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TRUNCATIONS OF PRINCIPAL GEOMETRIES *

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We investigate the class of principal pregeometries (free simplicial geometries with spanning simplex) which form an important subclass of the class of transversal pregeometries (free simplicial geometries). We give a coordinate-free method for imbedding a transversal pregeometry on a simplex as a free simplicial pregeometry which makes use only of the set-theoretic properties of a presentation of the transversal pregeometry. We introduce the notion of an (r, k)-principal set as a generalization of principal basis and prove the collection of (r, k)-principal sets of a rank k pregeometry, if non-empty, are the bases of another pregeometry whose structure is determined. An algorithm for constructing principal sets is given. We then characterize truncations of principal geometries in terms of the existence of a principal set. We do this by erecting a given pregeometry to a free simplicial pregeometry with spanning simplex. The erection is the freest of all erections of the given pregeometry.

1. Introduction

Our objective when we started to work on this paper was to obtain an intrinsic characterization of truncations of transversal geometries. Intrinsic characterizations of transversal geometries being known we thought it would be possible to extend them to this wider class of com-

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binatorial geometries. Geometrically, a transversal geometry is a set of points on a simplex in real affine space with the points freely situated on the faces of the simplex but where the vertices of the simplex need not all be points of the geometry (a free simplicial geometry). Thus a truncation of a transversal geometry can be regarded as a set of points on a simplex which has been freely projected into a smaller dimensional space. In trying to obtain an intrinsic characterization of these truncations, then, we are faced with the problem of taking a geometry which is claimed to be a truncation of a free simplicial geometry and erecting it into a higher dimensional space in such a way that it becomes a free simplicial geometry and the lower dimensional geometric properties are preserved. For transversal geometries in general we found this problem (from this point of view) intractable. Roughly the difficulty in carrying out the erection is due to the fact that in transversal geometries not all vertices of the simplex need be points of the geometry so that it is difficult to know how to reconstruct the simplex.

The subclass of transversal geometries which are free simplicial geometries where each vertex of the simplex is a point of the geometry is the class of principal transversal geometries or free simplicial geometries with spanning simplex. This class of geometries proved more tractable, for it is possible to isolate geometric properties of the vertices of the simplex when the simplex is freely projected into a smaller dimensional space. These properties can then be used to erect the truncated simplex into the space it came from and thus furnish a characterization of truncations of principal transversal geometries.

In Section 2 we discuss the basic concepts and theorems we need for the remainder of the paper. Section 3 gives a coordinate-free method to imbed a transversal geometry on a simplex which makes use only of the set-theoretic properties of a presentation of the transversal geometry. In Section 4 we introduce the notion of a principal set whose definition contains the basic geometric properties of the set of vertices of a free simplicial geometry with spanning simplex when the simplex is freely projected into a smaller dimensional space. We discuss properties of principal sets, give an algorithm for constructing them when they exist, and show that the principal sets are the bases of another geometry whose structure is determined. In Section 5 we carry out the erection of a geometry to a free simplicial geometry with spanning simplex and show the erection we perform is the freest of all possible erections. We conclude with some examples illustrating the theory developed.

Besides considering projections from points on the n-face of a free simplicial geometry (truncations of principal geometries), one can also consider projections from points on smaller dimensional faces. We plan to investigate this in a subsequent paper.

2. Basic concepts

In this section we discuss briefly the basic notions of combinatorial geometries that will be used in the paper.

A finite *pregeometry* [8] or *matroid* [13] $G(X)$ consists of a finite set of points X along with a closure operator $J(\cdot)$ defined on subsets of points which satisfies the exchange property: for any points x, y and any subset P of X, if $y \in J(P \cup \{x\})$, $y \notin J(P)$, then $x \in J(P \cup \{y\})$. A set $P \subseteq X$ is *closed* or is a *flat* of $G(X)$ if $J(P) = P$. A *geometry* is a pregeometry in which the empty set and points are closed sets. The lattice of flats of the pregeometry $G(X)$ is a geometric lattice, that is a semimodular point lattice. This lattice will also be denoted by $G(X)$. Since a geometric lattice satisfies the Jordan–Dedekind chain condition, each flat F has a well-defined rank $\rho(F)$. The rank of $P \subseteq X$ is defined by $\rho(P) = \rho(J(P))$. The rank of $G(X)$ is $\rho(X)$. The *hyperplanes* or *copoints* of $G(X)$ are the flats of rank $\rho(X) - 1$. A *proper flat* is a flat F with $\rho(F) \leqslant \rho(X) - 1$.

If F is a flat and $J(P) = F$, then P is said to *span* F. A set of points P is an *independent set* if $\rho(P) = |P|$. A *basis* of the flat F is an independent subset of F that spans F. An independent subset P of F is a basis of F if $\rho(F) = |P|$. A basis of $G(X)$ is simply a basis for the flat X. A set of points P is *dependent* if it is not independent; minimal dependent sets are called *circuits*. Every proper subset of a circuit is an independent set. A point x is a *loop* if $\{x\}$ is a dependent set. A point x is an *isthmus* of the flat F if x is in every basis of F. An isthmus of $G(X)$ is a point that is in every basis of $G(X)$.

If B is a basis of $G(X)$ and $x \notin B$, there exists a unique circuit C_x with $x \in C_x \subseteq B \cup \{x\}$; C_x is called a *fundamental circuit with respect to the basis B*. A flat F is *cyclic* if it has no isthmuses; equivalently, a flat F is cyclic if every element of F is contained in a circuit lying entirely in F. A pregeometry is determined up to isomorphism by its family of cyclic flats and their ranks [6]. If F is any flat of a pregeometry, then the *free part A* of F is the set of isthmuses of F; the set $F \setminus A$ is then a cyclic flat called the *cyclic part* of F.

A pregeometry can also be defined in terms of its independent sets, rank function, circuits or bases. For instance, a collection $\mathcal{C}$ of non-empty subsets of X is the set of circuits of a pregeometry on X if

(i) no member of $\mathcal{C}$ properly contains another,

(ii) (Circuit elimination property) $C_1, C_2 \in \mathcal{C}, x \in C_1 \setminus C_2, y \in C_1 \cap C_2$ imply the existence of $C_3 \in \mathcal{C}$ with $x \in C_3 \subseteq (C_1 \cup C_2) \setminus \{y\}$.

Likewise a collection $\mathcal{B}$ of subsets of X is the set of bases of a pregeometry on X if

(i) no member of $\mathcal{B}$ properly contains another,

(ii) $B_1, B_2 \in \mathcal{B}, x \in B_1 \setminus B_2$ imply the existence of $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

We can also define a pregeometry on X using the *submodular inequality* satisfied by the rank function ρ: $\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B)$.

Two pregeometries $G^1(X)$ and $G^2(Y)$ are *isomorphic* if there is a bijection $f: X \to Y$ such that A is an independent set of $G^1(X)$ if and only if $f(A)$ is an independent set in $G^2(Y)$.

There are several ways to obtain new pregeometries from given ones which we shall make use of. If $G(X)$ is a pregeometry with closure J and $X_0 \subseteq X$, then if we define for $A \subseteq X_0$, $J_0(A) = J(A) \cap X$, then J_0 is the closure operator of a pregeometry on X_0 which is denoted by $G(X_0)$ and called the *restriction* of $G(X)$ to X_0 or the *subgeometry* on X_0. Let $G^1(X)$ and $G^2(Y)$ be pregeometries on disjoint sets X and Y. Then the *direct sum* $G^1(X) \oplus G^2(Y)$ is the pregeometry $G(X \cup Y)$ whose independent sets are all unions of an independent set of $G^1(X)$ and an independent set of $G^2(Y)$. If $G(X)$ is a pregeometry of rank r and $k \leq r$, then those flats of $G(X)$ whose ranks are not equal to $k, k+1, ..., r-1$ form a geometric lattice of rank k which is denoted by $G_{(k)}(X)$ and called the *k-truncation* of $G(X)$. A set $A \subseteq X$ is an independent set of $G_{(k)}(X)$ if and only if A is an independent set in $G(X)$ and $|A| \leq k$. If $G(X)$ is the *free geometry* on X, $\mathcal{P}(X)$, where every set is independent, then $\mathcal{P}_{(k)}(X)$ is the pregeometry with every set of cardinality at most k independent.

If $G(X)$ is a pregeometry of rank k and $r \geq k$, then a pregeometry $G'(X)$ of rank r is an *r-erection* of $G(X)$ provided the k-truncation of $G'(X)$ is $G(X)$. While a k-truncation of a pregeometry is unique, an r-erection need not be.

