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# ON GRAPHS WITH A CONSTANT LINK, II

Morton BROWN
*University of Michigan, Ann Arbor, Mich. 48104, USA*

and

Robert CONNELLY
*Cornell University, Ithaca, N.Y. 14850, USA*

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**Introduction and summary of part I.** We study the following problem: For which finite graphs $L$ do there exist graphs $G$ such that the link (i.e., the neighborhood subgraph) of each vertex of $G$ is isomorphic to $L$? We give a complete solution for the cases (i) $L$ is a disjoint union of arcs, (ii) $L$ is a tree with only one vertex of degree greater than two, (iii) $L$ is a circle of prescribed length. Some other cases are also discussed. An interesting case is whether the situation is changed if we require $G$ also to be finite. It transpires (see for example, Corollaries VII. 3 and VII. 4) that this is indeed the case.

Part I of this paper will appear in [3]. It provides the basic definitions used in both part I and part II. Section III provides the basic tool, an identification procedure, that is used throughout the rest of the paper. Section IV sets up the basic building technique for the construction of more complicated graphs. It is shown how to build graphs such that the link of each vertex is an arc (of non-constant length), and how to control the proportional number of vertices with links of various lengths.

## V. Properties of graphs with arcs as links

Before we go on to build more complicated graphs, it is helpful at this stage to investigate some of the properties that graphs with arcs as links must have. In particular, we would like to prove that a certain class of types of Section IV is impossible. Although we do not obtain complete information as to what types are possible and what are not, we do obtain enough information for our use in later sections.

First, we wish to define certain invariants associated with a graph, and show some relationships. As in Section IV, we consider, in this section, only finite graphs that have an arc (of varying positive length) as the link of any vertex. Suppose $H$ is such a graph. Define $\rho_i = \rho_i(H)$

for $i = 1, 2, 3, \ldots$, as the number of vertices of $H$ which have an arc of length $i$ as a link. The main purpose of this section will be to derive certain relationships among the $\rho_i$.

Recall from Section II that we have associated a simplicial complex $K(H)$ with a graph $H$. Let us denote the Euler characteristic of $K(H)$ by $\chi = \chi(K(H))$. (Not to be confused with the chromatic number.) One very basic fact from algebraic topology is that

$$\chi = \sum_i (-1)^i \beta_i ,$$

where $\beta_i$ is the number of $i$-simplices in $K(H)$. We shall use various properties of $\chi$ which are well known and can be found in [7, 9]. In particular, $\chi$ is a topological invariant (in fact a homotopy invariant), the only connected 2-manifold with a boundary that has a positive Euler characteristic is the disk, and the Euler characteristic is the sum of the Euler characteristics of each of the components.

We should remark that, for our $H$'s, $K(H)$ is a 2-manifold with boundary since, if there were a 3-simplex in $K(H)$, some link in $H$ would not be an arc. Thus $\chi = \beta_0 - \beta_1 + \beta_2$. Let $\Sigma_i \rho_i = m$, the total number of vertices. Then $\beta_0 = m$, $\beta_1 = \frac{1}{2} \Sigma_i (i+1)\rho_i$ and $\beta_2 = \frac{1}{3} \Sigma_i i\rho_i$ since there are $i+1$ arcs adjacent to any fixed vertex whose link has arc length $i$, and there are $\rho_i$ of these vertices, where each edge is counted twice. Similarly, there are $i$ 2-simplices (or triangles) adjacent to any fixed vertex whose link has length $i$, there are $\rho_i$ of these vertices, and each 2-simplex is counted 3 times. We may also intepret $\Sigma_i (i\rho_i/m)$ as the average length of a link of a vertex in $H$. Using the previously mentioned formula for $\chi$, we obtain

$$(14) \qquad \chi = \sum_i \rho_i - \tfrac{1}{2} \sum_i (i+1)\rho_i + \tfrac{1}{3} \sum_i i\rho_i = \tfrac{1}{6} \sum_i (3-i)\rho_i$$
$$= \tfrac{1}{6}(+2\rho_1 + \rho_2 - \rho_4 - 2\rho_5 - \ldots) = \tfrac{1}{6} m \left(3 - \frac{1}{m} \sum_i i\rho_i\right).$$

Notice from the last formula above that, if $H$ is connected, and the average length of a link in $H$ is less than 3, then $H$ must be a triangulation of a 2-disk with all its vertices on the boundary – a maximal outer planar graph.

We state one of the basic inequalities needed later.

**Lemma V.1.** $2\chi \leqslant \rho_1$.

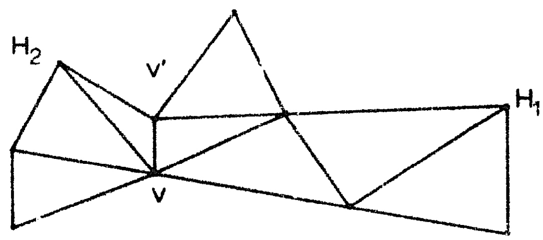



Fig. 1.

**Proof.** Since the Euler characteristic is the sum of the Euler characteristics of each component, we need only prove the above inequality when $H$ is connected. In this case the inequality is obvious unless $\chi > 0$ and in particular unless $\chi = 1$. Thus, we are reduced to the case when $K(H)$ is a triangulation of a 2-disk, and we wish to show that there are at least 2 vertices which have only one edge in their link. This is easy to show by induction on $m$, the total number of vertices in $H$. If $m = 3$, all three vertices have one edge in their link. If $m > 3$, consider any edge $\langle v, v' \rangle$ that is not in the boundary $\partial H$ — if every edge is in $\partial H$, then every vertex must have a link of length = 1 and thus $m = 3$. Since both $v$ and $v'$ are in the boundary of $H$, $\langle v, v' \rangle$ separates $H$ into 2 pieces, $H_1$ and $H_2$, with $H_1 \cap H_2 = \langle v, v' \rangle$, where $K(H_1)$ and $K(H_2)$ are triangulations of a 2-disk (see Fig. 1).

By the induction hypothesis both $H_1$ and $H_2$ must have 2 vertices whose link (in $H_1$ or in $H_2$) has length 1. We may assume that one of each of these two vertices is not $v$ or $v'$, since if the $lk(v, H_1)$ and $lk(v', H_1)$ have length 1, then $H_1$ must be a single 2-simplex, and the third vertex has length 1. Similarly for $H_2$. Thus, in all cases there is at least one vertex in each of $H_1$ and $H_2$ whose link has length 1 in $H$. Thus, the inductive step is proved. (The fact that $\rho_1 \geqslant 2$ for disks is mentioned in [2].)

**Corollary V.2.**

$$\rho_2 \leqslant \rho_1 + \sum_{i=4}^{\infty} (i-3)\rho_i = \rho_1 + \rho_4 + 2\rho_5 + \dots .$$

**Proof.** Apply formula (14).

Note that the above inequality is an equality if and only if each component is either an annulus or Möbius band with no vertices whose link has length 1, or is a disk with exactly 2 vertices whose link has length 1.

The above inequality also has the useful property that a constant can be factored out. The importance of this is the following: Suppose $H$ is a finite graph which is as in Section IV and has the type $(n_1, ..., n_k)$. Let $\lambda_i$ denote the number of $n_j$'s, $j = 1, ..., k$, which are equal to $i$, for $i = 1, 2, ...$ . It is easy to see thus that from the definition of a type there is a constant positive integer $\alpha$ such that there are $\rho_i = \alpha\lambda_i$ vertices in $H$ whose link has length $i$, for $i = 1, 2, ...$ . Thus, we have:

**Corollary V.3.** *Let $H$ be a finite graph with a type $(n_1, ..., n_k)$. Suppose for $i = 1, 2, ..., \lambda_i$ of the $n_i$ are equal to $i$. Then*

$$\lambda_2 \leqslant \lambda_1 + \sum_{i=4}^{\infty} (i-3)\lambda_i = \lambda_1 + \lambda_4 + 2\lambda_5 + \dots .$$

Note that this puts a definite restriction on the types that can occur. In particular, it says that a $Z$-regular graph with an arc of length 2 as its common link cannot occur. Perhaps this is a round about way to prove the fact, but, as we shall see, it has a great advantage in the more general situation.

Using the results of Section IV, we proceed to prove a sort of converse to Corollary V.2.

**Lemma V.4.** *Let $r_1, r_2, ...$ be a finite number of non-negative rational numbers such that*

(a) $\Sigma_i r_i = 1$;

(b) $r_2 \leqslant r_1 + \Sigma_{i=4}^{\infty}(i-3)r_i = r_1 + r_4 + 2r_5 + 3r_6 + \dots$ .

