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The Hardy Classes for Functions in the Class $MV[\alpha, k]$

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Suppose that $f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$ is regular in the unit disc D with $[f(z) f'(z)/z] \neq 0$ in D, and further let $\alpha \geqslant 0$ and $k \geqslant 2$. If $\int_0^{2\pi} |\operatorname{Re}\{(1-\alpha)z[f'(z)/f(z)] + \alpha(1 + z[f''(z)/f'(z)])\}|\, d\theta \leqslant k\pi$ for $z \in D$, then $f(z)$ is said to belong to the class $MV[\alpha, k]$. This class contains many of the special classes of regular and univalent functions. The authors determine the Hardy classes of which $f(z)$, $f'(z)$ and $f''(z)$ belong and obtain growth estimates of a_n.

1. INTRODUCTION

In a recent paper [3] we have determined the Hardy classes to which $f(z)$ and $f'(z)$ belong when $f(z)$ is in U_k, the class of functions of bounded argument rotation. Recently, Coonce and Ziegler [1] have investigated some interesting subclasses of U_k and it is the purpose of this paper to determine the Hardy classes of $f(z)$, $f'(z)$ and $f''(z)$ for these subclasses. In addition we will obtain some growth conditions on the coefficients of the Taylor expansion of $f(z)$.

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2. Preliminaries

Definition 1. Let $f(z) = z + a_2 z^2 + \cdots$ be regular in the unit disc D, with $(f(z)f'(z))/z \neq 0$ in D, and further let $\alpha \geq 0$ and $k \geq 2$. If

$$\int_0^{2\pi} \left| \operatorname{Re} \left\{ (1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + z \frac{f''(z)}{f'(z)} \right) \right\} \right| d\theta \leqslant k\pi,$$

for $z = re^{i\theta} \in D$, then $f(z)$ is said to be in the class $MV[\alpha, k]$.

This class of functions, which was introduced by Coone and Ziegler [1], contains many of the heavily researched classes of regular and univalent functions. In fact, for the classes of starlike functions S^*, convex functions K, alpha-convex functions M_α, functions of bounded argument rotation U_k and functions of bounded boundary rotation V_k, we have

$$\begin{aligned} MV[0, 2] &= S^*, \\ MV[1, 2] &= K, \\ MV[\alpha, 2] &= M_\alpha, \\ MV[0, k] &= U_k, \\ MV[1, k] &= V_k, \end{aligned}$$

In their paper, Coonce and Ziegler proved the following two results which we will need in this paper.

Theorem A. *If $f(z) \in MV[\alpha, k]$, then $f(z) \in MV[0, k]$.*

Theorem B. *If $\alpha > 0$, then $f(z) \in MV[\alpha, k]$ if and only if there exists $g(z) \in MV[0, k]$ such that*

$$f(z) = \left[\frac{1}{\alpha} \int_0^z g(\zeta)^{1/\alpha} \frac{d\zeta}{\zeta} \right]^\alpha. \tag{1}$$

For $\lambda > 0$, we say that a function $f(z) = z + a_2 z^2 + \cdots$, regular in D, belongs to the Hardy class H^λ if $\lim_{r\to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^\lambda \, d\theta$ exists (and is finite). In a recent paper [3] the authors investigated the Hardy classes for functions in the class $U_k = MV[0, k]$. In this paper we extend this result and determine the Hardy classes for functions in the class $MV[\alpha, k]$, when $\alpha \geqslant 0$.

In what follows, we denote by $g(\tau, k; z)$ any function of the form

$$g(\tau, k; z) \equiv [z(1 - e^{-i\tau} z)^{-2}]^{(2+k)/4} \, [s(z)]^{(2-k)/4},$$

where τ is a real constant and $s(z)$ is a starlike function. Note that $g(\tau, k; z) \in MV[0, k]$ and is an extremal function for many problems in this class. In addition, we denote by $f(\alpha, \tau, k; z)$ the function obtained from (1) by letting $g(z)$ be the function $g(k, \tau; z)$ and we interpret $f(0, \tau, k; z)$ to be the function $g(\tau, k; z)$.