Given a pregeometry $G(X)$ with closure operator J, let k be any non-negative integer. A subset D of X is *k-closed* [7] if and only if it contains the closures of all its j-element subsets for all $j \leq k$. The collection of k-closed subsets of $G(X)$ form a lattice which need not be geometric [7]. The *k-closure* of D, $J_k(D)$, is defined to be the smallest k-closed set containing D. If $p \in J_k(D)$, it need not be the case that $p \in J(D')$ for some k-element subset D' of D. In [7], Crapo shows that *a set D is k-closed* if and only if $D \cap F$ is closed for all flats F of rank k and proves the following interesting theorem.

Theorem 2.1. *A set $\mathcal{B}$ of subsets (called blocks) of the point set X of a rank k pregeometry $G(X)$ is the set of copoints of a $(k+1)$-erection of G if and only if*

(i) *each block contains at least one basis of $G(X)$;*

(ii) *each basis is contained in a unique block;*

(iii) *each block is $(k-1)$-closed.*

Crapo introduces a partial order on the collection consisting of the $(k+1)$-erections of the rank k pregeometry $G(X)$ and $G(X)$ as follows. Let $G^1(X)$ and $G^2(X)$ be two $(k+1)$-erections of $G(X)$ with sets of copoints $\mathcal{B}^1$ and $\mathcal{B}^2$, respectively. Then $G^1(X) \leq G^2(X)$ if and only if each copoint of $\mathcal{B}^1$ is contained in a copoint of $\mathcal{B}^2$. Crapo shows that using this partial order the collection of $(k+1)$-erections of $G(X)$ form a complete lattice with a least element which is called the *free $(k+1)$-erection* of $G(X)$. It is not difficult to see that an independent set of any $(k+1)$-erection of $G(X)$ is an independent set of the free $(k+1)$-erection. Thus the free $(k+1)$-erection contains all other $(k+1)$-erections as rank preserving weak map images (see [8] for definitions). It may happen that a pregeometry has no non-trivial erection; for example, three pairwise intersecting lines in the plane can not be erected to a rank 4 geometry.

An important class of pregeometries are the transversal pregeometries [9]. These are defined as follows: Let $(A_1, ..., A_n)$ be a family of subsets of a set X. A set T is a *transversal* (*partial transversal*) of $(A_1, ..., A_n)$ if there is a bijection (injection) $\sigma : T \to \{1, ..., n\}$ such that $x \in A_{\sigma(x)}$ $(x \in T)$. Hall's theorem [10] (see also [12]) asserts that $(A_1, ..., A_n)$ *has a transversal if and only if* $|\bigcup_{i \in I} A_i| \geq |I|$ for all $I \subseteq \{1, ..., n\}$. The collection of partial transversals of $(A_1, ..., A_n)$ are the independent sets of a pregeometry on X, called a *transversal pregeometry*. If a transversal pregeometry $G(X)$ has rank k, then there are k sets, say $A_1, ..., A_k$, such that the independent sets of $G(X)$ are precisely the partial transversals of $(A_1, ..., A_k)$ (see [5]). Such a family is called a *presentation* of $G(X)$. The sets $X \setminus A_1, ..., X \setminus A_k$ are flats of the pregeometry $G(X)$ (see [3]). If C is a circuit of $G(X)$, then $|\{i : C \cap A_i \neq \emptyset\}| = \rho(C) = |C| - 1$ (see, e.g., [4]).

Finally a *fundamental transversal pregeometry* [1] (see also [2]) is a pregeometry $G(X)$ for which there is a presentation $(A_i: 1 \leq i \leq k)$ and a basis B such that $|\{i: b \in A_i\}| = 1$ for each $b \in B$. Since B is a basis, $\sigma : B \to \{1, ..., k\}$ defined by $\sigma(b) = i$ if $b \in A_i$ is a bijection. A basis B with this property is called a *fundamental basis*. Since the sets $X \setminus A_i$ $(1 \leq i \leq k)$ are flats and have rank $k - 1$ (because $|(X \setminus A_i) \cap B| = k - 1$), they must be hyperplanes.

3. Imbedding transversal pregeometries on simplices

An *r-simplex* S in real affine space $\mathbf{R}^{r-1}$ is the convex hull $A(B)$ of a set $B = \{p_1, .., p_r\}$ of r affinely independent points which are termed *vertices*. A *k-face* $(1 \leq k \leq r)$ of an n-simplex is the convex hull of a set of k vertices. A point x in the r-simplex S is *on* the k-face determined by a set F of k vertices if x is in the convex hull $A(F)$ of F but not in the convex hull of any proper subset of F. If X is a finite set of points in $A(B)$, then using affine independence we obtain a combinatorial geometry $G(X)$ on X with closure operator J; if it is permitted to repeat points, then $G(X)$ may only be a pregeometry. Let a point x in X be on the face $A(F)$. Then x on $A(F)$ is said to be *freely situated* with respect to the pregeometry $G(X)$ provided the following is true: For each set $P \subseteq X$ with $x \notin P$, $x \in J(P)$ if and only if $A(F) \cap X \subseteq J(P)$. The pregeometry $G(X)$ is called *free simplicial with spanning simplex B* provided $B \subseteq X$ and every point of X is freely situated on its face. In such a pregeometry, only the vertices may be repeated (doubled). A pregeometry is termed *free simplicial* if it is a restriction of a free simplicial pregeometry with spanning simplex B (some of the vertices may be deleted).

Let $G(X)$ now represent any pregeometry on a set X with no loops. A basis B of $G(X)$ is a *principal basis* if subsets of B span all cyclic flats (equivalently all closures of circuits). A pregeometry with a principal basis is called a *principal pregeometry*. Brylawski [6] has proved the equivalence of the following three statements:

(i) $G(X)$ is isomorphic to a free simplicial pregeometry with spanning simplex B;

(ii) $G(X)$ is a principal pregeometry with principal basis B;

(iii) $G(X)$ is a fundamental transversal pregeometry with fundamental basis B.

From the fact that transversal pregeometries are precisely the restrictions of fundamental transversal pregeometries, it follows that the following two statements are equivalent:

(iv) $G(X)$ is a transversal pregeometry;

(v) $G(X)$ is isomorphic to a free simplicial geometry.

Some related results are due to Ingleton [11]. The method that Brylawski uses to identify a transversal pregeometry with a free simplicial geometry is algebraic (coordinates) and the connection between the geometric definition of a free simplicial geometry and the algebraic representation remains in the background. We offer here a coordinate-free *algorithm* for identifying a transversal pregeometry with a free simplicial geometry which, as in [6], uses only the set-theoretic properties of a presentation of the transversal pregeometry.

Let $G(X)$ be a transversal pregeometry on X of rank r with presentation $(A_i: i \in I)$, where $I = \{1, ..., r\}$. Set $F_i = X \setminus A_i$ $(i \in I)$. Let S be an r-simplex in $\mathbf{R}^{r-1}$ with vertex set $B = \{p_1, ..., p_r\}$. We label the k-face of S spanned the points $p_{i_1}, ..., p_{i_k}$ with the set $I \setminus \{i_1, ..., i_k\}$.

Algorithm for determining a free simplicial pregeometry isomorphic to the transversal pregeometry $G(X)$

(0) (Preliminary step) For each subset I_1 of I with $|I_1| = r - 1$, consider $\bigcap_{i \in I_1} F_i$. If this intersection is empty, then we adjoin to X a new point x_0 and put this new point in the set A_k if $I \setminus I_1 = \{k\}$ (equivalently, $x_0 \in F_i$ if and only if $i \in I_1$). The resulting set of points (X and the new points) is denoted by X^0 and the resulting pregeometry $G(X^0)$ is a transversal pregeometry with presentation $(A_1, ..., A_r)$ such that the restriction of $G(X^0)$ to X is the given pregeometry $G(X)$. (Note that we have not altered the notations for $A_1, ..., A_r, F_1, ..., F_r$ even though new points have been added to these sets.)

(1) Consider each $I_1 \subseteq I$ with $|I_1| = r - 1$. Identify each point in $\bigcap_{i \in I_1} F_i$ with the vertex of S whose label is I_1. In case $\bigcap_{i \in I_1} F_i$ contains more than one point, each of these points is identified with this vertex; this is the only time two different points of X can be identified with the same point of the simplex S.

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(k) Consider each subset $I_k \subseteq I$ with $|I_k| = r - k$ and the set of all points which are in $\bigcap_{i \in I_k} F_i$ for some I_k but not in $\bigcap_{i \in I_l} F_i$ for any I_l with $l < k$. Linearly order these points $x_1, ..., x_t$. For $1 \leq j \leq t$ there is a unique subset $I_k(j)$ of cardinality $r - k$ with $x_j \in \bigcap_{i \in I_k(j)} F_i$. Suppose $x_1, ..., x_{s-1}$ $(s \leq t)$ have already been identified with points of the simplex S. Pick a point *on* the face of S with label $I_k(s)$ in such a way that this point lies in an affine flat spanned by points of the simplex previously identified with points of X (this includes all such points obtained in steps (1), ..., $(k-1)$ and those identified with $x_1, ..., x_{s-1}$ in the present step) if and only if the affine flat contains the face of the simplex with label $I_k(s)$. For the same reason given at the end of step (2), such a point can be found.

