Laplacian on an open set $\Omega$ with Dirichlet boundary conditions, as a Wiener integral. The formula is

$$p(x, y, t) = E_x\{\Xi_\Omega(w, t) : w(t) = y\}\, p_0(x, y, t).$$

Here $E_x\{\cdot \mid w(t) = y\}$ is the conditional Wiener integral over the set of Brownian paths on $[0, t]$ with endpoints at $x$ and $y$, and $\Xi_\Omega(\cdot, t)$ is a function defined on the set of Brownian paths by

$$\Xi_\Omega(w, t) = \begin{cases} 1 & \text{if } \ w(\tau) \in \Omega \quad \text{for } \ 0 < \tau \leqslant t, \\ 0 & \text{otherwise.} \end{cases}$$

To see these tools in action, we now study the kernels for $e^{t\Delta_n}$ where $\Delta_n$ is given by Dirichlet boundary condition on $\partial(\Omega - K_n)$, and $\{K_n\}$ is a fading sequence of sets. By Proposition 6.1, this reestablishes part of Theorem 4.2.

THEOREM 6.3. *Let* $\Omega_n = \Omega \backslash K_n$ *be regions in* $\mathbb{R}^m$, *and let* $\Delta_n$ *be defined by Dirichlet boundary conditions on* $\partial\Omega_n$. *If* $\operatorname{cap}(K_n) \to 0$ *as* $n \to \infty$, *then* $e^{t\Delta_n} P_n \to e^{t\Delta}$ *strongly.*

*Proof.* To compare the kernels $p_n(x, y, t)$ and $p(x, y, t)$ of these operators, we use their Wiener integral representations. Thus

$$p_n(x, y, t) = E_x\{\Xi_{\Omega\backslash K_n}(w, t) \mid w(t) = y\}\, p_0(x, y, t),$$
$$p(x, y, t) = E_x\{\Xi_\Omega(w, t) \mid w(t) = y\}\, p_0(x, y, t).$$

Taking the difference and applying Cauchy's inequality,

$$\begin{aligned} &| p_n(x, y, t) - p(x, y, t)| \\ &\quad = E_x\{\Xi_\Omega(w, t)(1 - \Xi_{\mathbb{R}^m\backslash K_n}(w, t)) \mid w(t) = y\}\, p_0(x, y, t) \\ &\quad \leqslant [E_x\{\Xi_\Omega(w, t) \mid w(t) = y\}]^{1/2} [E_x\{(1 - \Xi_{\mathbb{R}^m\backslash K_n}(w, t)) \mid w(t) = y\}]^{1/2} p_0(x, y, t) \\ &\quad = p(x, y, t)^{1/2}\, p_0(x, y, t)^{1/2} [E_x\{(1 - \Xi_{\mathbb{R}^m\backslash K_n}(w, t)) \mid w(t) = y\}]^{1/2}. \end{aligned}$$

Hence,

$$| p_n(x, y, t) - p(x, y, t)|^2 \leqslant p(x, y, t)\, p_0(x, y, t)\, A_n(x, y, t)$$

where $A_n(x, y, t)$ is the probability that a Brownian path on $[0, t]$ with end points at $x$ and $y$ hits the set $K_n$.

Let $T_n(w) = \inf\{t > 0: w(t) \in K_n\}$ define a function on path space. Since $p_0(x, y, t)$ is the probability density for a particle starting at $x$ to reach $y$ at time $t$, we have $\int A_n(x, y, t)\, p_0(x, y, t)\, dy = P_x[T_n \leqslant t] \leqslant P_x[T_n < \infty]$ where $P_x[T_n \leqslant t]$ is the probability that a Brownian path starting at $x$ hits $K_n$ within the interval $(0, t]$, if $0 < t < \infty$.