We require the following lemmas:

LEMMA 1. *If $f'(z) \in H^\lambda$ $(0 < \lambda \leqslant 1)$, then $f(z) \in H^{\lambda/(1-\lambda)}$ (H^∞, the class of bounded functions, if $\lambda = 1$).*

LEMMA 2. *If $f(z) \in H^\lambda$ $(0 < \lambda \leqslant 1$ and $f(z) = z + a_2 z^2 + \cdots$, then $a_n = 0(n^{(1/\lambda - 1)})$.*

LEMMA 3. *If $g(z) \in MV[0, k]$, then $g(z) \in H^\lambda$ for all $\lambda < 2/(k+2)$. If in addition $g(z) \neq g(\tau, k; z)$, then there exists $\epsilon = \epsilon(g) > 0$ such that $g(z) \in H^{(2/(k+2))+\epsilon}$.*

LEMMA 4. *If $g(z) \in MV[0, k]$, then $g'(z) \in H^\lambda$ for all $\lambda < 2/(k+4)$. If in addition $g(z) \neq g(\tau, k; z)$, then there exists $\epsilon = \epsilon(g) > 0$ such that $g'(z) \in H^{(2/(k+4))+\epsilon}$.*

Lemma 1 is in [2, p. 88], Lemma 2 is in [2, p. 98] and Lemmas 3 and 4 are in [3].

3. H^p PROPERTIES $f(z) \in MV[\alpha, k]$

The ratio $2\alpha/(k+2)$ plays a very crucial role in determining the Hardy classes for functions in $MV[\alpha, k]$ and it is because of this that we have broken up our results into the following three theorems.

THEOREM 1. *If $2\alpha/(k+2) < 1$ and $f(z) \in MV[\alpha, k]$ then:*

(i) *$f(z) \in H^\lambda$, for all $\lambda < 2/(k + 2 - 2\alpha)$,*

(ii) *if $f(z) \neq f(\alpha, \tau, k; z)$, then there exists $\epsilon = \epsilon(f) > 0$ such that $f(z) \in H^{(2/(k+2-2\alpha))+\epsilon}$.*

Proof. Since the case $\alpha = 0$ has already been proved in Lemma 3 we restrict our study to the case $\alpha > 0$.

(i) Since $f(z) \in MV[\alpha, k]$, by Theorem B there exists a function $g(z) \in MV[0, k]$ such that

$$f(z) = \left[\frac{1}{\alpha}\int_0^z g(\zeta)^{1/\alpha}\frac{d\zeta}{\zeta}\right]^\alpha,$$

or

$$f'(z) = f(z)^{1-1/\alpha} g(z)^{1/\alpha}/z. \tag{2}$$

Since $f(z)/z \neq 0$ in D, if we let

$$F(z) = \left(\frac{f(z)}{z}\right)^{1/\alpha}, \tag{3}$$

then $F(z)$ is regular in D and satisfies

$$F'(z) = \frac{1}{\alpha}\frac{g(z)^{1/\alpha}}{z^{1/\alpha+1}} - \frac{1}{\alpha}\frac{f(z)^{1/\alpha}}{z^{1/\alpha+1}}.$$

If $0 \leqslant \lambda \leqslant 1$, then for $z = re^{i\theta}$ $(0 < r < 1)$ we obtain

$$I(r) \equiv \int_0^{2\pi} |F'(z)|^\lambda \, d\theta \leqslant \int_0^{2\pi} \left|\frac{1}{\alpha}\frac{g(z)^{1/\alpha}}{z^{1/\alpha+1}}\right|^\lambda d\theta + \int_0^{2\pi} \left|\frac{1}{\alpha}\frac{f(z)^{1/\alpha}}{z^{1/\alpha+1}}\right|^\lambda d\theta$$

$$\equiv \frac{I_1(r) + I_2(r)}{\alpha^\lambda r^{(\lambda+\lambda/\alpha)}}.$$

Since $g(z) \in MV[0, k]$, by Lemma 3

$$\lim_{r\to 1^-} I_1(r) \qquad \text{exists if} \qquad \lambda/\alpha < 2/(k+2). \tag{4}$$