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(r) Consider all points in X^0 not previously considered in steps (1), ..., $(r-1)$. (These points form a subset of the isthmuses of $G(X^0)$.) Linearly order these as $x_1, ..., x_t$ and as in previous steps we identify these points with points on the face of S with label $\emptyset$ (i.e., the interior of S).

The preceding algorithm identifies points of X^0 with points of the simplex S. The points of the simplex S so identified will also be denoted by X^0. The pregeometry on X^0 determined by affine independence will be denoted by $G'(X^0)$ and its closure operator by J'.

Theorem 3.1. *$G'(X^0)$ is a free simplicial pregeometry with spanning simplex B.*

Proof. Let $P \subseteq X^0$ with $x \in J'(P)$ but $x \notin P$. We choose P minimal with respect to these properties so that P is an independent set and $\{x\} \cup P$ is a circuit. Let it be that x is *on* the face F of the simplex. We show that x is freely situated on F. If x is a vertex, this is clearly so; thus we may assume x is not a vertex.

First suppose that x is the last point in $\{x\} \cup P$ that was chosen in the algorithm. If $F \subsetneq J'(P)$, then according to the algorithm when choosing x on F we would not have chosen x to be in $J'(P)$. Hence $F \subseteq J'(P)$.

Now suppose $y \neq x$ is the last point in $\{x\} \cup P$ that was chosen in the algorithm. If y were a vertex, then every point in $P \cup \{x\}$ (in particular, x) would be a vertex. Thus y is not a vertex. Suppose $F \subsetneq J'(P)$. We may choose a P with the above properties for which the element y (last element of $\{x\} \cup P$ chosen in the algorithm) is minimal with respect to the linear order of choosing points in the algorithm. Let y be on the face F^*. By the preceding argument (with y replacing x, F^* replacing F, $P^* = (P \setminus \{y\}) \cup \{x\}$ replacing P), we conclude that $F^* \subseteq J'(P^*) = J'(P)$. Since $B \cap F^*$ consists of the vertices of F^*, $(B \cap F^*) \cup \{y\}$ is a circuit of $G'(X)$. Applying the circuit elimination axiom to the two circuits $P \cup \{x\}$, $(B \cap F^*) \cup \{y\}$ where y is in both of these circuits but x is only in the first, we obtain a circuit C^* with

$$x \in C^* \subseteq (P \setminus \{y\}) \cup (B \cap F^*) \cup \{x\}.$$

Thus every point in C^* is chosen in the algorithm before y. But $x \in J'(C^* \setminus \{x\})$ and $y \notin C^* \setminus \{x\}$. Hence by the minimality restrictions we have imposed, $F \subseteq J'(C^* \setminus \{x\}) = J'(C^*)$. But since $\{x\}$, $F^* \subseteq J'(P)$,

$$J'((P \setminus \{y\}) \cup (B \cap F^*) \cup \{x\}) = J'(P);$$

hence $F \subseteq J'(C^*) \subseteq J'(P)$ and this is a contradiction. The theorem is proved.

Theorem 3.2. $G'(X^0) = G(X^0)$.

Proof. We prove the theorem by showing that every circuit of $G'(X^0)$ is a dependent set of $G(X^0)$ and that every circuit of $G(X^0)$ is a dependent set of $G'(X^0)$.

Let C be a circuit of $G'(X^0)$ with $|C| = k$ so that C has rank $\rho'(C) = k - 1$ in $G'(X^0)$. Let $x \in C$. Since, by Theorem 3.1, $G'(X^0)$ is a free simplicial geometry and since $x \in J'(C \setminus \{x\})$, $x \notin C \setminus \{x\}$, it follows that $F^x \subseteq J'(C \setminus \{x\}) = J'(C)$, where x is *on* the face F^x. Thus

$$\bigcup_{x \in C} F^* \subseteq J'(C).$$

Hence $\bigcup_{x \in C}(B \cap F^x) \subseteq J'(C)$, so that

$$k - 1 = \rho'(C) \geq \Big|\bigcup_{x \in C} (B \cap F^*)\Big|.$$

But $C \subseteq J'(\bigcup_{x \in C}(B \cap F^x))$ since $x \in J'(B \cap F^x)$ so that

$$\Big|\bigcup_{x \in C} (B \cap F^x)\Big| \geq \rho(C) = k - 1.$$

Hence $|\bigcup_{x \in C}(B \cap F^x)| = k - 1$ and C is contained in the $(k-1)$-face of the simplex with vertices $B \cap (\bigcup_{x \in C} F^x) = \{p_{i_1}, ..., p_{i_{k-1}}\}$. Let $J = I \setminus \{i_1, ..., i_{k-1}\}$ so that $|J| = r - k + 1$ and C is a subset of the $(k-1)$-face with label J. According to the algorithm, $C \subseteq \bigcap_{i \in J} F_i$ so that $C \cap (X^0 \setminus F_i) = C \cap A_i = \emptyset$ for all $i \in I$. Hence in the transversal pregeometry $G(X^0)$, $\rho(C) \leq r - |I| = k - 1$ and C is dependent.

Now consider a circuit C in $G(X^0)$ with $|C| = \rho(C) - 1 = k$. Since

$G(X^0)$ is a transversal pregeometry with presentation $(A_1, \ldots, A_r)$, if $K = \{i: C \cap A_i = \emptyset\}$, then $|K| = r - k + 1$. Hence $C \subseteq F_i = E \setminus A_i$ for all $i \in K$, so that $C \subseteq \bigcap_{i \in K} F_i$ and C is a subset of the $(k-1)$-face of S whose label is K. This means that C is a dependent set of $G'(X^0)$. The proof is completed.

Corollary 3.3. *A transversal pregeometry is isomorphic to a free simplicial pregeometry.*

Proof. This follows from Theorems 3.1 and 3.2 and the fact that $G(X)$ is the restriction of $G(X^0)$ to X while $G'(X)$ is the restriction of $G'(X^0)$ to X.

We conclude this section by remarking that there is considerable latitude in carrying out the algorithm. Because of Theorem 3.2 all yield isomorphic free simplicial geometries.

4. Principal sets

In this section we investigate a generalization of the notion of a principal basis, that of a principal set. As we will see in the next section, a principal set is precisely the result of a principal basis when a principal geometry is truncated. Before defining principal sets we would like to have one simple property of principal bases. We have defined a principal basis of a pregeometry to be a basis whose subsets span all cyclic flats. As a consequence of the next lemma we have that subsets of principal bases also span the intersection of cyclic flats.

Lemma 4.1. *If B is a basis of a pregeometry $G(X)$ and subsets of B span flats $F_1, \ldots, F_t$, then a subset of B spans $F_1 \cap \ldots \cap F_t$.*

Proof. Suppose $t = 2$. We have that $\rho(F_i) = |B \cap F_i|$ for $i = 1, 2$. Since $B \cap (F_1 \cup F_2)$ spans $J(F_1 \cup F_2)$ and B is an independent set, $\rho(F_1 \cup F_2) = |B \cap (F_1 \cup F_2)|$. By the submodular inequality,

$$\begin{aligned}\rho(F_1 \cap F_2) &\leqslant \rho(F_1) + \rho(F_2) - \rho(F_1 \cup F_2)\\ &= |B \cap F_1| + |B \cap F_2| - |B \cap (F_1 \cup F_2)|\\ &= |B \cap (F_1 \cap F_2)|\\ &\leqslant \rho(F_1 \cap F_2).\end{aligned}$$

We conclude that $\rho(F_1 \cap F_2) = |B \cap (F_1 \cap F_2)|$ and thus $B \cap (F_1 \cap F_2)$ spans $F_1 \cap F_2$.

The rest of the lemma follows by induction.

Let now $G'(X)$ be a pregeometry of rank r on set X with principal basis A. Thus $G'(X)$ can be regarded as a free simplicial geometry with spanning simplex A. Consider an integer k with $1 \leqslant k \leqslant r$ and the pregeometry $G(X) = G'_{(k)}(X)$ which is the k-truncation of $G'(X)$. With respect to $G(X)$, A has the following properties:

(i) $|A| = r$;

(ii) if F is any cyclic flat or the intersection of cyclic flats, then $A \cap F$ spans F;

(iii) every k-element subset of A is independent.

By convention, the empty intersection of cyclic flats of $G(X)$ is the collection of isthmuses of $G(X)$.