We now use a connection between Brownian motion and potential theory. Recall that if $\operatorname{cap}(K_n) > 0$ then there is a positive measure $\mu_n$ supported by $K_n$, of greatest possible total mass, whose potential $U_n(x) = \int d\mu_n(y)/|x - y|^{m-2} \leqslant 1$, and then $\operatorname{cap}(K_n) = \int d\mu_n$. The connection with probability theory is the following formula (see [7]).

$$U_n(x) = P_x[T_n < \infty] \qquad (\text{if } m \geqslant 3). \tag{6.1}$$

Given this, the rest is easy. If $S$ is any bounded measurable subset of $\Omega$,

$$\begin{aligned}\int_S \int |p(x, y, t) - p_n(x, y, t)|^2\, dy\, dx &\leqslant (2\pi t)^{-m/2} \int_S U_n(x)\, dx \\ &= (2\pi t)^{-m/2} \int_S \left( \int \frac{d\mu_n(y)}{|x - y|^{m-2}} \right) dx \\ &\leqslant (2\pi t)^{-m/2} \alpha_S \operatorname{cap}(K_n),\end{aligned}$$

where

$$\alpha_S = \sup_{x \in \mathbb{R}^m} \int_S \frac{dx}{|x - y|^{m-2}} < \infty.$$

If $\operatorname{cap}(K_n) \to 0$, we deduce that, for each fixed $t$, any subsequence of $p_n(x, y, t)$ has a further subsequence which converges a.e. on $\Omega \times \Omega$ to $p(x, y, t)$. Proposition 6.2 now completes the proof, for the case $m \geqslant 3$. If $m = 2$, formula (6.1) breaks down, but the theorem here can be deduced from the three dimensional case by Hadamard's method of descent. ∎

The next problem we shall consider deals with sets $K_n \subset \mathbb{R}^3$ consisting of lots of small balls, in a situation intermediate between the two extremes considered in Section 4. Thus $K_n$ will consist of $n$ balls, of radius $r_n$, and we shall suppose that $nr_n = \alpha$ is kept constant as $n \to \infty$.

Kac showed in [6] that if balls are placed randomly in a bounded region $\Omega \subset \mathbb{R}^3$ in this fashion, with a uniform distribution on $\Omega$, then the eigenvalues of $-\Delta_n$ are shifted up, in the limit, by the quantity $2\pi\alpha/\text{vol}\,\Omega$, this limit holding in probability on the set of all placements. (The measure space is described precisely below.)

In the situation we now consider, $\Omega$ is a possibly unbounded region in $\mathbb{R}^3$, and the centers of the balls are placed in $\Omega$ according to a given probability distribution having continuous density $\rho \geqslant 0; \int_\Omega \rho(x)dx = 1$.

Our space of random obstacles will be $X = \Omega \times \Omega \times \cdots$, with probability measure the product of the probability measures $\mu(S) = \int_S \rho(x)\,dx$ on each factor $\Omega$. If $\xi = (\xi_1, \xi_2, \ldots) \in X$, we let $\Delta_n^{(\xi)}$ be the Laplacian on $\Omega \backslash K_n$ with Dirichlet boundary conditions, where $K_n = K_n^{(\xi)}$ consists of $n$ balls of radius $r_n = \alpha/n$ and centers at $\xi_1, \ldots, \xi_n$. We let $P_n^{(\xi)}$ be projection of $L^2(\Omega)$ into $L^2(\Omega\backslash K_n)$.

THEOREM 6.4. *For all* $u \in L^2(\Omega)$

$$\| e^{t\Delta_n^{(\xi)}} P_n^{(\xi)} u - e^{t(\Delta - 2\pi\alpha\rho(x))} u \|_{L^2(\Omega)} \to 0$$

*in probability on* $X$.

Applying Proposition 6.1, in the case of bounded $\Omega$ we deduce the convergence of the eigenvalues and eigenfunctions of $\Delta_n^{(\xi)}$ to those of $\Delta - 2\pi\alpha\rho(x)$. Clearly the convergence of the eigenfunctions cannot be much smoother than the $L^2$ convergence obtained in this manner.

*Proof of theorem.* Denote the kernel of $e^{t\Delta_n^{(\xi)}} P_n^{(\xi)} = P_n{}^t(\xi)$ by $p_n(x, y, t; \xi)$. What we are to prove is that $P_n{}^t(\xi) \to e^{t(\Delta - 2\pi\alpha\rho(x))}$ strongly, in probability, as $n \to \infty$.