*Then there is a finite graph $H$ with $\rho_i$ vertices with a link of length $i$, $i = 1, 2, ...$, where $\rho_i/m = r_i$, and $m = \Sigma_i \rho_i$ is the total number of vertices in $H$.*

**Proof.** (Note all the vertices of $H$ have an arc of positive length as their link.) We first make the crucial observation that the set $C = \{(r_1, r_2, ...)$: there is a finite graph $H$ with $r_i = \rho_i(H)/m(H)\}$ is a rational convex set, where the $H$ in the definition of $C$ has the link of each vertex an arc of varying length, and $\rho_i(H)$ is the number of vertices of $H$ with link of length $i$, and $m(H) = \Sigma_i \rho_i(H)$. Thus, suppose $(r_1, r_2, ...), (r'_1, r'_2, ...) \in C$, and $p$ and $q$ are two non-negative integers not both 0. We wish to show

$$\frac{p}{p+q}(r_1, r_2, \dots) + \frac{p}{p+q}(r'_1, r'_2, \dots) \in C.$$

Let $H$ be the graph corresponding to $(r_1, r_2, ...)$ and $H'$ the graph corresponding to $(r'_1, r'_2, ...)$. Let $aH$ denote $a$ disjoint copies of $H$ and $bH'$ $b$ disjoint copies of $H'$ (disjoint from $aH$ as well), where $a = pm(H')$ and $b = qm(H)$. Then

$$\rho_i(aH \cup bH') = a\rho_i(H) + b\rho_i(H') = pm(H')\rho_i(H) + qm(H)\rho_i(H'),$$

$$m(aH \cup bH') = am(H) + bm(H') = (p+q)\,m(H)\,m(H').$$

Thus

$$r_i(aH \cup bH') = \frac{pm(H')\rho_i(H) + qm(H)\rho_i(H')}{(p+q)\,m(H)\,m(H')} = \frac{p}{p+q}r_i + \frac{q}{p+q}r'_i.$$

Thus $C$ is convex, and to complete the proof we need only show that the extreme points of the set defined by (a) and (b) are in $C$. It is easy to see that all the extreme points have at most 2 non-zero coordinates, and, if $r_2 = 0$, (b) can be ignored. Thus it is easy to check that the following is a list of the extreme points of the set defined by (a) and (b): $(1,0,0,0)$, $(0,0,1,0, ...)$, $(0,0,0,1, ...)$; and when (b) is an equality: $(\frac{1}{2}, \frac{1}{2})$, $(0,\frac{1}{2},0,\frac{1}{2})$, $(0,\frac{2}{3},0,0,\frac{1}{3},0,0, ...)$, $(0,\frac{3}{4},0,0,0,\frac{1}{4},0, ...)$, ... .

Now suppose $H$ is a finite graph as in Section IV with type $(n_1, n_2, ..., n_k)$, and $\lambda_i$ of the $n_i$'s are $i$. It is clear that $\Sigma_i \lambda_i = k$, and from our earlier comments $\lambda_i/k = \rho_i(H)/m(H) = r_i(H)$. Thus, if we know the type, we can compute the $r_i$'s. Hence, in particular: For the triangle of type $(1,1,1)$, $\lambda_1 = 3$; $(r_1, r_2, ...) = (1,0,0, ...)$. By (8) of Section IV, $\exists (n)^2$, $n \geq 3$, $\lambda_n = 2$;

$$(r_1, ...) = (0,0, ..., \underset{\uparrow\, n\text{th position}}{1},0, ...).$$

By (7) of Section IV, $\exists\,(1,2)^2$, $\lambda_1 = 2$, $\lambda_2 = 2$;

$$(r_1, r_2, ...) = (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, ...).$$

By (10) of Section IV, $\exists\,(\underbrace{2, 2, ...,}_{n-3\text{ 2's}} n)$, $n \geq 3$, $\lambda_2 = n-3$, $\lambda_n = 1$;

$$(r_1, r_2, ...) = (0, (n-3)/(n-2), 0, 0, ..., \underset{\uparrow\, n\text{th position}}{1/(n-2)}, 0, ...).$$

Thus we see that all the extreme points are in $C$.

Although Lemma V.4 is interesting in its own right, the real purpose will be seen later in the construction of $Z$-regular graphs with the disjoint union of arcs as its common link. Similarly, the following two lemmas are used in the investigation of $Z$-regular graphs with an $m$-ad as its common link, although the next lemma does give some more information as to which types (of Section IV) are not possible.

**Lemma V.5.** *Let $H$ be a finite graph such that the link of each vertex is an arc (of variable length $> 0$). Suppose that no connected component of $H$ is a triangle. Let $\rho_i$, $i = 1, 2, \ldots$, be the number of vertices whose link has length $i$. Then $2\chi \leqslant \rho_2 + \rho_3$.*

**Proof.** As with Lemma V.1, since the Euler characteristic of two disjoint graphs is the sum of the Euler characteristics of each graph, we need only to prove the inequality when $H$ is connected. In this case the inequality is obvious unless $\chi > 0$ and in particular unless $\chi = 1$. Thus, we may assume that $K(H)$ is a triangulation of a 2 disk with all vertices on the $\partial K(H)$ (a maximal outer planar graph), and we wish to show that there are at least 2 vertices whose link has length 2 or 3, if $H$ is not a triangle. In fact, we shall prove the following more general statement: If $H$ is a graph, $\neq$ a triangle, $\ni K(H)$ is a triangulation of a 2 disk, then there exist at least 2 vertices, $v, w \in H$, such that $lk(v, H)$ and $lk(w, H)$ have length 2 or 3, and $v$ and $w$ are *not* adjacent along $\partial H$ (i.e., if $\langle v, w\rangle \in H$, $\langle v, w\rangle \notin \partial H$). We show this by 2 cases.

*Case* 1: Every interior edge of $H$ separates $H$ into two components one of which is a triangle.

Here it is easy to check that there are only three isomorphism classes of such $H$'s (the subgraph of $H$ determined by the vertices whose link has length greater than 1 is again either a disk with all its edges on the new boundary (i.e., a triangle) or a single edge), and here it is clear

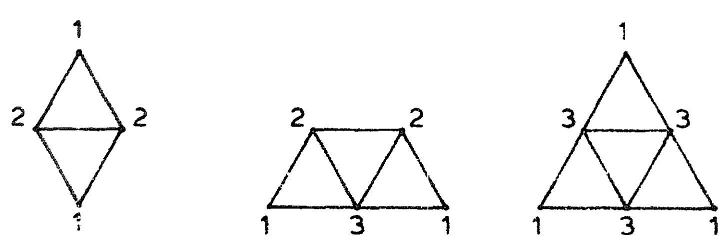



Fig. 2.

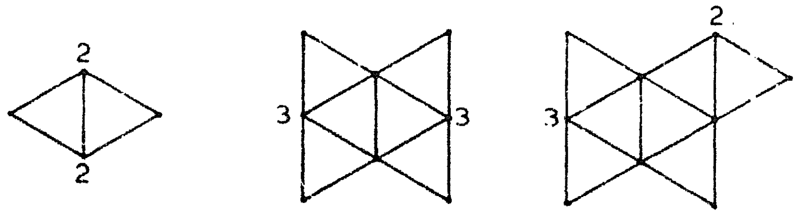



Fig. 3.

that the more general statement holds (see Fig. 2).

*Case* 2: There is an interior edge $e$ of $H$ which separates $H$ into two components neither of which is a triangle.

This follows easily by induction on the number of vertices and Case 1: Namely, let $H_1$ and $H_2$ be the components which are separated by $e$ (so that $H_1 \cap H_2 = e$, $H_1 \cup H_2 = H$). By induction on the more general statement (and implicitly Case 1), both $H_1$ and $H_2$ have a vertex, not on $e$, whose link is of length either 2 or 3, and thus they have this same property in $H$, and are not adjacent (let alone along $\partial H$).

**Remark V.6.** Again it is easy to see that equality holds if and only if each component of $H$ consists of a disk, a Möbius band, or annulus, where each disk has precisely 2 vertices which have links of length 2 or 3, and the annuli and Möbius bands have no vertices whose link is of length 2 or 3.

Also, the examples shown in Fig. 3 demonstrate explicitly that the coefficients on $\rho_2$ and $\rho_3$ cannot be improved.

**Corollary V.7.** $2\rho_1 \leq 2\rho_2 + 3\rho_3 + \rho_4 + 2\rho_5 + \dots (n-3)\rho_n + \dots$ *(if $H$ has no triangles).*

**Proof.** Apply formula (14) for $\chi$.

In the construction of $Z$-regular graphs with an $m$-ad as a common link it is important to know about graphs $H$ with an arc as their (variable) link when each component of $\partial H$ is a circle of even length. In particular, we are not interested (in this case) when some component of $H$ is a triangle. Thus, as with Lemma V.5, if $H$ has no triangles as a component, no two vertices adjacent along $\partial H$ can both have links of length 1. Thus, we know that at most half of the vertices have link of length 1. Thus, if $H$ has no triangles, $r_1 \leq \frac{1}{2} \leq r_2 + r_3 + \dots$ is automatic.

The next lemma again provides a kind of converse to Corollary V.7, with previous inequalities in mind, and is a key to the construction of $Z$-regular graphs with an $m$-ad as common link.