Property (ii) is a consequence of the definition of principal basis, the fact that every proper cyclic flat of $G(X)$ is a cyclic flat of $G'(X)$, and Lemma 4.1. Property (iii) is a direct consequence of the definition of k-truncation. In the presence of (iii), (ii) is equivalent to:

(ii′) A contains a basis of every cyclic flat and A contains the free parts of the intersections of cyclic flats.

We also note that if F is a proper flat of $G(X)$ which is either cyclic or the intersection of cyclic flats, then $F \cap A$ is a basis of F.

For any pregeometry $G(X)$ of rank k and integer $r \geqslant k$, a set A having properties (i), (ii) and (iii) is called an *(r, k)-principal set* of $G(X)$. For a pregeometry $G(X)$ we denote by $\mathrm{Fr}(G(X))$, or simply $\mathrm{Fr}(G)$, the set of all points which are in the free part of the intersection of some collection of cyclic flats. By our convention the set of isthmuses of $G(X)$ is a subset of $\mathrm{Fr}(G)$. Also every (r, k)-principal set contains $\mathrm{Fr}(G)$, according to (ii′). Before proving several properties of (r, k)-principal sets, we derive the following lemma.

Lemma 4.2. *Let x be a point of the pregeometry $G(X)$ with $x \notin \mathrm{Fr}(G)$. Then there is a unique smallest cyclic flat F_x which contains x.*

Proof. Suppose $F_1, ..., F_t$ are all the cyclic flats containing x (there must be at least one since x can not be an isthmus of $G(X)$). Then the flat $F_1 \cap ... \cap F_t = F \cup A$, where F is the cyclic part. Since $x \notin \mathrm{Fr}(G)$, $x \notin A$ and thus $x \in F$. Then $F = F_x$ is clearly the unique smallest cyclic flat containing x.

Theorem 4.3. *Let A be an (r, k)-principal set of a pregeometry $G(X)$. Let $x \in A$ and suppose that $x \notin \mathrm{Fr}(G)$. Let $y \in X \setminus A$. Then $(A\setminus\{x\}) \cup \{y\}$ is an (r, k)-principal set if and only if $F_y = F_x$.*

Proof. Since $y \notin A$ and $\mathrm{Fr}(G) \subseteq A$, F_y is defined.

(a) Suppose $(A\setminus\{x\}) \cup \{y\}$ is an (r, k)-principal set. First consider the case $F_x = X$. Suppose that $F_y \neq X$, so that F_y is a proper cyclic flat and thus $x \notin F_y$. Then $F_y \cap A = F_y \cap (A\setminus\{x\})$ is a basis of F_y. But also $F_y \cap ((A\setminus\{x\}) \cup \{y\})$ is a basis of F_y, and since $y \in F_y$ this is a contradiction. Thus $F_y = X = F_x$. Now consider the case where F_x is a proper cyclic flat. Since $F_x \cap A$ is a basis of F_x containing x and since $F_x \cap ((A\setminus\{x\}) \cup \{y\})$ is also a basis of F_x, we conclude that $y \in F_x$ and thus that $F_y \subseteq F_x$. By symmetry we also conclude that $F_x \subseteq F_y$, so that $F_x = F_y$.

(b) Now suppose $F_x = F_y$. We need to prove that $A' = (A\setminus\{x\}) \cup \{y\}$ satisfies the defining properties (ii) and (iii) of principal sets. We first show every k-element subset of A' is independent. First consider the case $F_x = F_y = X$. Then the only circuits containing y have cardinality $k + 1$. Hence in this case every k element subset of A' is independent. Now consider the case where $F_x = F_y$ is a proper cyclic flat. Suppose there were a circuit C with $y \in C \subseteq A'$ and $|C| \leq k$. Then $J(C)$ is a proper cyclic flat containing y so that $F_y \subseteq J(C)$. Now $A \cap J(C)$ is a basis of $J(C)$. Since $C\setminus\{y\}$ is a subset of A and is also a basis of $J(C)$, $A \cap J(C) = C\setminus\{y\}$. Since $A \cap F_y$ is a basis of F_y and $F_y \subseteq J(C)$, we conclude that $A \cap F_y \subseteq A \cap J(C) = C\setminus\{y\}$. But then $x \in A \cap F_x = A \cap F_y$ and thus $x \in C$. This is a contradiction, and every k-element subset of A' is independent.

We have left to show that subsets of A' span all proper cyclic flats. This will guarantee that subsets of A' span the intersection of cyclic flats since $x \notin \mathrm{Fr}(G)$. Thus let F be a proper cyclic flat. If $x \notin F$, the conclusion is obvious, so assume $x \in F$. We need only show that $y \in F$, for then since $A \cap F$ is a basis of F and every k-element subset of A' is independent, $|A' \cap F| = |A \cap F|$ and $A' \cap F$ is a basis of F. So consider $F_x \cap F$. Since $x \notin \mathrm{Fr}(G)$, $F_x \subseteq F$. Since $F_x = F_y$, $y \in F_x$. Thus $y \in F$. The proof of the theorem is now complete.

Lemma 4.4. *Let A_1 and A_2 be (r, k)-principal sets of the pregeometry $G(X)$. For proper cyclic flats $F_1, \ldots, F_t$,*

(i) $$\left|\left(\dot{\bigcup_{i=1}^{t}} F_i\right) \cap A_1\right| = \left|\left(\dot{\bigcup_{i=1}^{t}} F_i\right) \cap A_2\right|;$$

(ii) $$\left|\left(X \setminus \dot{\bigcup_{i=1}^{t}} F_i\right) \cap A_1\right| = \left|\left(X \setminus \dot{\bigcup_{i=1}^{t}} F_i\right) \cap A_2\right|.$$

Proof. Since $F_1, \ldots, F_t$ are proper cyclic flats, for each $\emptyset \neq K \subseteq \{1, \ldots, t\}$, $(\bigcap_{i\in K} F_i) \cap A_j$ is a basis of $\bigcap_{i\in K} F_i$ $(j = 1, 2)$ and hence

$$\left|\left(\bigcap_{i\in K} F_i\right) \cap A_1\right| = \left|\left(\bigcap_{i\in K} F_i\right) \cap A_2\right|.$$

From the inclusion-exclusion principle we conclude (i) holds. From (i) and the fact that $|A_1| = |A_2|$ we conclude (ii) also holds.

Theorem 4.5. *Let A_1, A_2 be (r, k)-principal sets of the pregeometry $G(X)$. Let $x \in A_1 \setminus A_2$. Then there exists $y \in A_2 \setminus A_1$ such that both $(A_1 \setminus \{x\}) \cup \{y\}$ and $(A_2 \setminus \{y\}) \cup \{x\}$ are (r, k)-principal sets.*

Proof. Since $x \notin A_2$, $x \notin \mathrm{Fr}(G)$ and thus F_x is defined. First suppose $F_x = X$. Applying (ii) of Lemma 4.4 to the collection of all proper cyclic flats of $G(X)$, we conclude there exists $y \in A_2 \setminus A_1$ with y in no proper cyclic flat of $G(X)$. For this y, $F_y = X$. By Theorem 4.3, $(A_1 \setminus \{x\}) \cup \{y\}$ and $(A_2 \setminus \{y\}) \cup \{x\}$ are both (r, k)-principal sets.

Now suppose F_x is a proper cyclic flat. Since $F_x \cap A_1$ is a basis of F_x and since $x \in F_x \cap (A_1 \setminus A_2)$, there exists $y \in F_x \cap (A_2 \setminus A_1)$. Since $y \notin A_1$, $y \notin \mathrm{Fr}(G)$ and F_y is defined. Since $F_y \subseteq F_x$, F_y is a proper cyclic flat. We conclude that

$$\bigcup_{y\in(A_2\setminus A_1)\cap F_x} F_y \cap (A_2 \setminus A_1) = (A_2 \setminus A_1) \cap F_x.$$

From (i) of Lemma 4.4 and the above relation we obtain

$$\begin{aligned}\left|\bigcup_{y\in(A_2\setminus A_1)\cap F_x} F_y \cap (A_1 \setminus A_2)\right| &= \left|\bigcup_{y\in(A_2\setminus A_1)\cap F_x} F_y \cap (A_2 \setminus A_1)\right| \\ &= |(A_2 \setminus A_1) \cap F_x| \\ &= |(A_1 \setminus A_2) \cap F_x|.\end{aligned}$$

On the other hand, since for each y under consideration $F_y \subseteq F_x$, we have that:

$$\bigcup_{y \in (A_2 \setminus A_1) \cap F_x} F_y \cap (A_1 \setminus A_2) \subseteq F_x \cap (A_1 \setminus A_2).$$

We conclude these last two sets are equal and thus that there exists a $y_0 \in (A_2 \setminus A_1) \cap F_x$ with $x \in F_{y_0} \cap (A_1 \setminus A_2)$. For this y_0, $F_x \subseteq F_{y_0}$ so that $F_x = F_{y_0}$. From Theorem 4.3 we obtain that $(A_1 \setminus \{x\}) \cup \{y_0\}$ and $(A_2 \setminus \{y_0\}) \cup \{x\}$ are (r, k)-principal sets and the theorem is proved.