Let $q(x, y, t)$ denote the kernel of $e^{t(\Delta - 2\pi\rho(x))}$. Then $q$ is given by the Feynman–Kac formula

$$q(x, y, t) = E_x\{\Xi_\Omega(w, t)\, e^{-\int_0^t 2\pi\alpha\rho(w(\tau))d\tau} \mid w(t) = y\}\, p_0(x, y, t). \qquad (6.2)$$

We wish to compare this with the formula

$$p_n(x, y, t; \xi) = E_x\{\Xi_\Omega(w, t)\, \Xi_{\mathbb{R}^3\backslash K_n(\xi)}(w, t) \mid w(t) = y\}\, p_0(x, y, t). \qquad (6.3)$$

Thus we want to take a look at $\Xi_{\mathbb{R}^3\backslash K_n(\xi)}(w, t)$.

Following Kac, we introduce the "Wiener sausage," $W_\delta(t_1, t_2) = \{x \in \mathbb{R}^3 : | w(t) - x | \leqslant \delta \text{ for some } t,\ t_1 \leqslant t \leqslant t_2\}$. It is known [6, 11, 12] that in probability on path space

$$(\text{vol}(W_\delta(t_1, t_2))/\delta) \to 2\pi(t_2 - t_1) \qquad \text{as} \quad \delta \to 0. \qquad (6.4)$$

Now to say that a Brownian path $w(\tau)$ misses $K_n$ for $0 \leqslant \tau \leqslant t$ is the same as saying that no point $\xi_1, \ldots, \xi_n$ lies in the Wiener sausage $W_{r_n}(0, t)$.

A point picked at random in $\Omega$ has probability $\int_{W_{r_n}(0,t)} \rho(x)\,dx$ of belonging to our Wiener sausage. By (6.4) it is reasonable to approximate this quantity by $2\pi r_n \int_0^t \rho(w(\tau))\,d\tau$. Let us give a careful estimate of the error.

What (6.4) says is that for any $t_1$, $t_2$ and any subsequence of $\{r_n\}$ there is a further subsequence $r_j$, $j = n_1, n_2, \ldots$, and a set of paths of Wiener measure 1 such that

$$\operatorname{vol}(W_{r_j}(t_1, t_2)/r_j) \to 2\pi(t_2 - t_1) \quad \text{as } j \to \infty. \tag{6.5}$$

By a diagonal argument it follows that such a subsequence and set of paths can be chosen so that (6.5) holds for all $t_1$, $t_2$ rational and therefore for all $t_1$, $t_2$. Let $\{r_j\}$ be such a subsequence, and let $w(\tau)$ be such a path. With these fixed, we break up our time interval $[0, t]$; say $0 = t_1 < t_2 < \cdots < t_\nu < t_{\nu+1} = t$.

Now $2\pi r_j \int_0^t \rho(w(\tau))\,d\tau = 2\pi r_j \sum_{l=1}^{\nu} \rho(w(t_l))(t_{l+1} - t_l) + 2\pi r_j \alpha_{\nu,j}$ with $|\alpha_{\nu,j}| \leqslant t \sup_l \operatorname{osc}_{t_l \leqslant \sigma \leqslant t_{l+1}} \rho(w(\sigma))$. Similarly

$$2\pi r_j \sum_{l=1}^{\nu} \rho(w(t_l))(t_{l+1} - t_l) = \sum_{l=1}^{\nu} \int_{W_{r_j}(t_l, t_{l+1})} \rho(x)\,dx + 2\pi r_j \beta_{\nu,j}$$

with

$$|\beta_{\nu,j}| \leqslant (\sup|\rho|) \sum_{l=1}^{\nu} |((\operatorname{vol} W_{r_j}(t_l, t_{l+1}) - 2\pi r_j(t_{l+1} - t_l))/2\pi r_j)|$$

$$+ \sum_{l=1}^{\nu} (\operatorname{vol} W_{r_j}(t_l, t_{l+1})/2\pi r_j) \operatorname*{osc}_{t_l \leqslant \sigma \leqslant t_{l+1}} \rho(w(\sigma)).$$