**Lemma V.8.** *Let $r_1, r_2, \ldots$ be a finite number of non-negative rational numbers such that:*

(a) $$\sum_i r_i = 1;$$

(b) $$r_2 \leqslant r_1 + \sum_{i=4}^{\infty} (i-3) r_i = r_1 + r_4 + 2r_5 + \ldots;$$

(c) $$r_1 \leqslant \sum_{i=2}^{\infty} r_i = r_2 + r_3 + r_4 + \ldots;$$

(d) $$2r_1 \leqslant 2r_2 + 3r_3 + \sum_{i=4}^{\infty} (i-3) r_i = 2r_2 + 3r_3 + r_4 + 2r_5 + \ldots.$$

*Then there is a finite graph $H$ with $\rho_i$ vertices with as link an arc of length $i$, $i = 1, 2, \ldots$, where $\rho_i/m = r_i$, and $m = \Sigma_i \rho_i$ is the total number of vertices in $H$. Furthermore, each component of $\partial H$ is a circle of even length.*

**Proof.** We use the same notation here as in Lemma V.4, and in fact the outline of the proof here is the same as that for Lemma V.4.

We first observe that $C = \{(r_1, r_2, \ldots)$: there is a finite graph $H$ with $r_i = \rho_i(H)/m(H)$, and every component of $\partial H$ has even length$\}$ is a rational convex set. The proof of this is much the same as the proof in the first part of Lemma V.4. We need only make the additional observation that the disjoint union of graphs whose boundary components have even length is a graph whose boundary components have even length.

Thus, to complete the proof we need to show that the extreme points of the convex set defined by (a), (b), (c) and (d) are in $C$. It is clear that at most 4 coordinates of any extreme point are non-zero, and, in fact, if (b) and either (c) or (d) are equalities, then $r_1 = r_2$ and $r_3 = r_4 = r_5 = \ldots = 0$. Thus, at most 3 coordinates of any extreme point are non-zero. We obtain the following list of extreme points:

(e) $(0,0,1,0,0,\ldots)$, $(0,0,0,1,0,\ldots)$, … ;

(f) $(\frac{1}{2},\frac{1}{2},0,0,\ldots)$, $(0,\frac{1}{2},0,\frac{1}{2},0,0,\ldots)$, $(0,\frac{2}{3},0,0,\frac{1}{3},0,\ldots)$, …, when (b) is an equality;

(g) $(\frac{1}{2},0,\frac{1}{2})$, $(\frac{1}{2},0,0,0,\frac{1}{2})$, $(\frac{1}{2},0,0,0,0,\frac{1}{2})$, ..., when (c) is an equality;
(h) $(\frac{1}{3},0,0,\frac{2}{3},0,0,\ldots)$, when (d) is an equality (other possible points violate (c)):
(i) $(\frac{1}{2},0,\frac{1}{4},\frac{1}{4},0,0,\ldots)$, $(\frac{1}{2},0,0,\frac{1}{4},0,\frac{1}{4},\ldots)$, $(\frac{1}{2},0,0,\frac{2}{6},0,0,\frac{1}{6},0)$, ..., $(\frac{1}{2},0,0,(n-5)/(2n-8),\ldots,1/(2n-8)\ldots)$, when (c) and (d) are equalities.

The points in (e) and (f) were discussed in Lemma V.4 and are in $C$ since $\exists(n,n)$, $n \geqslant 3$, $\exists(1,2)$ and $\exists(2,\ldots,2,n-3)^2$, $n > 3$, and each graph has all boundary components of even length.

For points in (g) we know by (9) of Section IV that $\exists(1,n)$, $n = 3$, $n \geqslant 5$, and hence $\lambda_1 = 1$, $\lambda_n = 1$, so

$$(r_1, r_2, \ldots, r_n, \ldots) = (\tfrac{1}{2},0,0,\ldots,\underset{\uparrow\, n\text{th slot}}{\tfrac{1}{2}},0,0,\ldots).$$

For the point in (h) the graph of (13) of Section IV $\Rightarrow \exists(1,4,4)^2$ and $\lambda_1 = 2$, $\lambda_4 = 4$ so $(r_1,\ldots) = (\frac{1}{3},0,0,\frac{2}{3},0,\ldots)$.

For points in (i) by (12) of Section IV $\Rightarrow \exists(1,4,1,3) \Rightarrow \lambda_1 = 2$, $\lambda_3 = 1$, $\lambda_1 = 1$, so $(r_1, r_2, \ldots) = (\frac{1}{2},0,\frac{1}{4},\frac{1}{4},0,\ldots)$ which is the first point of (i). By (11) of Section IV,

$$\exists\,(\underbrace{1,4,1,4,\ldots,1,4,}_{n-5\text{ times}}1,n)^2,\ n \geqslant 6, \text{ and } \lambda_1 = 2(n-4),\ \lambda_4 = 2(n-5), \lambda_n = 2$$

so

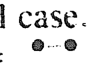
$$(r_1,\ldots) = (\tfrac{1}{2},0,0,(n-5)/(2n-8),0,0,\ldots,\underset{\uparrow\, n\text{th slot}}{1/(2n-8)},0,\ldots,0,\ldots),$$

which are the other points of (i).

Note all the graphs mentioned above have boundary components of even length so all the points of (e), ..., (i) are in $C$.

## VI. Cutting graphs apart

We now come to one of the main applications of Section V. We show how the graphs of Section V arise "naturally" from $Z$-regular graphs. The process, roughly speaking, is to cut open the $Z$-regular graphs along vertices, edges, or whatever is handy. We first describe the inspiration for the process when we "cut" along vertices, for a special case.

Suppose $G$ is a $Z$-regular graph with a common link $L =$ . Let $L_1 =$  and $L_2 =$ . Then locally $G$ looks as shown in Fig 4.

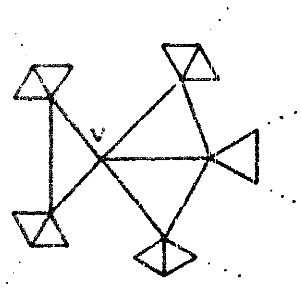



Fig. 4.

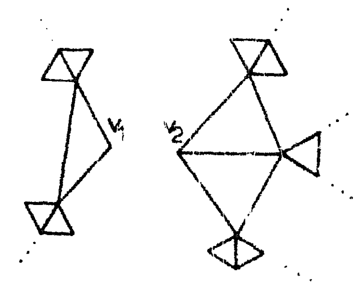



Fig. 5.

Now it is very natural to "cut" $G$ at $v$ to obtain a new graph that looks as the graph in Fig. 5, where two new vertices are created, $v_1$ and $v_2$, in place of the old $v$, and $lk(v_i, H_1) \cong L_i$, $i = 1, 2$, where $H_1$ is the cut graph. If we proceed to cut at all the vertices of $G$, then we will have created a graph $H$ where half of the vertices have a link isomorphic to $L_1$ and the other half a link isomorphic to $L_2$.

In order to generalize the above we first make a simple generalization of links. Namely, we wish to define the link of an edge in a graph $G$. If $\langle v, v'\rangle \in G$ is an edge, we define $lk(\langle v, v'\rangle, G)$ as the subgraph of $G - \{v\} - \{v'\}$ determined by those vertices adjacent to both $v$ and $v'$. It is easy to check that

$$lk(\langle v, v'\rangle, G) = lk(v, lk(v', G)) = lk(v', lk(v, G)).$$

Next, we need some notation to deal with $Z$-regular graphs. Let $G$ be a $Z$-regular graph with common link $L$. We regard $L$ as a fixed graph, and for each $v \in G$ we have a graph isomorphism $\theta_v : lk(v, G) \to L$. If $\langle v, v'\rangle \in G$ is an edge, then it is clear that

$$\theta_v^{-1} \mid lk(\theta_v(v'), L) : lk(\theta_v(v'), L) \to lk(v', lk(v, G)) = lk(\langle v, v'\rangle, G)$$

is an isomorphism. Thus we obtain an isomorphism

$$\theta_{v',v} : lk(\theta_v(v'), L) \to lk(\theta_{v'}(v), L),$$

where

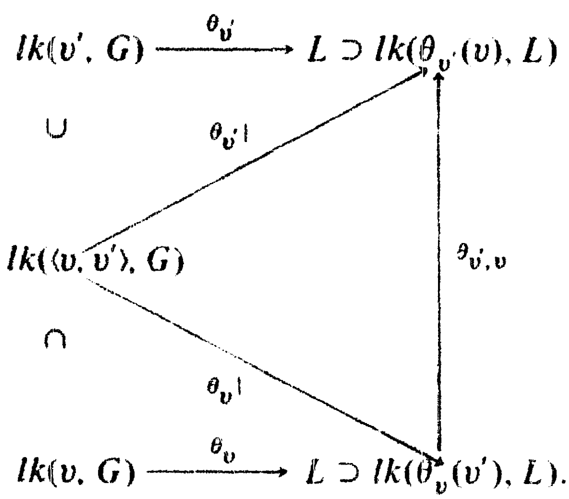
$$\theta_{v',v} = \theta_{v'} \cdot \theta_v^{-1} \mid lk(\theta_v(v'), L)$$

Note

$$\theta_{v',v}^{-1} = \theta_{v,v'}.$$

$$\theta_{v',v}\theta_v \mid lk(\langle v, v'\rangle, G) = \theta_{v'} \mid lk(\langle v, v'\rangle, G),$$

and, if $v$, $v'$ and $v''$ form a triangle in $G$, then $\theta_{v',v}\theta_{v,v''} = \theta_{v',v''}$, wherever both sides are defined. We have the following commutative diagram:




It will be seen shortly that the isomorphisms $\theta_{v',v}$ are important in the nature of $G$.