As a final theorem concerning general properties of (r, k)-principal sets, we have the following.

Theorem 4.6. *Let $G(X)$ be a pregeometry which has at least one (r, k)-principal set. Let $P(G)$ be the intersection of all (r, k)-principal sets and let $C(G)$ equal $X \setminus (\bigcup F$: F a proper cyclic flat).*

(i) *If there is an (r, k)-principal set which contains $C(G)$, then $P(G) = C(G) \cup \mathrm{Fr}(G)$.*

(ii) *Otherwise, $P(G) = \mathrm{Fr}(G)$.*

Proof. By definition, $\mathrm{Fr}(G) \subseteq P(G)$. From (ii) of Lemma 4.4 applied to the collection of all proper cyclic flats, we conclude that one (r, k)-principal set contains $C(G)$ if and only if all do. By using Theorem 4.3, we obtain that if there is an (r, k)-principal set not containing $C(G)$, then given an element $y \in C(G)$ there is an (r, k)-principal set not containing y. Thus it suffices to show that if $x \notin \mathrm{Fr}(G)$ but F_x is a proper cyclic flat, then there is an (r, k)-principal set not containing x.

Consider such an x. Let A be an (r, k)-principal containing x; since F_x is a proper cyclic flat, $A \cap F_x$ is a basis of F_x. Since F_x is a cyclic flat, there exists $y \in F_x \setminus (A \cap F_x)$ such that the fundamental circuit C of y with respect to the basis $A \cap F_x$ of F_x contains x. If $A \cap F_x$ were not a subset of C, then $J(C)$ would be a cyclic flat containing x properly contained in F_x. Thus $C = (A \cap F_x) \cup \{y\}$. Since $y \in F_x$, $F_y \subseteq F_x$. Since $A \cap F_y$ is a basis of F_y and $A \cap F_y \subseteq A \cap F_x$, the fundamental circuit of y with respect to the basis $A \cap F_y$ of F_y must be C. We conclude that $A \cap F_x = A \cap F_y$ and thus that $F_x = F_y$. By Theorem 4.3, $(A \setminus \{x\}) \cup \{y\}$ is an (r, k)-principal set and this principal set does not contain x. This proves the theorem.

Theorem 4.5 has as a consequence that the collection of all (r,k)-principal sets of a pregeometry $G(X)$, if non-empty, are the bases of another pregeometry on X. The isthmuses of this pregeometry are determined by Theorem 4.6.

The previous theorems contain information concerning properties of (r, k)-principal sets. We now describe an algorithm which produces an (r, k)-principal set of the pregeometry $G(X)$ of rank k if one exists.

Algorithm for (r, k)-principal sets

(0) Set A_0 equal to $\mathrm{Fr}(G)$. If some j-element subset of A_0 ($j \leqslant k$) is dependent, we stop and the construction fails. If not, we proceed to step (1).

...

(t) Let F be a cyclic flat of rank t. Then $|F \cap A_{t-1}| \leqslant t$ If $|F \cap A_{t-1}| < t$, we select $t - |F \cap A_{t-1}|$ points in F in such a way that these points along with the points in $F \cap A_{t-1}$ are a basis of F. We do this for each cyclic flat of rank t. The set of points so selected along with the points in A_{t-1} form a set A_t. If some j-element subset of A_t ($j \leqslant k$) is dependent, we stop and the construction fails. Otherwise we proceed to step ($t + 1$).

...

(k) If $|A_{k-1}| > r$, the construction fails. If $|A_{k-1}| = r$, the construction is complete and successful. If $|A_{k-1}| < r$ and X is not a cyclic flat, the construction fails. If $|A_{k-1}| < r$, X is a cyclic flat and $|X \setminus C(G)| < r - |A_{k-1}|$, the construction fails. Otherwise we select $r - |A_{k-1}|$ points from $X \setminus C(G)$ so that these points along with the points in A_{k-1} form a set A_k; the construction is then complete and successful.

Theorem 4.7. *The pregeometry $G(X)$ of rank k has an (r, k)-principal set if and only if the previous algorithm can be successfully completed in which case A_k is an (r, k)-principal set.*

Proof. We have two statements to prove.

(a) *A_k is an (r, k)-principal set.* Surely $|A_k| = r$ and since $\mathrm{Fr}(G) \subseteq A_k$, subsets of A_k span all proper cyclic flats and their intersections. Since every k-element subset of A_{k-1} is independent and since no point of $A_k \setminus A_{k-1}$ is in a proper cyclic flat, every k-element subset of A_k is independent. Since $r \geq k$, A_k contains a basis of X. Thus A_k is an (r, k)-principal set.

(b) *If G has an (r, k)-principal set, then the algorithm can be successfully completed.* By definition, $\mathrm{Fr}(G) = A_0$ is a subset of every (r, k)-principal set. We prove by induction that there is an (r, k)-principal set equal to A_k. Suppose we have an (r, k)-principal set A which contains A_{t-1} $(t \leq k)$. Let $x \in A_t \setminus A$ and let F be the cyclic flat of rank t which puts x into A_t. Then $F_x \subseteq F$. If F_x had rank less than t, then since $F_x \cap A_{t-1}$ is a basis of F_x, $(F_x \cap A_{t-1}) \cup \{x\}$ would be a dependent subset of F, which is contrary to the way in which the algorithm is carried out. Thus the rank of F_x is t and $F_x = F$. Since then $F_x \cap A_{t-1}$ is not a basis of F_x, there is an element $y \in F_x \cap (A \setminus A_{t-1})$. For this y, $F_y = F_x$; otherwise F_y is a flat of rank less than t spanned by $A \cap F_y = A_{t-1} \cap F_y$ and $(A_{t-1} \cap F_y) \cup \{y\}$ is a dependent subset of A of cardinality at most $t \leq k$. By Theorem 4.3, $(A \setminus \{x\}) \cup \{y\}$ is an (r, k)-principal set containing $A_{t-1} \cup \{x\}$. We may repeat this argument until we obtain an (r, k)-principal set containing A_t. This completes the induction and the theorem is proved.

Theorem 4.8. *Let the pregeometry $G(X)$ of rank k have an (r, k)-principal set. Then the collection of all (r, k)-principal sets are the bases of a rank r pregeometry $G'(X)$. This pregeometry $G'(X)$ is the direct sum of the free geometry on the set* $\mathrm{Fr}(G)$ *and all the pregeometries $G_F(F)$ obtained as follows:*

Let F be a cyclic flat and let F' equal the union of all the cyclic flats of $G(X)$ which are properly contained in F and $\mathrm{Fr}(G)$. *Then*

$$G_F(F) = \mathcal{F}_{\rho(F)-\rho(F')}(F \setminus F').$$

Proof. The only comment that need be made, given the algorithm and the previous theorems, is that if F is a cyclic flat of rank t, then $P \cup (A_{t-1} \cap F)$ is a basis of F if and only if P is a subset of $F \setminus F'$ with $\rho(F) - \rho(F')$ elements.

We conclude this section with two remarks.

(1) If the algorithm can be successfully completed through the $(k-1)$st step, then G has an (r, k)-principal set for some $r \geq k$. The minimum such r is given by $\max\{|A_{k-1}|, k\}$. In particular $G(X)$ is a principal pregeometry if and only if $|A_{k-1}| \leq k$.

(2) Brylawski [6] gives a method for determining a principal basis whose verification depends on a certain inequality which he proves holds for principal pregeometries. Our algorithm when restricted to the case $r = k$, appears to be different.

5. Erections of pregeometries with principal sets

In the last section we have seen that if a pregeometry $G(X)$ is the k-truncation of a rank r principal pregeometry (a free simplicial pregeometry with spanning simplex), then $G(X)$ has an (r, k)-principal set. In this section we show that this property characterizes truncations of principal pregeometries. We do this by erecting a pregeometry with an (r, k)-principal set to a free simplicial geometry with spanning simplex of rank r. The erected pregeometry that we construct is actually the free erection of $G(X)$ to rank r in the sense of Crapo [7].

Let $G(X)$ be a pregeometry of rank k with closure operator J and rank function ρ, and let A be an (r, k)-principal set of $G(X)$ with $r > k$. We construct a collection of subsets of X, called *blocks*, which we show are the hyperplanes of a pregeometry $G^{(k+1)}(X)$ with $(G^{(k+1)})_{(k)}(X) = G(X)$ and with A an (r, k)-principal set of $G^{(k+1)}(X)$. The blocks are of two types:

(a) $J_{k-1}(P)$ for each k-element subset P of A, provided $J_{k-1}(P)$ is cyclic. These blocks are called *cyclic blocks*.