Since $f(z) \in MV[\alpha, k]$, by Theorem A $f(z) \in MV[0, k]$ and hence, by Lemma 3,

$$\lim_{r\to 1^-} I_2(r) \qquad \text{exists if} \qquad \lambda/\alpha < 2/(k+2). \tag{5}$$

From (4) and (5) we see that $\lim_{r\to 1^-} I(r)$ exists provided that $\lambda < 2\alpha/(k+2)$, i.e.,

$$F'(z) \in H^\lambda \qquad \text{for all} \quad \lambda < 2\alpha/(k+2). \tag{6}$$

Since $2\alpha/(k+2) < 1$ we can use Lemma 1 to obtain $F(z) \in H^\lambda$ for all $\lambda < 2\alpha/(k+2-2\alpha)$, and hence, from (3), we obtain

$$f(z) \in H^\lambda \qquad \text{for all} \quad \lambda < 2/(k+2-2\alpha). \tag{7}$$

(ii) If $f(z) \neq f(\alpha, \tau, k; z)$, then $g(z) \neq g(\tau, k; z)$ and by Lemma 3 condition (4) can be replaced by

$$\lim_{r\to 1^-} I_1(r) \qquad \text{exists if} \qquad \lambda/\alpha \leqslant 2/(k+2) + \epsilon, \tag{4'}$$

where $\epsilon = \epsilon(g) > 0$. In light of (7) we replace (5) by the stronger condition

$$\lim_{r\to 1^-} I_2(r) \qquad \text{exists if} \qquad \lambda/\alpha < 2/(k+2-2\alpha). \tag{5'}$$

If we set a new positive ϵ less than the

$$\min\{\epsilon, 1/\alpha - 2/(k+2), 2/(k+2-2\alpha) - 2/(k+2)\}$$

then (4′) and (5′) will be satisfied if $\lambda/\alpha \leqslant 2/(k+2) + \epsilon$ and (6) will be replaced by

$$F'(z) \in H^\lambda \qquad \text{for all } \lambda \leqslant 2\alpha/(k+2) + \alpha\epsilon. \tag{6'}$$

Since $2\alpha/(k+2) + \alpha\epsilon < 1$ we can use Lemma 1 to obtain $F(z) \in H^\lambda$, for all $\lambda \leqslant 2\alpha/(k+2-2\alpha) + \epsilon$, for a possibly different value of ϵ. Combining this with (3) we have our result.

THEOREM 2. *If* $2\alpha/(k+2) > 1$, *and* $f(z) \in MV[\alpha, k]$, *then* $f(z) \in H^\infty$ (*i.e.*, $f(z)$ *is bounded*).

Proof. From (6) we have $F'(z) \in H^\lambda$ for $\lambda < 2\alpha/(k+2)$, and since $2\alpha/(k+2) > 1$ we have $F'(z) \in H^1$. Hence by Lemma 1 $F(z)$ is bounded and consequently, by (3) $f(z)$ is also bounded.

Note that this result is also proved in [1] using a different method.

THEOREM 3. *If* $2\alpha/(k+2) = 1$ *and* $f(z) \in MV[\alpha, k]$, *then*

(i) $f(z) \in H^\lambda$ *for all* $\lambda > 0$,

(ii) *if* $f(z) \neq f(\alpha, \tau, k; z)$, *then* $f(z) \in H^\infty$.

Proof. (i) From (6) we obtain $F'(z) \in H^\lambda$ for $\lambda < 1$ which, by Lemma 1 and (3), yields $f(z) \in H^\lambda$ for all $\lambda > 0$.