By (6.5), the first of these terms tends to 0 as $r_j \to 0$, and the second is dominated by a constant times $\sup_l \operatorname{osc}_{t_l \leqslant \sigma \leqslant t_{l+1}} \rho(w(\sigma))$. Finally,

$$\sum_{l=1}^{\nu} \int_{W_{r_j}(t_l, t_{l+1})} \rho(x)\,dx = \int_{W_{r_j}(0,t)} \rho(x)\,dx + 2\pi r_j \gamma_{\nu,j}$$

with

$$|\gamma_{\nu,j}| \leqslant (\sup|\rho|)(1/2\pi r_j) \sum_{l \neq k} \operatorname{vol}(W_{r_j}(t_l, t_{l+1}) \cap W_{r_j}(t_k, t_{k+1}))$$

$$= (\sup|\rho|)(1/2\pi r_j)\Big(\sum_l \operatorname{vol} W_{r_j}(t_l, t_{l+1}) - \operatorname{vol} W_{r_j}(0, t)\Big).$$

By (6.5), this term tends to 0 as $r_j \to 0$.

To continue our argument, the probability that a random obstacle $K_j$ does not intersect the Brownian path $w$ on $[0, t]$ is

$$\left(1 - \frac{2\pi\alpha}{j}\left[\int_0^t \rho(w(\tau))\,d\tau - \alpha_{\nu,j} - \beta_{\nu,j} - \gamma_{\nu,j}\right]\right)^j$$

provided $r_j = \alpha/j$, $j = n_1, n_2, \ldots$ . As we let $j \to \infty$, the limits inf and sup of this lie between

$$e^{-2\pi\alpha\int_0^t \rho(w(\tau))\,d\tau}\, e^{-2\rho\alpha\delta_\nu} \qquad \text{and} \qquad e^{-2\pi\alpha\int_0^t \rho(w(\tau))\,d\tau}\, e^{2\pi\alpha\delta_\nu}$$

where $\delta_\nu = C \sup_l \operatorname{osc}_{t_l \leqslant \sigma \leqslant t_{l+1}} \rho(w(\sigma))$. Since we can choose the partition $t_1 < \cdots < t_\nu$ fine enough so that $\delta_\nu$ is as small as desired, the limit is exactly

$$e^{-2\pi\alpha\int_0^t \rho(w(\tau))\,d\tau}.$$

Thus we have

$$\int_X \Xi_{\mathbb{R}^3 \setminus K_n(\xi)}(w, t)\, d\xi \to e^{-2\pi\alpha\int_0^t \rho(w(\tau))\,d\tau}$$

as $n \to \infty$, in probability with respect to Wiener measure on path space. By (6.2) and (6.3), we deduce that

$$\int_X p_n(x, y, t; \xi)\, d\xi \to q(x, y, t),$$

and hence that $\int_X P_n{}^t(\xi)\, d\xi \to e^{t(\Delta - 2\pi\alpha\rho(x))}$ strongly, for each $t \geqslant 0$.

This formula for the mean behavior of the semigroups $P_n{}^t(\xi)$ as $n \to \infty$ is enough to establish convergence in probability. In fact, the following lemma easily completes the proof of Theorem 6.4.

LEMMA 6.5. *Let $A_n(\xi)$ be strongly measurable functions whose values are self-adjoint operators on a separable Hilbert space $H$ with a uniform bound $\| A_n(\xi)\| \leqslant M$. Suppose*

$$\int_X A_n(\xi)\, d\xi \to A \quad \textit{weakly},$$

*and*

$$\int_X A_n(\xi)^2\, d\xi \to A^2 \quad \textit{weakly}.$$

*Then $A_n(\xi) \to A$ strongly, in probability.*

*Proof.* We must show that every subsequence of $A_n(\xi)$ has a further subsequence which converges a.e. to $A$, strongly. Fix $u \in H$. Then

$$\begin{aligned}\int_X \| A_n(\xi)u - Au \|^2\, d\xi &= \int_X ((A_n(\xi) - A)^2 u, u)\, d\xi \\ &= \int_X ((A_n(\xi)^2 - A_n(\xi)\,A - AA_n(\xi) + A^2)u, u)\, d\xi \to 0.\end{aligned}$$