(b) The maximal sets of the form $H \cup \{x\}$, where H is a hyperplane of $G(X)$ and $x \in X \setminus H$, provided these sets are not contained in any block of (a). We note that if $J_{k-1}(P)$ is not cyclic, then $J_{k-1}(P) = H \cup \{x\}$ where H is a hyperplane and $x \notin H$.

The blocks can be algorithmically constructed as follows.

(0) Construct $J_{k-1}(P)$ for each k-element subset P of A.

(1) Construct $H \cup \{x\}$, where H is a cyclic hyperplane of $G(X)$ and $x \notin H$, and take only those not contained in sets of step (0).

...

(j) Construct $H \cup \{x\}$, where H is a hyperplane of $G(X)$ with exactly $j - 1$ isthmuses, and take only those not contained in sets of steps (0), ..., ($j - 1$).

...

Lemma 5.1. *This algorithm gives the blocks as described in* (a) *and* (b) *and no block properly contains another.*

Proof. We need only show that no block constructed in step (j) properly contains a block constructed in any of the previous steps or another block constructed in step (j).

Suppose that $J_{k-1}(P)$ is properly contained in $H_1 \cup \{x_1, ..., x_i\} \cup \{x\}$ where $H_1 \cup \{x_1, ..., x_i\}$ is a hyperplane of $G(X)$ with H_1 its cyclic part. Let P_1 be a $(k-1)$-element subset of P not containing x. Then $J_{k-1}(P_1) = J(P_1)$ is a hyperplane of $G(X)$ contained in $H_1 \cup \{x_1, ..., x_j\}$. Hence $J(P_1) = H_1 \cup \{x_1, ..., x_j\}$ and $|J_{k-1}(P)| \geqslant |H \cup \{x_1, ..., x_j\} \cup \{x\}|$ which is a contradiction.

Now suppose that $K_1 \cup \{y_1, ..., y_i\} \cup \{y\}$ is properly contained in $H_1 \cup \{x_1, ..., x_j\} \cup \{x\}$ where $K_1 \cup \{y_1, ..., y_i\}$ and $H_1 \cup \{x_1, ..., x_j\}$ are hyperplanes of $G(X)$ with K_1 and H_1 their respective cyclic parts, and where $i \leqslant j$. Because $i \leqslant j$, $\rho(H_1) \leqslant \rho(K_1)$. Regarded as a set, $H_1 \cup \{x_1, ..., x_j\} \cup \{x\}$ has x as an isthmus. Hence $K_1 \subseteq H_1$. Hence $\rho(K_1) \leqslant \rho(H_1)$, so that $H_1 = K_1$ and $i = j$. But then

$$|K_1 \cup \{y_1, ..., y_i\} \cup \{y\}| = |H_1 \cup \{x_1, ..., x_j\} \cup \{x\}|$$

and this is a contradiction.

Lemma 5.2. *The blocks are* $(k-1)$*-closed.*

Proof. The blocks of the form (a) are $(k-1)$-closed by definition. Consider a block of the form $H \cup \{x\}$ where H is a hyperplane of $G(X)$ and $x \notin H$. Write $H = H_1 \cup \{x_1, ..., x_j\}$ where H_1 is the cyclic part of H. If $j = 0$, $\{x_1, ..., x_j\}$ is regarded as empty. Let $B = \{x_1, ..., x_j, x\}$ so that B is the set of isthmuses of the block $H \cup \{x\}$.

Suppose $H \cup \{x\}$ were not $(k-1)$-closed. Then there must exist a $y \notin H \cup \{x\}$ and a circuit C with $y \in C \subseteq H \cup \{x\} \cup \{y\}$ and $|C| \leqslant k$. Suppose there were $z \in B \setminus C$. Then

$$C' = C \setminus \{y\} \subseteq (H_1 \cup B) \setminus \{z\} = H_1 \cup (B \setminus \{z\})$$

which is a flat of $G(X)$ of rank at most $k-1$. Hence

$$y \in H_1 \cup (B \setminus \{z\}) \subseteq H \cup \{x\},$$

a contradiction. Thus $B \subseteq C'$. Since H_1, $J(C)$ are proper cyclic flats,

$$|A \cap (H_1 \cup J(C))| = |A \cap H_1| + |A \cap J(C)| - |A \cap H_1 \cap J(C)|$$
$$= \rho(H_1) + |C| - 1 - \rho(H_1 \cap J(C)).$$

But we claim that $(C \setminus \{y\}) \setminus B = D$ is a basis of $H_1 \cap J(C)$; it surely is an independent set. Suppose there were $w \in (H_1 \cap J(C)) \setminus D$ such that $\{w\} \cup D$ is an independent set. Since B consists of isthmuses of $H \cup \{x\}$, $\{w\} \cup D \cup B$ is an independent subset of $J(C)$, of cardinality equal to $|C|$ and this is a contradiction. Hence D is a basis of $H_1 \cap J(C)$ and

$$\rho(H_1 \cap J(C)) = |D| = |C| - 1 - |B|.$$

We calculate that

$$|A \cap (H_1 \cup J(C))| = \rho(H_1) + |C| - 1 - (|C| - 1 - |B|)$$
$$= \rho(H_1) + |B| = k.$$

But now $J_{k-1}(A \cap (H_1 \cup J(C)))$ is a block of the type of Lemma 5.1 since H_1, $J(C)$ are both cyclic. This block contains $H \cup \{x\} \cup \{y\}$ and this contradicts the fact that $H \cup \{x\}$ is a block.

Lemma 5.3. *For a k-element subset P of A with $J_{k-1}(P)$ a block of type* (a),

$$J_{k-1}(P) = K,$$

where $K = \bigcup(F\colon F$ a cyclic flat, $\rho(F) \leqslant k-1, F \cap P$ spans $F)$.

Proof. The outline of the proof is as follows. It is clear that $K \subseteq J_{k-1}(P)$. We show that we have equality by proving that K is $(k-1)$-closed, and we do this by showing that $K \cap H$ is a flat for every hyperplane H of $G(X)$.

(i) Let H be a proper cyclic flat of $G(X)$. Let $x \in H \cap K$. Then $x \in F$, for some proper cyclic flat F with $F \cap P$ spanning F. Thus $x \in H \cap F$, the intersection of two cyclic flats, so that x depends on $(H \cap F) \cap A$. Hence $(H \cap K) \cap A$ spans at least all points in $H \cap K$. Now let y depend on $H \cap K$, so that y depends on $(H \cap K) \cap A$. Since H is a proper cyclic flat, $|A \cap H| = \rho(H) \leqslant k-1$. But $(A \setminus P) \cap K = \emptyset$, for otherwise $|F \cap A| > \rho(F)$ for some flat F used in the definition of K. Thus $(H \cap K) \cap A$ is a subset of P of cardinality at most $k-1$. Since H is a flat, y is surely in H. Since y depends on a proper subset of P, $y \in F$ for some F used in defining K so that $y \in K$. Hence $y \in H \cap K$, and $H \cap K$ is a flat of $G(X)$.

(ii) Let H be a hyperplane with at least one isthmus, say $H = H_1 \cup B$, where B is the non-empty set of isthmuses of H. Suppose $y \notin H \cap K$ but y depends on $H \cap K = (H_1 \cap K) \cup (B \cap K)$. By (i), $H_1 \cap K$ is a flat. Thus there is a circuit C with $y \in C \subseteq (H \cap K) \cup \{y\}$ such that $C \cap (B \cap K) \neq \emptyset$. But then $C \subseteq H \cup \{y\}$ so that $y \in H$ and $C \subseteq H$. Now C is a circuit in H containing at least one of the isthmuses in B and this is a contradiction.

Lemma 5.4. *Each block contains a basis of $G(X)$.*

Proof. The blocks constructed in (a) contain a k-element subset of A and thus a basis of $G(X)$. The blocks constructed in (b) are unions of a hyperplane of $G(X)$ with a point not in the hyperplane and hence contain a basis of $G(X)$.

Lemma 5.5. *Every basis of $G(X)$ is contained in exactly one block.*

Proof. (1) That every basis is in some block follows easily from the definition of blocks. For, if B is a basis, consider $B \setminus \{x\}$ for some $x \in B$. Then $J(B \setminus \{x\})$ is a hyperplane H not containing x and then $H \cup \{x\}$ is either a block or is contained in one.

(2) We show that every basis is contained in at most one block by showing that two distinct blocks intersect in a set of rank at most $k-1$.