(ii) If $f(z) \neq f(\alpha, \tau, k; z)$, then condition (4) can be replaced by $\lim_{r\to 1^-} I_1(r)$ exists if $\lambda \leqslant 1 + \epsilon$, and in light of what we have shown in (i) we can replace condition (5) by $\lim_{r\to 1^-} I_2(r)$ exists for all $\lambda > 0$. Combining these results we conclude that $F'(z) \in H^{1+\epsilon}$, and consequently $F(z)$ and $f(z)$ are bounded.

Note that Theorem 3 implies that all functions in $MV[\alpha, k]$, $2\alpha = k + 2$ are bounded with the exception of functions of the form $f(\alpha, \tau, k; z)$.

4. H^p Properties of $f'(z)$

In general, if a regular function belongs to some Hardy class its derivative need not belong to any Hardy class. We show in the next two theorems that this is not true for functions in $MV[\alpha, k]$.

THEOREM 4. *If* $2\alpha/(k+2) < 1$ *and* $f(z) \in MV[\alpha, k]$, *then*

(i) $f'(z) \in H^\lambda$ *for all* $\lambda < (2/(k+4-2\alpha))$,

(ii) *if* $f(z) \neq f(\alpha, \tau, k; z)$, *then there exists* $\epsilon = \epsilon(f) > 0$ *such that* $f(z) \in H^{2/(k+4-2\alpha)+\epsilon}$.

Proof. The case $\alpha = 0$ is handled by Lemma 4, so we restrict our study to the case $\alpha > 0$.

(i) Since $(f(z)f'(z))/z \neq 0$ in D, we can write $f'(z)$ as

$$f'(z) = \frac{f(z)}{z}\left[\frac{zf'(z)}{f(z)}\right],$$

and if $0 < \lambda \leqslant 1$, we have

$$I(r) \equiv \int_0^{2\pi} |f'(z)|^\lambda \, d\theta \equiv \int_0^{2\pi} \left|\frac{f(z)}{z}\left[\frac{zf'(z)}{f(z)}\right]\right|^\lambda d\theta,$$

for $z = re^{i\theta} (0 < r < 1)$. By applying Hölder's inequality with conjugate indices p and q we obtain

$$I(r) \leqslant \left[\int_0^{2\pi} \left|\frac{f(z)}{z}\right|^{\lambda p} d\theta\right]^{1/p} \left[\int_0^{2\pi} \left|\frac{zf'(z)}{f(z)}\right|^{\lambda q} d\theta\right]^{1/q}. \tag{8}$$

Since $f(z) \in MV[\alpha, k]$, by Theorem A we have $f(z) \in MV[0, k]$, i.e.,

$$\int_0^{2\pi} \left|\operatorname{Re}\frac{zf'(z)}{f(z)}\right| d\theta \leqslant k\pi.$$

Hence $\operatorname{Re}(zf'(z)/f(z))$ is an h^1 function and by a theorem of Kolmogorov [2, p. 57] $(zf'(z)/f(z)) \in H^\eta$ for all $\eta < 1$. Hence the second integral in (8) is bounded as $r \to 1^-$ provided that

$$\lambda q < 1. \tag{9}$$

By Theorem 1 the first integral in (8) will be bounded as $r \to 1^-$ if

$$\lambda p < 2/(k+2-2\alpha). \tag{10}$$

The inequalities (9) and (10) will be satisfied if $\lambda < 2/(k + 4 - 2\alpha)$, $p = (k + 4 - 2\alpha)/(k + 2 - 2\alpha)$ and $q = (k + 4 - 2\alpha)/2$. Hence $f'(z) \in H^\lambda$ for all $\lambda < 2/(k + 4 - 2\alpha)$.

(ii) If $f(z) \neq f(\alpha, \tau, k; z)$, then in the proof of (i) we can replace (10) by

$$\lambda p \leqslant 2/(k + 2 - 2\alpha) + \epsilon, \tag{10'}$$

and conditions (9) and (10′) will be satisfied if $\lambda \leqslant 2/(k + 4 - 2\alpha) + \epsilon$, for a possibly different value of ϵ.