Let us stop now and make a definition which will allow us to generalize the notion of "cutting" a graph. Let $L_i$, $i = 1, 2, \ldots$, be a collection of subgraphs of a graph $L$. We say $\{L_i\}$ is a *cutting collection* iff

(i) $L_i \cap L_j$ consists of only vertices for $i \neq j$ (no common edges);

(ii) if $\langle v, v'\rangle \in L$ is an edge and $v, v' \in L_i$, then $\langle v, v'\rangle \in L_i$ for each $i$ (i.e., $L_i$ is the subgraph determined by its vertices);

(iii) suppose $x, y \in L$ are two vertices such that there is a graph isomorphism $\theta: lk(x, L) \to lk(y, L)$. Then, for any such $i$, $\theta$, $x$ and $y$ such that $x \in L_i$, there is a unique $j$ such that $y \in L_j$ and

$$\theta(lk(x, L_i)) = lk(y, L_j).$$

The reason for these conditions will be seen in a moment. There are several examples of a graph $L$ and a cutting collection $\{L_i\}$.

**Example VI. 1.** Let $L$ be the disjoint union of arcs $L_i$ of positive length (as in Corollary VI. 6).

**Example VI.2.** Let $L$ be the union of arcs $L_i$, where $\bigcap_i L_i = \{p\}$, a single point, and $L_i \cap L_j = \{p\}$ if $i \neq j$, and here we suppose there are

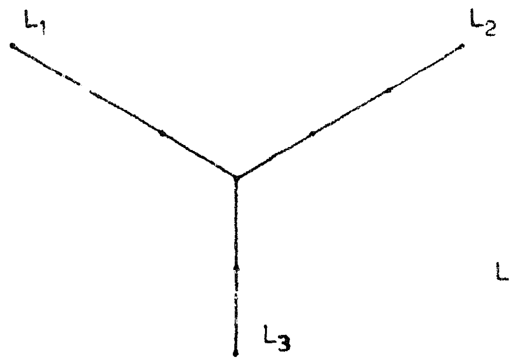



Fig. 6.

at least 3 $L_i$'s. Here $L$ is an $m$-ad, and each $L_i$ is one of its arms (see Fig. 6).

**Example VI.3.** Let $L$ be any finite tree. Consider those vertices of $L$ which have degree not equal to 2 (i.e., 1,3,4, ...). Let $\{L_i\}$ be those arcs in $L$ which have such vertices as endpoints and no such vertices between.

**Example VI.4.** (Generalizing Example VI.3.) Let $L$ be a finite graph such that every circle has at least 2 vertices of degree greater than 2 a distance greater than one apart, or no vertices of degree greater than 2. Again, consider those vertices of $L$ which have degree not equal to 2, and let $\{L_i\}$ be the collection of arcs (and they will be arcs) which have such vertices as endpoints (but no $L_i$ has any such vertex as a non-endpoint), and circles which are disjoint components of $L$ (see Fig. 7).

Now our generalizations.

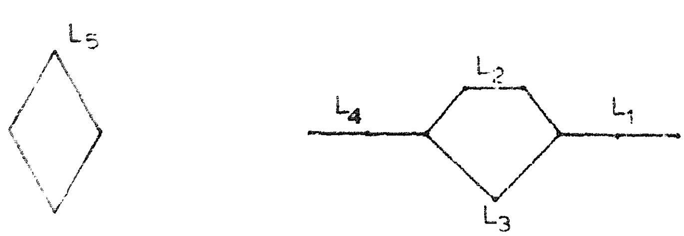



Fig. 7.

**Lemma VI.5.** *Let $G$ be a Z-regular graph, with common link $L$. Let $\{L_i\}$ be a cutting collection for $L$, with $m$ elements. Then there is a graph $H$ and a (simplicial-nondegenerate) map $\pi : H \to G$ such that for each vertex $v \in G$, $\pi^{-1}(v)$ consists of $m$ vertices $v_1, \ldots, v_m$ such that*

$$\theta_v \circ \pi \mid lk(v_i, H) : lk(v_i, H) \to L_i$$

*is an isomorphism. ($\theta_v$ is as previously defined and regarded as fixed once we are given the Z-regularity of $G$.)*

**Proof.** To define $H$ we first define the vertices: For each vertex $v \in G$ we define $m$ vertices $v_1, \ldots, v_m$ of $H$ (as we must). Next we must, for two vertices $v$ and $v'$ of $G$, define when $v_i$ and $v'_j$ form an edge in $H$. We say $\langle v_i, v'_j\rangle$ is an edge in $H$ iff $\theta_{v',v}(lk(\theta_v(v'), L_i)) = lk(\theta_{v'}(v), L_j)$, $\theta_{v'}(v) \in L_j$ and $\theta_v(v') \in L_i$. Note $v' \in lk(v, G)$ in order for there to be an edge between $v_i$ and $v'_j$. Thus, there is a well defined non-degenerate map $\pi : H \to G$ defined by saying $\pi(v_i) = v$, for all $v \in G$, and extending to the edges. Note that $H$ is well defined since $\theta_{v',v}^{-1} = \theta_{v,v'}$.

Since $\theta_v$ is an isomorphism it is sufficient to show
(A) $\pi \mid lk(v_i, H)$ is one to one;
(B) $\theta_v \circ \pi(lk(v_i, H)) \subset L_i$;
(C) $\theta_v \circ \pi(lk(v_i, H)) \supset L_i$.
To show (A) suppose $v'_j, v'_k \in lk(v_i, H)$ so $\pi(v'_j) = \pi(v'_k) = v'$, and, thus, $\langle v_i, v'_j\rangle$ and $\langle v_i, v'_k\rangle$ are edges in $H$. Thus, it must be, by the definition of $H$, that

$$\theta_{v',v}(lk\,\theta_v(v'), L_i) = lk(\theta_{v'}(v), L_j) = lk(\theta_{v'}(v), L_k),$$
$$\theta_{v'}(v) \in L_j \cap L_k,$$
$$\theta_v(v') \in L_i.$$

But by the uniqueness part of condition (iii) in the definition of a cutting collection, it follows that $j = k$. Thus, $\pi$ is one to one, since it is one to one on the vertices.

To show (B) we first show it for vertices. Let $v'_j \in lk(v_i, H)$ and $\pi(v'_j) = v'$. Then $\theta_v(v') \in L_i$ in order for $\langle v_i, v'_j\rangle$ to be an edge in $H$. Thus $\theta_v\pi(v'_j) \in L_i$. Now, if $e \in lk(v_i, H)$ is an edge, then the vertices of the edge $\theta_v\pi(e)$ are in $L_i$; so by condition (ii) of a cutting collection, $\theta_v\pi(e)$ is an edge of $L_i$. Thus, (B) holds.

To show (C) we first show it for vertices. Let $w \in L_i$ be a vertex in $L_i$. Then $v' = \theta_v^{-1}(w) \in lk(v, G)$, and by the existence part of (iii) for a cutting collection, there is a $j$ such that

$$\theta_{v'}(v) \in L_j$$
$$\theta_v(v') \in L_i\,,$$
$$\theta_{v',v}(lk(\theta_v(v'), L_i)) = lk(\theta_{v'}(v), L_j).$$

Thus, $\langle v_i, v_j' \rangle$ is an edge of $H$, and $w = \theta_v(v') \in \theta_v \pi(lk(v_i, H))$.

Lastly, to show (C) for edges let $v$, $v'$, $v''$ be a triangle in $G$, such that $\theta_v(\langle v', v'' \rangle) \in L_i$ is a typical edge. Thus, there is a $j$ and a $k$ such that

$$\theta_{v',v}(lk(\theta_v(v'), L_i)) = lk(\theta_{v'}(v), L_j),$$
$$\theta_{v'',v}(lk(\theta_v(v''), L_i)) = lk(\theta_{v''}(v), L_k).$$

Thus

$$\theta_{v',v}\theta_v(v'') = \theta_{v'}(v'') \in lk(\theta_{v'}(v), L_j) \subset L_j,$$
$$\theta_{v'',v}\theta_v(v') = \theta_{v''}(v') \in lk(\theta_{v''}(v), L_k) \subset L_k\,,$$

and, thus,

$$\theta_{v'',v'}(lk(\theta_{v'}(v''), L_j)) \cap lk(\theta_{v''}(v'), L_k)$$

contains at least the vertex $\theta_{v''}(v)$ since $\theta_{v'}(v) \in lk(\theta_{v'}(v''), L_j)$ and $\theta_{v'',v'}\theta_{v'}(v) = \theta_{v''}(v) \in lk(\theta_{v''}(v'), L_k)$. Thus, by condition (i) and (iii) of a cutting collection,

$$\theta_{v'',v'}(lk(\theta_{v'}(v''), L_j) = lk(\theta_{v''}(v'), L_k),$$

and, thus, $\langle v_j', v_k'' \rangle$ is an edge in $H$, and, thus, $v_i$, $v_j'$ and $v_k''$ form a triangle in $H$ and

$$\theta_v \pi(\langle v_j', v_k'' \rangle) = \langle \theta_v(v'), \theta_v(v'') \rangle;$$

and (C) is shown for edges.