Case (i). Let $K_1 = J(P_1)$ and $K_2 = J(P_2)$ be two cyclic blocks where P_1, P_2 are distinct k-element subsets of A. Since $K_i \cap A = P_i$ $(i = 1, 2)$, $|K_1 \cap K_2 \cap A| \leq k-1$. Now consider $x \in K_1 \cap K_2$. Since $x \in K_1$, $x \in F_1$ where F_1 is a proper cyclic flat spanned by $F_1 \cap P_1$. Likewise $x \in F_2$ where F_2 is a proper cyclic flat spanned by $F_2 \cap P_2$. Hence $x \in F_1 \cap F_2$ where $F_1 \cap F_2$ is a proper flat spanned by $(F_1 \cap F_2) \cap A \subseteq (K_1 \cap K_2) \cap A$. Thus every point in $K_1 \cap K_2$ is spanned by the set $(K_1 \cap K_2) \cap A$. Hence $\rho(K_1 \cap K_2) \leq k-1$.

Case (ii). Let K_1, K_2 be distinct blocks with $K_2 = H \cup B$, a block of type (a) having B as its non-empty set of isthmuses. Suppose the rank of $K_1 \cap K_2 = (K_1 \cap H) \cup (K_2 \cap B)$ were equal to k. Then a k-element independent subset of $K_1 \cap K_2$ equals the union of B with a basis of $H \cap K_1$. Since K_2 has rank k, a basis of $H \cap K_1$ must be a basis of H. But $H \cap K_1$ is a flat, since H is a proper flat of $G(X)$ and K_1 is a block. Hence $H \cap K_1 = H$ so that $H \subseteq K_1$ and $K_2 \subseteq K_1$. This is a contradiction.

We are now prepared to prove the main results of this section.

Theorem 5.6. *Let $G(X)$ be a pregeometry of rank k with an (r, k)-principal set A. Then the blocks of* (a) *and* (b) *are precisely the set of hyperplanes of a rank $k + 1$ pregeometry $G^{(k+1)}(X)$ where $(G^{(k+1)})_{(k)}(X) = G(X)$ and where A is an $(r, k + 1)$ principal set of $G^{(k+1)}(X)$.*

Proof. That there exists a rank $k + 1$ pregeometry $G^{(k+1)}(X)$ for which the blocks of (a) and (b) are the hyperplanes and where the k-truncation of $G^{(k+1)}(X)$ is $G(X)$ is an immediate consequence of the theorem of Crapo and Lemmas 5.2, 5.4 and 5.5. We have left to show that A is an $(r, k + 1)$-principal set of $G^{(k+1)}(X)$. For this we need to show that A spans all cyclic flats of $G^{(k+1)}(X)$ and their intersections, and that every $(k + 1)$-element subset of A is independent in $G^{(k+1)}(X)$.

Every cyclic flat F of $G^{(k+1)}(X)$ of rank less than k is a cyclic flat of $G(X)$ and hence is spanned by $A \cap F$. Now consider a cyclic hyperplane H of $G^{(k+1)}(X)$. Then $H = J_{k-1}(P)$, a block of type (a), where P is a k-element subset of A. By Lemma 5.3, such hyperplanes are unions of proper cyclic flats of $G(X)$ and the result follows. But likewise the intersection of cyclic flats of $G^{(k+1)}(X)$ are unions of intersections of proper cyclic flats of $G(X)$, so that for any such intersection D, $D \cap A$ spans D.

Now consider a $(k + 1)$-element subset A' of A. If A' were not an independent set in $G^{(k+1)}(X)$, then A' spans a hyperplane H of $G^{(k+1)}(X)$. Write $H = H_1 \cup B$ where H_1 is the cyclic part of H. But then

$$|H_1 \cap A'| \leq |H_1 \cap A| = \rho(H_1)$$

and

$$|B \cap A'| \leq |B| ,$$

so that $|H \cap A'| \leq \rho(H_1) + |B| = k$. But this is a contradiction since $A' \subseteq H$. This completes the proof of the theorem.

Theorem 5.7. *Let $G(X)$ be a pregeometry of rank k. Then $G(X)$ is the k-truncation of a principal pregeometry of rank r $(\geq k)$ if and only if $G(X)$ has an (r, k)-principal set.*

Proof. We have already observed that the k-truncation of a rank r principal pregeometry has an (r, k)-principal set.

Conversely, if $G(X)$ has an (r, k)-principal set A, then successive application of Theorem 5.6 produces a rank r pregeometry $G^*(X)$ whose k-truncation is $G(X)$ for which A is an (r, r)-principal set, that is, A is a principal basis.

Before getting to the next theorems concerning the free erection of a pregeometry, we introduce some terminology. Let $G(X)$ be a rank k pregeometry on X with an (r, k)-principal set A. Set $G_1(X) = G^{(k+1)}(X)$, a pregeometry for which A is an $(r, k+1)$-principal set, and define $G^{(k+2)}(X) = G_1^{(k+2)}(X)$, a pregeometry for which A is an $(r, k+2)$-principal set. In general, if $G_{l-k}(X) = G^{(l)}(X)$, a pregeometry for which A is an (r, l)-principal set, is defined for $k+1 \leq l < r$, then we define $G^{(l+1)}(X) = G_{l-k}^{(l+1)}(X)$, a pregeometry for which A is an $(r, l+1)$-principal set. Note that $G^{(r)}(X)$ is the pregeometry $G^*(X)$ of the proof of Theorem 5.7.

Lemma 5.8. *Given $k+1 \leq l \leq r$ and the pregeometry $G^{(l)}(X)$, if y is an element of a proper flat F of $G^{(l)}(X)$ with y not an isthmus of F, then there exists a proper cyclic flat F' of $G(X)$ with $y \in F' \subseteq F$.*

Proof. If F is a proper flat of $G(X)$, there is nothing to prove. Thus we may assume F is a rank t flat of $G^{(l)}(X)$ with $k+1 \leq t < l$. We can assume F is a cyclic flat of $G^{(l)}(X)$ since the removal of the isthmuses of F leaves a cyclic flat. By Lemma 5.3, the cyclic hyperplanes of $G^{(k+1)}(X)$ are unions of proper cyclic flats of $G(X)$ and the cyclic hyperplanes of $G^{(l)}(X)$ are unions of cyclic hyperplanes of $G^{(l-1)}(X)$. Hence, by induction all cyclic flats of $G^{(l)}(X)$ are unions of proper cyclic flats of $G(X)$.

Lemma 5.9. *Let $G(X)$ be a pregeometry of rank k with an (r, k)-principal set A. Let $k+1 \leq l \leq r$ and let $G^0(X)$ be any pregeometry on X of rank l with $(G^0)_{(k)}(X) = G(X)$. The following are true:*

(i) *Let $0 \leq t \leq l-1$ and let F be a flat in $G^{(l)}(X)$ which is spanned by a t-element subset P of A. Then if F^0 is the flat in $G^0(X)$ spanned by P, $F \subseteq F^0$.*

(ii) *For $x \in F^0$, $x \in F^0 \setminus F$ if and only if $x \cup D$ is a circuit in $G(X)$ for every k-element subset D of P.*

Proof. The result is clear if $t \leq k-1$, so that we may assume that $l-1 \geq t \geq k$. Let P be a t-element subset of A. Let $y \in F$, the flat in $G^{(l)}(X)$ spanned by P, with $y \notin P$. Since $y \notin P$, y is not an isthmus of F. By Lemma 5.8, there is a proper cyclic flat F' of $G(X)$ with $y \in F' \subseteq F$; the flat F' is spanned in $G(X)$ by a subset of P. But then $F' \subseteq F^0$, the flat in $G^0(X)$ spanned by P, since $(G^0)_{(k)}(X) = G(X)$. Hence $y \in F^0$. Since $P \subseteq F^0$, $F \subseteq F^0$.

Now suppose $x \in F^0$ but $x \notin F$ (thus $x \notin P$). Since $x \notin F$, $\{x\} \cup D$

cannot be a circuit of $G(X)$ for any $D \subsetneq P$ with $|D| < k$. Since each $D \subseteq P$ with $|D| = k$ is a basis of $G(X)$, $x \cup D$ is a circuit for all such D. The converse is immediate from Lemma 5.8.

The next theorem contains as a special case the fact that the pregeometry $G^{(k+1)}(X)$ is the free erection of $G(X)$ to rank $k + 1$.