THEOREM 5. *If* $2\alpha/(k + 2) \geqslant 1$, *then* $f'(z) \in H^\lambda$ *for all* $\lambda < 1$.

Proof. We first consider the case $2\alpha/(k + 2) > 1$. By Theorem 2 $f(z)$ is bounded, say $|f(z)| \leqslant M$, and thus

$$\int_0^{2\pi} |f'(z)|^\lambda \, d\theta = \int_0^{2\pi} \left| \left[\frac{f(z)}{z}\right] \left[\frac{zf'(z)}{f(z)}\right]^\lambda \right| d\theta \leqslant \left[\frac{M}{r}\right]^\lambda \int_0^{2\pi} \left| \frac{zf'(z)}{f(z)} \right|^\lambda d\theta. \tag{11}$$

By our discussion in Theorem 4 we know that this last integral will be bounded as $r \to 1^-$ if $\lambda < 1$.

If $2\alpha/(k + \alpha) = 1$, by applying Hölder's inequality on the second integral in (11) and using Theorem 3 part (i) we obtain our result.

5. H^p PROPERTIES OF $f''(z)$

As was mentioned in §4 a regular function will not in general have a derivative belonging to some Hardy class. We next show that for functions in $MV[\alpha, k]$, $\alpha > 0$, even the second derivative belongs to some Hardy classes.

THEOREM 6. *If* $0 < (2\alpha/(k + 2)) < 1$ *and* $f(z) \in MV[\alpha, k]$, *then*:

(i) $f''(z) \in H^\lambda$ *for all* $\lambda < 2/(k + 6 - 2\alpha)$,

(ii) *if* $f(z) \neq f(\alpha, \tau, k; z)$, *then there exists* $\epsilon = \epsilon(f) > 0$ *such that* $f''(z) \in H^{2/(k+6-2\alpha)+\epsilon}$.

Proof. (i) If $0 \leqslant \lambda \leqslant 1$, then from the identity

$$\alpha z f''(z) \equiv f'(z) \left[(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf''(z)}{f'(z)} + 1\right)\right] - \left[(1 - \alpha) f'(z) \frac{zf'(z)}{f(z)} + \alpha f'(z)\right],$$

we obtain

$$\begin{aligned} I(r) &\equiv \int_0^{2\pi} | \alpha z f''(z)|^\lambda \, d\theta \\ &\leqslant \int_0^{2\pi} \left| f'(z) \left[(1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right) \right] \right|^\lambda d\theta \\ &\quad + | 1 - \alpha |^\lambda \int_0^{2\pi} \left| f'(z) \left[\frac{zf'(z)}{f(z)} \right] \right|^\lambda d\theta + \alpha^\lambda \int_0^{2\pi} | f'(z) |^\lambda \, d\theta \\ &\equiv I_1(r) + | 1 - \alpha |^\lambda I_2(r) + \alpha^\lambda I_3(r). \end{aligned}$$

We apply Hölder's inequality, with conjugate indices p and q, to $I_1(r)$ and obtain

$$I_1(r) \leqslant \left[\int_0^{2\pi} | f'(z) |^{\lambda p} \, d\theta \right]^{1/p} \left[\int_0^{2\pi} \left| (1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right) \right|^{\lambda q} d\theta \right]^{1/q}. \tag{12}$$

By Theorem 4 the first integral will be bounded as $r \to 1^-$ if

$$\lambda p < 2/(k + 4 - 2\alpha). \tag{13}$$

From Definition 1 we see that

$$\operatorname{Re} J(\alpha, f) \equiv \operatorname{Re} \left[(1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right) \right]$$

is an h^1 function. Hence by a theorem of Kolmogorov [2, p. 57], $J(\alpha, f) \in H^\eta$ for all $\eta < 1$, and thus the second integral in (12) will be bounded as $r \to 1^-$ if

$$\lambda q < 1. \tag{14}$$

If we apply Hölder's inequality to $I_2(r)$ and use the fact that $[zf'(z)/f(z)] \in H^\eta$ for all $\eta < 1$, we obtain the same conditions (13) and (14).