We can now draw several corollaries and combine them with the results of Section V. Recall that our graphs may be infinite.

**Corollary VI.6.** *Let $G$ be a Z-regular graph with $\alpha$ vertices and common link $L$, where $L$ is the disjoint union of arcs, and $\lambda_i$ of the arcs have length $i$, for $i = 1, 2, \ldots$ . Then there is a graph $H$, with $\alpha$ classes of vertices, where $\lambda_i$ of the vertices have link an arc of length $i$ in each class.*

We now see the point of the discussion of graphs $H$ in Section V, where we found conditions on the proportions of vertices with links an arc of varying length. We can, thus, combine Corollary V.2 and Corollary VI.6 and factor out the $\alpha$ to obtain:

**Corollary VI.7.** *Let $G$ be a finite Z-regular graph, with common link $L$, where $L$ is the disjoint union of arcs, and $\lambda_i$ of the arcs have length $i$. Then*

$$\lambda_2 \leq \lambda_1 + \sum_{i=4}^{\infty} (i-3)\lambda_i = \lambda_1 + \lambda_4 + 2\lambda_5 + \ldots .$$

The surprising fact is that the converse to Corollary VI.7 holds, which we shall prove in the next section. Namely, if the above inequality holds for integers $\lambda_1, \lambda_2, \ldots,$ then we can construct a finite $Z$-regular graph with common link a disjoint union of arcs, where $\lambda_i$ of them have length $i$. This will be discussed in Section VII.

We come to the situation where the common link $L$ is an $m$-od, for $m \geq 3$, and we wish to apply Lemma VI. 5 to the cutting collection as in Example VI.2. Before we do this, however, we wish to observe that there is a very natural 1-factor (a subgraph of $G$ with the same vertices, but the degree of each vertex in the subgraph is 1) in $G$. Namely, let $p$ be the center of $L$, the vertex where all the arms meet (the only point of $L$ whose degree is greater than 2). Let $v \in G$ be a vertex. Then call $\bar{v} = \theta_v^{-1}(p)$. Thus, $\langle v, \bar{v}\rangle$ is an edge in $G$. Since $lk(\theta_v(\bar{v}), L)$ consists of $m$ vertices, then $\theta_{\bar{v}}(v) = p$ as well, since $lk(\theta_{\bar{v}}(v), L)$ must also consist of $m$ vertices. (Since $\theta_{\bar{v},v}$ is an isomorphism.) Thus $\bar{\bar{v}} = v$, and the collection of edges $\langle v, \bar{v}\rangle$ forms a one-factor $F$ in $G$. Since $\pi : H \to G$ is non-degenerate (no edge is mapped to a vertex) and is locally one-to-one ($\pi \mid lk(v_i, H)$ is one-to-one), then $\pi^{-1}(F)$ is a one-factor for $H$. Note that, since the link of each vertex in $H$ is an arc, $\partial H$ is defined, and we further say that $\pi^{-1}(F) \subset \partial H$. This is easy to see since, if $v \in G$, then $\theta_v(\bar{v})$ is an endpoint of $L_i$, and, thus, if $\langle v_i, \bar{v}_j\rangle \in \pi^{-1}(F)$, then $\theta_v \pi(\bar{v}_j)$ is an endpoint of $L_i$, and, thus $\langle v_i, \bar{v}_j\rangle \in \partial H$. Thus, $\partial H$ has the property that each component has even or $\infty$ length, and, in particular, $H$ can have no triangles as a component. Thus, we have the following:

**Corollary VI.8.** *Let G be a Z-regular graph, with α vertices and common link L, an m-ad, $m \geq 3$, where $\lambda_i$, $i = 1, 2, \ldots$, of the arms have length i. Then there is a graph H with α classes of vertices, where $\lambda_i$ of the vertices have link an arc of length i, and ∂H contains no components of odd length.*

Combining with Corollary V.2, Corollary V.7 and the remarks after Corollary V.7, we obtain:

**Corollary VI.9.** *Let G be a finite Z-regular graph with common link L, an m-ad, $m \geq 3$, where $\lambda_i$, $i = 1, 2, \ldots$, of the arms have length i. Then the following inequalities hold*

(b)′ $$\lambda_2 \leq \lambda_1 + \sum_{i=4}^{\infty} (i-3)\lambda_i = \lambda_1 + \lambda_4 + 2\lambda_5 + \ldots ;$$

(c)′ $$\lambda_1 \leq \sum_{i=1}^{\infty} \lambda_i = \lambda_2 + \lambda_3 + \ldots ;$$

(d)′ $$2\lambda_1 \leq 2\lambda_2 + 3\lambda_3 + \sum_{i=4}^{\infty} (i-3)\lambda_i = 2\lambda_2 + 3\lambda_3 + \lambda_4 + 2\lambda_5 + \ldots .$$

Again we shall see that the converse to Corollary VI.9 holds as with Corollary VI.7. In the next section we shall see how to apply Lemma V.8 to do this.

## VII. Building *Z*-regular graphs with disjoint arcs and *m*-ads as common links

We now wish to apply the techniques of Section III and the results of Section IV to show that all the graphs that were not ruled out by results of Section VI can actually be built.

It should be noted that this section describes just the inverse operation of what is happening in Section VI. In fact, if we cut apart the graphs we are about to build, we will simply get the disjoint union of the graphs we started with. We first describe the simpler building process for identifying along vertices.

**Lemma VII.1.** *Let $L_1, L_2, \ldots, L_m$ be a finite collection of graphs. Let*

*H be a graph whose vertices can be partitioned as m disjoint collections of α vertices (α possibly infinite but constant), where the link of any vertex in the ith collection is isomorphic to $L_i$. Then there is a Z-regular graph G whose common link is the disjoint union of the $L_i$'s. Furthermore, if α and each $L_i$ is finite, G can be taken to be finite.*

**Proof.** We shall prove the above by induction on $m$. The lemma is obvious for $m = 1$. Let $S_i$ denote the $i$th collection of vertices of $H$. So each vertex in $S_i$ has a link isomorphic to $L_i$, for $i = 1, ..., m$. Let $H_1$ denote $2\alpha$ disjoint copies of $H$, and let $H'_1$ denote another copy of $H_1$ disjoint from $H_1$. Let us index the copies of $H$ in $H_1$ and $H'_1$ from $-\alpha + 1$ to $\alpha$, i.e., $-\alpha + 1, -\alpha + 2, ..., -1, 0, 1, ..., \alpha$ if $\alpha$ is finite and by all the integers if $\alpha$ is infinite. (We only consider the case when $\alpha$ is countably infinite.) Also index the vertices in $S_1$ and $S_2$ from 1 to $\alpha$. Let $v_{1-i,j}$ and $v_{i,j}$, $i = 1, 2, ..., \alpha$, $j = -\alpha + 1, ..., \alpha$, be the $i$th vertex in $S_1$ and $S_2$ respectively, in the $j$th copy of $H$ in $H_1$. Also let $v'_{1-i}$ and $v'_i$, $i = 1, 2, ..., \alpha$, $j = -\alpha + 1, ..., \alpha$, be the $i$th vertex in $S_2$ and $S_1$ respectively (note the order is reversed here) in the $j$th copy of $H$ in $H'_1$. (If $\alpha$ is infinite, $i = 1, 2, ..., j = ... -1, 0, 1, 2, ...$ in both cases.) We wish to define an identification

$$\varphi : \bigcup_{i,j} v_{i,j} \to \bigcup_{i,j} v'_{i,j}$$

such that

(a) $\varphi(v_{i_1,j})$ and $\varphi(v_{i_2,j})$ are in different components of $H'_1$, if $i_1 \neq i_2$.

(b) $\varphi$ is an isomorphism (one to one and onto).

(c) If $v \in S_1$ and $w \in S_2$ in some copy of $H$ in $H_1$, then $\varphi(v) \in S_2$ and $\varphi(w) \in S_1$ in some copy of $H$ in $H'_1$.

To do this we simply define $\varphi(v_{i,j}) = v'_{i,i+j}$, where if $\alpha$ is finite the second index is taken mod $2\alpha$ – there is exactly one representative among $-\alpha + 1, ..., \alpha$, the range of $i$ and $j$. $i$ and $j$ range over all integers if $\alpha$ is infinite. (a), (b) and (c) are clear.