Thus we can pass to a subsequence for which $A_n(\xi)u \to Au$ on a set of measure one. Passing to further subsequences, and using a diagonal argument, we have $A_n(\xi)u \to Au$ a.e. for a countable dense set of vectors $u$, hence for all $u$ because of the uniform bound. ▮

As a final observation we merely describe what happens to the Dirichlet problem on $\Omega \backslash K_n$ if $K_n$ consists of $n$ balls of radius $r_n = \alpha/n$ placed so that their centers are randomly distributed on a smooth surface $S \subset \Omega \subset \mathbb{R}^3$, the probability density being $\rho(x)$ times surface measure on $S$. Then we have $e^{t\Delta_n} \to e^{t(\Delta - 2\pi\alpha\mu)}$ in probability, strongly, with a similar convergence for other functions of these operators. Here $\mu$ is a measure supported by $S$, namely $\rho(x)$ times surface measure.

Now a little care is needed to define $T = \Delta - 2\pi\alpha\mu$ since as a multiplication operator on $L^2(\Omega)$, $\mu$ has only 0 in its range. This difficulty can be overcome by defining $T$ with the aid of the closed quadratic form $b(u, v) = -\int_\Omega \nabla u \cdot \nabla v - 2\pi\alpha \int_S uv\rho$, $\mathscr{D}(b) = \mathring{H}_1(\Omega)$. Theorem 6.3.4 of [8] applies and we can show that if $u \in D(T)$ then $u \in H^2(\Omega \backslash S)$ and on $S$ $u$ satisfies the "transmission conditions,"

$$\left.\begin{aligned} u_+(x) &= u_-(x) \\ \frac{\partial u_+}{\partial \nu_+} + \frac{\partial u_-}{\partial \nu_-} &= 2\pi\alpha\rho(x)\, u_+(x) \end{aligned}\right\} \qquad x \in S,$$

where $u_\pm$ are the values of $u$ on either side of $S$ and $(\partial u_+/\partial \nu_+) + (\partial u_-/\partial \nu_-$ their normal derivatives. It is interesting that in this case the limiting behavior of a sequence of Dirichlet problems is given by a different type of boundary value problem.

## References

1. S. Agmon, "Lectures on Elliptic Boundary Value Problems," D. van Nostrand, Princeton, New Jersey, 1965.
2. J. T. Beale, Scattering frequencies of resonators, *Comm. Pure Appl. Math.* **26** (1973), 549–564.
3. L. Carleson, "Selected Problems on Exceptional Sets," Van Nostrand, New York, 1967.
4. R. Courant and D. Hilbert, "Methods of Mathematical Physics," Vol. I, Interscience, New York, 1953.
5. G. Hunt, Some theorems concerning Brownian motion, *Trans. Amer. Math. Soc.* **81** (1956), 294–319.
6. M. Kac, Probabilistic methods in some problems of scattering theory, *Rocky Mountain J. Math.* **4** (1974), 511–538.
7. M. Kac, "Aspects Probabilistes de la Théorie du Potentiel," Univ. de Montreal, Montreal, 1970.
8. T. Kato, "Perturbation Theory for Linear Operators," Springer–Verlag, New York, 1966.

9. P. Lax and R. Phillips, "Scattering Theory," Academic Press, New York, 1967.
10. P. Lax and R. Phillips, Scattering theory for the acoustic equation in an even number of space dimensions, *Indiana Univ. Math. J.* **22** (1972), 101–134.
11. F. Spitzer, Electrostatic capacity, heat flow, and Brownian motion, *Z. Wahrscheinlichkeitstheorie* **3** (1964), 110–121.
12. W. Whiteman, Thesis, Cornell University, 1964.
13. J. Rauch and M. Taylor, Electrostatic screening, *J. Math. Phys.*, to appear.
14. M. Taylor, Scattering length and perturbations of $-\Delta$ by positive potentials, to appear.