The integral $I_3(r)$ will be bounded as $r \to 1^-$ provided that

$$\lambda < 2/(k + 4 - 2\alpha). \tag{15}$$

Conditions (13), (14) and (15) will be satisfied if $\lambda < 2/(k + 6 - 2\alpha)$, and hence $f''(z) \in H^\lambda$ for all $\lambda < 2/(k + 6 - 2\alpha)$.

(ii) If $f(z) \neq f(\alpha, \tau, k; z)$, the inequalities (13) and (15) can be replaced by stronger results involving ϵ which yield our result.

Note that for $\alpha = 0$ this coincides with the result [3] for the class V_k. Using Theorem 5 and the technique in Theorem 6 we immediately obtain the corresponding result for $2\alpha/(k + 2) > 1$.

THEOREM 7. *If* $2\alpha/(k+2) \geqslant 1$, *then* $f''(z) \in H^{\lambda}$ *for all* $\lambda < 1/2$.

If we apply Theorems 6 and 7 to the class of alpha-convex functions $M_\alpha \equiv MV[\alpha, 2]$, we obtain the following corollaries.

COROLLARY 1. *If* $f(z) \in M_\alpha$, $0 < \alpha < 2$, *then*

(i) $f''(z) \in H^{\lambda}$ *for all* $\lambda < 1/(4-\alpha)$,

(ii) *if* $f(z) \neq f(\alpha, \tau, 2; z)$, *then there exists* $\epsilon = \epsilon(f) > 0$ *such that* $f''(z) \in H^{1/(4-\alpha)+\epsilon}$.

COROLLARY 2. *If* $f(z) \in M_\alpha$, $\alpha \geqslant 2$, *then* $f''(z) \in H^{\lambda}$ *for all* $\lambda < 1/2$.

6. COEFFICIENT ESTIMATES

By combining Theorem 1 with Lemma 2 we obtain the following theorem:

THEOREM 8. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in MV[\alpha, k]$, $2\alpha/(k+1) < 1$, *then*

$$a_n = \begin{cases} 0(n^{1/\lambda - 1}) & \text{for } \lambda < 2/(k+2-2\alpha) \quad \text{if } k \leqslant 2\alpha, \quad \text{and} \\ 0(1) & \text{if } k < 2\alpha < k+2. \end{cases}$$

If in addition $f(z) \neq f(\alpha, \tau, k; z)$, *then*

$$a_n = \begin{cases} 0(n^{(k-2\alpha)/2}) & \text{if } 2\alpha < k, \quad \text{and} \\ 0(1) & \text{if } k \leqslant 2\alpha < k+2. \end{cases}$$

From Theorems 2 and 3 and the Cauchy estimate we obtain the following result.

THEOREM 9. *If* $f(z) = z + \sum_{2}^{\infty} a_n z^n \in MV[\alpha, k]$ *and* $2\alpha/(k+2) > 1$, *or* $2\alpha/(k+2) = 1$ *and* $f(z) \neq f(\alpha, \tau, k; z)$, *then* $\lim_{n\to\infty} a_n = 0$.

In [1] it is shown that $f(z) \in MV[\alpha, k]$ is univalent if and only if $k \leqslant 2 + 2\alpha$. Combining this result with Theorems 8 and 9 we obtain the following Theorem:

THEOREM 10. *If* $f(z) = z + \sum_{2}^{\infty} a_n z^n \in MV[\alpha, k]$ *is univalent and* $f(z) \neq f(\alpha, \tau, k; z)$, *then*

(i) $\lim_{n\to\infty} (a_n/n) = 0, \quad 2\alpha < k$,

(ii) $\lim_{n\to\infty} a_n = 0, \quad k \leqslant 2\alpha$.

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