Let $H_2 = H_1 \cup_\varphi H'_1$. $\varphi$ is true by (a) and (b) and Corollary 1. Thus, we see that the link in $H_2$ of any vertex that was in $S_1$ or $S_2$ in some copy of $H$ in $H'_1$ or $H_1$ will now be the disjoint union of $L_1$ and $L_2$. The link of any other vertex will remain unchanged. Thus in $H_2$ we now have $m-1$ classes of $4\alpha^2$ vertices $T_2, T_3, ..., T_m$ where the vertices of $T_2$ are the vertices of $H_1$ and $H'_1$ that were in some $S_1$ or $S_2$ and identified. The vertices of $T_i$, $i \geq 3$, are simply the old vertices in the disjoint union of the $S_i$'s. Thus, now the induction hypothesis applies and we see that there is a $Z$-regular graph $G$ with the disjoint union of $L_1, L_2, \ldots$, and $L_m$ as its common link, and $G$ is finite if $\alpha$ is finite. (In fact, the final graph will have $4^{2^{m-1}-1}\alpha^{2^{m-1}}$ vertices if $\alpha$ is finite.)

We next consider the case when each $L_i$ is an arc. Here we consider the infinite case and the finite case separately.

**Corollary VII.2.** *Let $H$ be a (countable infinite) graph, where the link of each vertex is an arc. Suppose that there are only finitely many lengths of $lk(v, H)$. Let $\lambda_i$, $i = 1, 2, \ldots$, be any sequence of nonnegative integers such that $\lambda_i \neq 0$ if and only if there is a $v \in H$ such that the length of $lk(v, H)$ is $i$. (Only finitely many $\lambda_i$ are thus nonzero.) Then there is an (infinite) $Z$-regular graph $G$ with the finite disjoint union of arcs $L$ as its common link, where $\lambda_i$ of the arcs have length $i$.*

**Proof.** By taking countably many copies of $H$ we may assume that if for some $i$ there is a $v$ such that the length of $lk(v, H)$ is $i$, then there are infinitely many such $v$. Then it is a simple matter to partition the vertices of $H$ into $\Sigma_i \lambda_i$ disjoint classes with countably many vertices in each class, and $\lambda_i$ of the classes have the property that the link of any vertex in any of those classes has length $i$. Thus, Lemma VII.1 applies.

Applying the results of Section IV, we obtain more specific information.

**Corollary VII.3.** *Let $L$ be a graph consisting of the finite disjoint union of arcs, where $\lambda_i$, $i = 1, 2, \ldots$, of the arcs have length $i$. Suppose also that if $\lambda_2 > 0$, then either $\lambda_1, \lambda_4, \lambda_5, \ldots$ are $> 0$ (i.e., $\lambda_2(1-\lambda_1-\lambda_4-\lambda_5 \ldots) \leq 0$). Then there is a (possibly infinite) $Z$-regular graph with $L$ as its common link.*

**Proof.** If $\lambda_2 = 0$, by (8) and (3) of Section III, we know that any $L$ is possible. By (7) and (10), if $\lambda_2 > 0$, then there is a graph with each vertex having a link of length 2 or $i$, where $i \neq 3$. Thus, again by Corollary VII.2 any such graph $L$ is possible as a common link as long as $\lambda_3$ is not the only $\lambda_i$ besides $\lambda_2$ which is not zero.

We shall learn later that all the $L$'s that are left out above are all impossible as common links even for an infinite graph. (This is easy to check anyway.) Thus the above corollary gives a complete characterization of graphs $L$ (which are the disjoint union of arcs) which can occur as the common link of a possibly infinite $Z$-regular graph.

The situation, however, is quite different in the finite case.

**Corollary VI.4.** *Let $L$ be the finite disjoint union of arcs where $\lambda_i$, $i = 1, 2, \ldots$, of the arcs have length $i$. Suppose that*

$$\lambda_2 \leqslant \lambda_1 + \sum_{i=4}^{\infty} (i-3)\lambda_i = \lambda_1 + \lambda_4 + 2\lambda_5 + \ldots .$$

*Then there is a finite $Z$-regular graph with $L$ as its common link.*

**Proof.** By Lemma V.4, there is a finite graph $H$ with each $lk(v, H)$ an arc, where $\rho_i$ of the vertices of $H$ have as link an arc of length $i$, and $\rho_i/\Sigma\rho_i = \lambda_i/\Sigma\lambda_i$. Thus, $\rho_i = (\Sigma\rho_i/\Sigma\lambda_i)\lambda_i$, and possibly taking more disjoint copies of $H$ we may assume that $\Sigma\rho_i/\Sigma\lambda_i$ is an integer. Thus, we can partition the $\rho_i$ vertices of $H$ whose link has length $i$ into $\lambda_i$ disjoint collections, each collection having $\alpha = \Sigma\rho_i/\Sigma\lambda_i$ vertices. Now we apply Lemma VII.1 to these $\Sigma\lambda_i$ collections.

Finally, we have an example (in fact many examples) of problem (b) mentioned in the introduction. Namely any graph in the difference between the two sets of Corollary VII.3 and Corollary VII.4. For instance, if $L$ = , then $\lambda_1 = 1$, $\lambda_2 = 2$, and since $\lambda_2(1 - \lambda_1 - \lambda_4 \ldots) = 2 \cdot 0 \leqslant 0$, an inifinite $Z$-regular graph with $L$ as common link is possible, but since $\lambda_2 > \lambda_1 + \lambda_4$, no such finite graph is possible. Combining Corollary VI.7 and Corollary VII.4, we have:

**Theorem VII.5.** *Let $L$ be a finite disjoint union of arcs where $\lambda_i$, $i = 1, 2, \ldots$, of the arcs have length $i$. Then there is a finite $Z$-regular graph with $L$ as its common link if and only if*

$$\lambda_2 \leqslant \lambda_1 + \sum_{i=4}^{\infty} (i-3)\lambda_i = \lambda_1 + \lambda_4 + 2\lambda_5 + \ldots .$$

This completes the study of the case when $L$ is the disjoint union of arcs. We now proceed to the case when $L$ is an $m$-ad ($m \geqslant 3$). First, some preliminary remarks concerning the notion of orienting certain edges of a graph.

If $\langle v, v' \rangle \in G$ is an edge in some graph $G$, it will be convenient later to order the vertices $v$, $v'$ of the edge $\langle v, v'\rangle$. Namely, we simply decide which of $v$ or $v'$ is the first vertex and which is a second vertex. Thus, if $\varphi$ is a graph map (simplicial), and if one edge is identified with another by $\varphi$, we say $\varphi$ preserves the orientation on that edge, if $\varphi$ of the first vertex in one edge is the first vertex in the second edge and similarly for second vertices.

We wish to do a process similar to the process described in Lemma VII.1, only for $m$-ads instead of the disjoint union of arcs. However, the situation is somewhat more complicated. First, we recall that a *one-factor* $F$ for a graph $H$ is a subgraph consisting of the disjoint union of edges (and their vertices), where each vertex of $H$ is in one (and only one) of the edges of $F$. For us an *oriented one-factor* $F$ for a graph $H$ is a one-factor $F$, where each edge is given some preferred orientation.

Suppose we have a sequence of graphs $L_1, L_2, ..., L_m$, and in each $L_i$ there is a preferred vertex $v_i \in L_i$. By $L_1 \vee L_2 \vee ... \vee L_m$, we will mean the graph obtained by taking first *disjoint* copies of $L_1$ and $L_2$ and identifying $v_1$ and $v_2$ then a disjoint copy of $L_3$ and identifying $v_3$ to $v_1$, $v_3$, etc., up to $L_m$. For instance, if each $L_i$ is an arc and $v_i$ is an endpoint, then $L_1 \vee L_2 \vee ... \vee L_m$ is an $m$-ad (with $m$ arms). Note that even if the $L_i$ are not disjoint, the intersection of $L_i$ and $L_j$, $i \neq j$, "in" $L_1 \vee ... \vee L_m$ is just the preferred vertex.

Now we are in a position to define the basic tool needed for building $Z$-regular graphs with an $m$-ad as common link. Let $H$ be a graph with an oriented one-factor $F$. Let $L$ be another graph and $m$ a positive integer. Let there be a partition of the edges of $F$ into $\alpha$ disjoint classes $S_1, S_2, ...,$ where each $S_i$ consists of $m$ ordered oriented edges of $F$, $\langle v_{i,1}, w_{i,1}\rangle, \langle v_{i,2}, w_{i,2}\rangle, ..., \langle v_{i,m}, w_{i,m}\rangle$, where $v_{i,j}$ is the first vertex in the $j$th edge in the $i$th class. We say $S_1, S_2, ...$ form a *partitioned oriented one-factor* (POOF) *of index* $m$ *for* $L$ *iff*

(i) for each $i$, $lk(v_{i,1}, H) \vee lk(v_{i,2}, H) \vee ... \vee lk(v_{i,m}, H)$ is isomorphic to $L$, where $w_{i,j}$ is the preferred vertex in $lk(v_{i,j}, H)$;

(ii) for each $i$, $lk(w_{i,1}, H) \vee lk(w_{i,2}, H) \vee ... \vee lk(w_{i,m}, H)$ is isomorphic to $L$, where $v_{i,j}$ is the preferred vertex in $lk(w_{i,j}, H)$.