Theorem 5.10. *Let $G(X)$ be a pregeometry of rank k with an (r, k)-principal set A. Let $k < l \leqslant r$. Let $G^0(X)$ be any pregeometry of rank l whose k-truncation is $G(X)$. Then given any flat F of $G^{(l)}(X)$ of rank t $(0 \leqslant t \leqslant l)$, there exists a flat F^0 of $G^0(X)$ of rank t with $F \subseteq F^0$. Thus $G^{(l)}(X)$ is the freest pregeometry of rank l which truncates to $G(X)$.*

Proof. By Lemma 5.9, any flat F of $G^{(l)}(X)$ which is spanned by a subset of A satisfies the conclusion of the theorem. Also any proper flat F of $G(X)$ is a flat of $G^0(X)$, and the conclusion follows trivially. Since the flats of $G^{(l)}(X)$ to which Lemma 5.9 applies includes all cyclic flats of $G^{(l)}(X)$, we need only consider flats F of $G^{(l)}(X)$ of rank t $(k \leqslant t < l)$ which have at least one isthmus. Consider such a flat F and write $F = F_1 \cup B$ where B is the non-empty free part of F and the rank of F_1 in $G^{(l)}(X)$ is $t_1 < t$. Then, by Lemma 5.9, $F_1 \subseteq F_1^0$ where F_1^0 is a flat of $G^0(X)$ whose rank does not exceed t_1. If ρ^0 denotes the rank function of $G^0(X)$, we have that $F \subseteq F_1^0 \cup B$ so that

$$\rho^0(F) \leqslant \rho^0(F_1^0 \cup B) \leqslant \rho^0(F_1^0) + |B|$$

$$\leqslant t_1 + |B| = t.$$

Thus F spans in $G^0(X)$ a flat of rank at most t, and the theorem is proved.

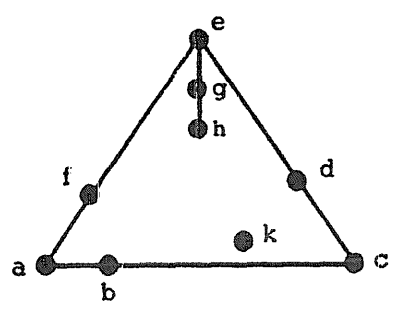

Fig. 1.

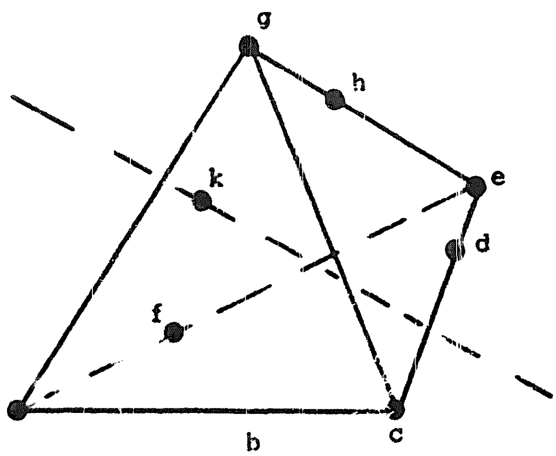

Fig. 2.

Corollary 5.11. *Let $G(X)$ be a pregeometry of rank k with A_1, A_2 two (r, k)-principal sets of $G(X)$. Then the rank $k+1$ geometries $G^1(X)$, $G^2(X)$ that are constructed from the principal sets A_1, A_2 respectively are identical.*

We conclude this section with some examples that illustrate the theory developed.

Example 5.12. Let $G(X)$ be the affine geometry of rank 3 depicted in Fig. 1.

In Fig. 1 only the 3 point lines are illustrated. There are no cyclic flats of ranks 0 or 1. The cyclic flats of rank 2 are the sets of points of the 3 point lines: $L_1 = \{a, b, c\}$, $L_2 = \{c, d, e\}$, $L_3 = \{a, f, e\}$, $L_4 = \{e, g, h\}$. We use the algorithm of Section 4 to determine any principal sets. The set $A_0 = \{a, c, e\}$. Since there are no cyclic flats of rank 1, $A_0 = A_1$. To construct A_2 we need to span cyclic flats of rank 2. The lines L_1, L_2, L_3 are already spanned by subsets of A_1. Thus we need only choose one other point from L_4 other than e, say g. Thus $A_2 = \{a, b, e, g\}$. Then A_2 is a $(4, 3)$-principal set and G has no $(3, 3)$-principal set. The set $A_3 = A_2 \cup \{k\}$ is a $(5, 3)$-principal set. Except for replacing g by h $(F_g = F_k)$ these are the only principal sets of $G(X)$.

Using the $(4, 3)$-principal set A_2, we can construct a principal geometry $G^{(4)}(X)$ of rank 4 whose 3-truncation $G^{(4)}(X)$ is depicted in Fig. 2 as an affine geometry in 3-space. The point k is in the interior of the simplex spanned by $\{a, c, e, g\}$.

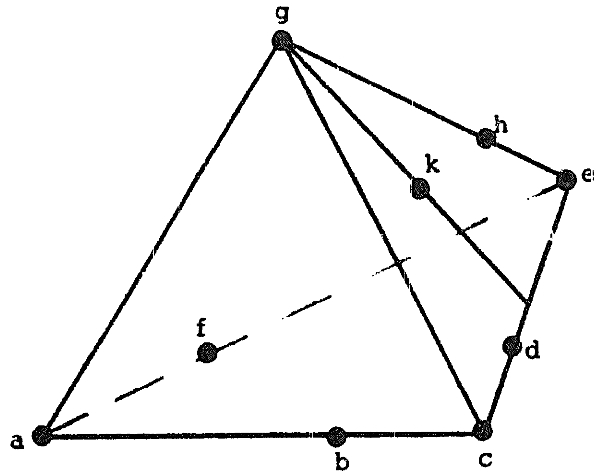

Fig. 3.

There are other erections of $G(X)$ in which A_2 is a (4, 4)-principal set or principal basis, but these are less free. These are accomplished by placing the point k in the interior of one of the 3-faces of the simplex so that k is on no three point lines. In Fig. 3 this is illustrated for one of the 3-faces.

Example 5.13. The rank 4 geometry illustrated in Fig. 4 is an example of a geometry which has no $(r, 4)$-principal set for any $r \geqslant 4$.

In the algorithm for constructing principal sets, $A_1 = \{a, b, c, d, e, f\}$ and the points a, b, c, d form a dependent set, so that the algorithm fails. Thus no free simplicial geometry of any rank can truncate to this geometry.

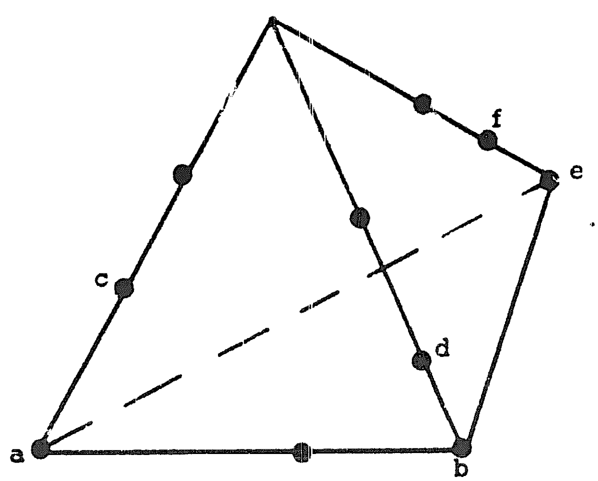

Fig. 4.

References

[1] J.A. Bondy and D.J.A. Welsh, Some results on transversal matroids and constructions for identically self-dual matroids, Quart. J. Math. Oxford (2) 22 (1971) 435–451.
[2] R.A. Brualdi, On fundamental transversal matroids, Proc. Am. Math. Soc. 45 (1974) 151–156.
[3] R.A. Brualdi and G.W. Dinolt, Characterizations of transversal matroids and their presentations, J. Combin. Theory 12 (1972) 268–286.
[4] R.A. Brualdi and J.H. Mason, Transversal matroids and Hall's theorem, Pacific J. Math. 41 (1972) 601–613.
[5] R.A. Brualdi and E.B. Scrimger, Exchange systems, matchings, and transversals, J. Combin. Theory 5 (1968) 244–257.
[6] T.H. Brylawski, An affine representation for transversal geometries, to appear.
[7] H.H. Crapo, Erecting geometries, in: R.C. Bose and T. Dowling (eds.), Proc. 2nd North Carolina Conference on Combinatorial Mathematics, 74–99.
[8] H.H. Crapo and G.C. Rota, Combinatorial Geometries, preliminary edition (M.I.T. Press, Cambridge, Mass., 1970).
[9] J.R. Edmonds and D.R. Fulkerson, Transversals and matroid partition, J. Res. Natl. Bur. Standards 69B (1965) 147–153.
[10] P. Hall, On representatives of subsets, J. London Math. Soc. 10 (1935) 26–30.
[11] A.W. Ingleton, A geometrical characterization of transversal independence structures, Bull. London Math. Soc. 3 (1971) 47–51.
[12] L. Mirsky, Transversal Theory (Academic Press, New York, 1971).
[13] W.T. Tutte, Lectures on matroids, J. Res. Natl. Bur. Standards 69B (1965) 1–47.