Notice that this approach is in a sense orthogonal to that in Lemma VII.1 since we consider $\alpha$ disjoint classes of $m$ objects, rather than $m$ classes of $\alpha$ objects. We shall see that this is somewhat more convenient, especially since it will be handy not to have to assume $lk(v_{ij}, H)$ is isomorphic to $lk(w_{ij}, H)$, where $\langle v_{ij}, w_{ij}\rangle$ is an edge in the POOF.

The graphs shown in Fig. 8 are an example of a POOF of index 3.

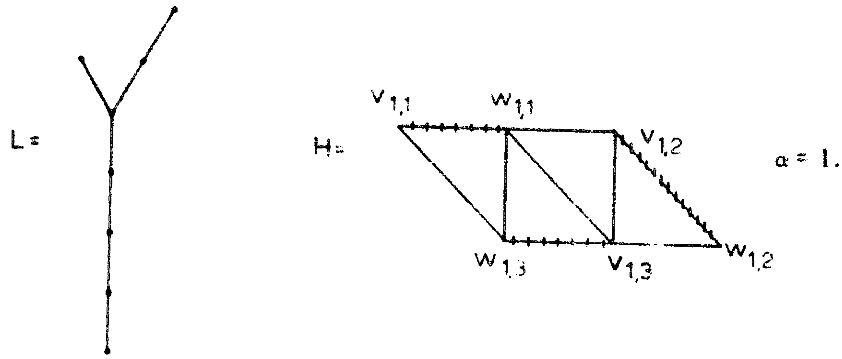



Fig. 8.

To justify the notion of a POOF we show how to use them to build certain $Z$-regular graphs.

**Lemma VII.6.** *Let $S_1, S_2, \ldots$ be a POOF of index $m$ for $L$ in a graph $H$. Then there is a $Z$-regular graph $G$ with common link $L$. Furthermore, if $H$ is finite, $G$ can be taken to be finite.*

**Proof.** Roughly speaking we would like to simply identify all the edges of each $S_i$ in an orientation preserving manner, but this identification may not be true, and so it may create extra edges, loops and generally wreak havoc. Thus, we must create disjoint copies and cross identify as in Lemma VII.1.

As with Lemma VII.1, we may use induction on $m$, the index of the POOF. If $m = 1$, $H$ is obviously $Z$-regular with common link $L$. Suppose there are $\alpha$ $S_i$'s ($\alpha$ could be countably infinite) and recall the notation above.

Let $H_1$ be $2\alpha$ disjoint copies of $H$ and $H'_1$ be another copy of $H_1$ disjoint from $H_1$. As before let us index the copies of $H$ in $H'_1$ from $-\alpha + 1$ to $\alpha$. I.e., from $-\alpha + 1, \ldots, -1, 0, 1, \ldots, \alpha$, if $\alpha$ is finite and by all the integers (positive, negative or zero), if $\alpha$ is infinite. Let $e_{1-i,j} = \langle v_{i,1}, w_{i,1}\rangle$ and $e_{i,j} = \langle v_{i,2}, w_{i,2}\rangle$, $i = 1, 2, \ldots, \alpha$, $j = -\alpha + 1, \ldots, -1, 0, 1, \ldots, \alpha$, be the first and second edges respectively, of $S_i$ in the $j$th copy of $H$ in $H_1$. Let $e'_{1-i,j} = \langle v_{i,2}, w_{i,2}\rangle$, $e'_{i,j} = \langle v_{i,1}, w_{i,1}\rangle$, $i = 1, 2, \ldots, \alpha$, $j = -\alpha + 1, \ldots, \alpha$, be the *second* and first edges respectively, of $S_i$ in the $j$th copy of $H$ in $H'_1$. (If $\alpha$ is infinite, $i = 1, 2, \ldots, j = \ldots -1, 0, 1, 2, \ldots$ in both cases.) We now wish to define an identification

$$\varphi : \bigcup_{i,j} e_{i,j} \to \bigcup_{i,j} e'_{i,j}$$

such that

(a) $\varphi$ preserves the orientation of each edge, and $\varphi$ of a first edge in a $S_i$ in $H_1$ is a second edge in another copy of $S_i$ in $H'_1$, and vice-versa.

(b) $\varphi(e_{i_1,j})$ and $\varphi(e_{i_2,j})$ are in different components of $H'_1$ if $i_1 \neq i_2$.

(c) $\rho$ is an isomorphism (one-one and onto).

To do this we simply define $\varphi(e_{i,j}) = e'_{i,i+j}$ in an orientation preserving manner, where if $\alpha$ is finite the second index is taken mod $2\alpha$ – there is exactly one representative among $-\alpha + 1, ..., \alpha$ the range of $i$ and $j$, if $\alpha$ is finite. $i$ and $j$ range over all integers if $\alpha$ is infinite.

(a) follows from the way the $e_{i,j}$'s and $e'_{i,j}$'s were indexed, and (b) and (c) are obvious from the definition of $\varphi$.

Let $H_2 = H_1 \cup_\varphi H'_1$. $\varphi$ is true by (b), (c) and Corollary 1. Notice that for each $i$, $H_i$ contains $4\alpha^2$ copies of an edge which is obtained by identifying a first edge of one $S_i$ with a second edge in a disjoint copy of $S_i$. Notice, also, there are $4\alpha^2$ copies of the $j$th edge, $j \geq 3$, in $S_i$ in $H_2$. Thus, we can define a POOF of index $m-1$ for $L$ in $H_2$. Namely, we define the one-factor of $H_2$ as $\overline{F}$, where $F$ is the one-factor of $H_1 \cup H'_1$, and we orient $\overline{F}$ the "same way" as $F$, which makes sense for the edges that were identified since $\varphi$ is orientation preserving. We partition $\overline{F}$ into $4\alpha^2$ sets as follows: For each $S_i$ in $H$, we shall define $4\alpha$ collections of edges $T_{i,1}, T_{i,2}, ...$, each collection corresponding to a copy of $H$ in $H_1$ or $H'_1$ (and thus a copy of $S_i$). Each $T_{i,j}$ is simply the image under the quotient map of the appropriate copy of $S_j$ minus the first edge. Notice that the vertices of the first edge in $T_{i,j}$ have links isomorphic to $lk(v_{i,1}, H) \vee lk(v_{i,2}, H)$ and $lk(w_{i,1}, H) \vee lk(w_{i,2}, H)$ respectively, and the links of other vertices remains unchanged. Thus, it is easy to check that the $T_{i,j}$'s form a POOF of index $m-1$ for $L$. Thus, by the induction hypothesis, a $Z$-regular graph $G$ with $L$ as its common link exists and is finite, if $\alpha$ is finite. (In fact, $G$ will have $2 \cdot 4^{2^{m-1}-1}\alpha^{2^{m-1}}$ vertices, if $\alpha$ is finite.)

Now that we know that the existence of a POOF for $L$ implies the existence of a $Z$-regular graph with $L$ as its common link, we must investigate how to create POOFs for $L$. Although often it is easy to find them directly we shall find it convenient to make a general statement.

**Lemma VII.7.** *Let $H$ be a finite graph with a one-factor $F$. Let $L_1, L_2, ..., L_m$ be (finite) graphs with preferred vertices $t_i \in L_i$, $i = 1, 2, ..., m$. Suppose the vertices of $H$ can be partitioned into $m$ classes of $\alpha$ vertices, where if $v$ is in the $i$th class, then $lk(v, H)$ is isomorphic to $L_i$ with $lk(v, F)$ as preferred vertex (i.e., the isomorphism carries $lk(v, F)$ onto $t_i$).*

*Then there is a finite graph (2 disjoint copies of H) $H_1$ with a POOF of index m for $L_1 \vee L_2 \vee \ldots \vee L_m$ (with identification along the preferred vertices of $L_i$).*

**Proof.** Let $H_1$ be two disjoint copies of $H$. Orient the edges of $F$ one way in the first copy of $H$ and the opposite way in the second copy of $H$, to obtain an oriented one-factor $F_1$ for $H_1$. Consider the following bipartite (simple) graph $H_2$ which may have multiple edges. $H_2$ will have $2m$ vertices, $v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_m$, where each $v_i$ corresponds to the $i$th class of first vertices in $H_1$ and $w_i$ to the $i$th class of second vertices in $H_1$. Then we draw an edge from $v_i$ to $w_j$ for each edge in $F_1$ in $H_1$ which has a first vertex in the $i$th class and a second vertex in the $j$th class. Simply speaking, $H_2$ is the graph obtained by identifying all the first vertices of the $i$th class, for $i = 1, 2, \ldots, n$, and all the second vertices of the $j$th class, for $j = 1, 2, \ldots, m$, in the graph $F_1$.

Note that the degree of each vertex in $H_2$ is $\alpha$ (the result of taking two copies and orienting the second copy oppositely), and, thus, by a theorem of Köning and Hall (see [11]) – which can be made to follow from an algorithm of Ford and Fulkerson – there are $\alpha$ disjoint one-factors, $\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_\alpha$, in $H_2$. Each oriented edge of each $\bar{S}_i$ corresponds to an oriented edge in $H_1$, and, thus, if we let $S_i$, $i = 1, \ldots, \alpha$, denote the $m$ such edges in $H_1$, we obtain a POOF of degree $m$ for $L = L_1 \vee L_2 \vee \ldots \vee L_m$ along the preferred vertices $t_i \in L_i$.

**Corollary VII.8.** *Let H be a finite graph, where the link of each vertex is an arc (of variable length) and each component of $\partial H$ has even length. Suppose $\lambda_i\alpha$ of the vertices of H have links of length i, $i = 1, 2, \ldots$, where $\lambda_i$ and $\alpha$ are positive integers, and $m = \Sigma\lambda_i \geq 3$. Let L be the m-ad which has $\lambda_i$ arms of length i. Then there is a finite Z-regular graph with L as common link.*

**Proof.** Since each component of $\partial H$ has even length, by choosing every other edge on each boundary component, we can find a one-factor $F$ for $H$ such that $F \subset \partial H$. Thus, if $v \in H$, $lk(v, F)$ is an endpoint of $lk(v, H)$ and Lemma VII.7 applies. Thus, there is a POOF of index $m$ for $L$ (for $H_1$ 2 disjoint copies of $H$). We then apply Lemma VII.6.

As before we can apply the results of Section V to obtain more specific results.

**Corollary VII.9.** *Let L be the m-ad, $m \geq 3$, where $\lambda_i$ of the arms of L have length i, $i = 1, 2, \dots$ . Suppose*

(b) $$\lambda_2 \leq \lambda_1 + \sum_{i=4}^{\infty} (i-3)\lambda_i = \lambda_1 + \lambda_4 + 2\lambda_5 + \dots .$$

(c) $$\lambda_1 \leq \sum_{i=2}^{\infty} \lambda_i = \lambda_2 + \lambda_3 + \lambda_4 + \dots .$$

(d) $$2\lambda_1 \leq 2\lambda_2 + 3\lambda_3 + \sum_{i=4}^{\infty} (i-3)\lambda_i = 2\lambda_2 + 3\lambda_3 + \lambda_4 + 2\lambda_5 + \dots .$$

*Then there is a finite Z-regular graph with L as its common link.*

**Proof.** By Lemma V.8, there is a finite graph $H$ with the link of each vertex an arc, where $\rho_i$ of the vertices have link an arc of length $i$, where $\rho_i/\Sigma\rho_i = \lambda_i/\Sigma\lambda_i$ and each component of $\partial H$ has even length. By replacing $H$ by an appropriate number of disjoint copies, we may assume that $\Sigma\rho_i/\Sigma\lambda_i$ is an integer. Then apply Corollary VII.8.

We now combine Corollary VI.9 and Corollary VII.9 to obtain:

**Theorem VII.10.** *Let L be a finite m-ad, $m \geq 3$, where $\lambda_i$, $i = 1, 2, \dots$, of the arms of L have length i. Then there is a finite Z-regular graph with L as its common link if and only if*

(b) $$\lambda_2 \leq \lambda_1 + \sum_{i=4}^{\infty} (i-3)\lambda_i = \lambda_1 + \lambda_4 + 2\lambda_5 + \dots .$$

(c) $$\lambda_1 \leq \sum_{i=2}^{\infty} \lambda_i = \lambda_2 + \lambda_3 + \dots .$$

(d) $$2\lambda_1 \leq 2\lambda_2 + 3\lambda_3 + \sum_{i=4}^{\infty} (i-3)\lambda_i = 2\lambda_2 + 3\lambda_3 + \lambda_4 + 2\lambda_5 + \dots .$$

This completes the study of *finite* $Z$-regular graphs with an $m$-ad as common link. We now investigate the infinite case. We wish to prove a statement similar to Corollary VII.3. However, the situation is complicated by the necessity for the one-factor.

Suppose $L_1, L_2, \dots$ is a collection of graphs, with a preferred vertex $t_i \in L_i$, $i = 1, 2, \dots$ . For us we will only consider the case when each $L_i$ is an arc of length $i$, and $t_i$ is one of its end points. If $\lambda_i$ is an integer let $\lambda_i L_i$ denote the graph $L_i \vee L_i \vee ... \vee L_i$, $\lambda_i$ times along the preferred vertex $t_i$. Given a sequence of integers $\lambda_1$, $\lambda_2$, ..., where all but a finite number are 0 ($0L_i = \emptyset$), we wish to be able to know when we can construct a POOF of index $\Sigma\lambda_i$ for $L = \lambda_1 L_1 \vee \lambda_2 L_2$ ... ($\vee$ taken along the preferred vertices), and, thus, get a $Z$-regular graph with $L$ as common link. Suppose $H$ is a (possibly infinite) graph, where the link of each vertex is isomorphic to one of the $L_i$, but only finitely many $L_i$'s are needed. Suppose, further, that $H$ has a one-factor $F$ such that for each edge $\langle v, w\rangle \in F$, $w \in lk(v, H)$ is the preferred vertex (and, thus, $v \in lk(w, H)$ is the preferred vertex, also). Let $R_1, R_2, ..., R_p$ be a sequence of vectors which describe which pairs $L_i$, $L_j$ appear as links for $v$ and $w$ for $\langle v, w\rangle \in F$. I.e., if $\langle v, w\rangle \in F$ and $lk(v, H)$ is isomorphic to $L_i$ (with $w$ as preferred vertex) and $lk(w, H)$ is isomorphic to $L_j$ (with $v$ as preferred vertex), then there is an $R_k = (0, ..., \frac{1}{2}, 0, ..., \frac{1}{2}, 0, ...)$ or $(0, ..., 1, 0, 0)$ with nonzero entries only in the $i$ and $j$ slots, and if such an $R_k$ appears in the sequence, there is some edge $\langle v, w\rangle \in F$ such that the links of $v$ and $w$ correspond to $i$ and $j$ as above. We shall call $R_1, R_2, ..., R_p$ the *vector sequence for H, with respect to F.*

**Lemma VII.11.** *Let $L_1, L_2, ...$ be a (finite) collection of graphs with preferred vertices $t_i \in L_i$, $i = 1, 2, ...$ . Let $H$ be a (possibly infinite) graph with a one-factor $F$. Suppose for each vertex $v \in H$, $lk(v, H)$ is isomorphic to one of the $L_i$, with $lk(v, F)$ corresponding to $t_i$. Suppose that $R_1, R_2, ..., R_p$ is the vector sequence for $H$ with respect to $F$. Let $r_1, r_2, ..., r_p$ be rational numbers such that $r_j > 0$, $j = 1, 2, ..., p$, and $\Sigma_{j=1}^{p} r_j = 1$. Let $\lambda_i$, $i = 1, 2, ...$ be non-negative integers such that the kth coordinate of $R = \Sigma_{j=1}^{p} r_j R_j$ is $\lambda_k / \Sigma_i \lambda_i$. Let $L = \lambda_1 L_1 \vee \lambda_2 L_2 \vee ...$ along preferred vertices.*

*Then there is a graph $H_1$ with a POOF for $L$ of index $\Sigma_i \lambda_i$.*

**Proof.** Let $\alpha$ be a positive finite integer large enough so that each $\alpha r_j$, $j = 1, 2, ..., p$, and $\alpha/\Sigma\lambda_i$ is an integer. Let $H'$ denote countably many copies of $H$ (in case there are not enough edges for the next operations). Let $F'$ denote the corresponding one-factor for $H'$. Thus, it is possible to partition the edges of $F'$ into countably many collections $S_1, S_2, S_3, ...$ such that there are $\alpha$ edges in each $S_i$, and the edges of each $S_i$ are partitioned into $p$ collections of $\alpha r_j$ edges, where if $\langle v, w\rangle$ is in the $k$th collection, then $L_i$ and $L_j$ correspond (with preferred vertices) to the links of $v$ and $w$, where $i$ and $j$ are the nonzero entries of $R_k$. (If $i = j$, the $i$th entry of $R_k$ is 1, and all the others are 0.) This is easy to do since

Table 1

| Graph | Graph type | Vector sequence |
|---|---|---|
| $A_n$, $n \geqslant 3$ | $(n, n)$ | $(0,0,\ldots,1)$ ← $n$th slot |
| $B_n$, $n \neq 1,4$ | $(1, n)$ | $(\frac{1}{2},0,0,\ldots,\frac{1}{2})$ ← $n$th slot |
| $C_n$, $n > 5$ | $(\underbrace{1,4,\ldots,1,4}_{n-5 \text{ times}},1,n)$ | $(\frac{1}{2},0,0,\frac{1}{2})$, $(\frac{1}{2},0,0,\ldots,\frac{1}{2})$ ← $n$th slot |
| $C_3$ | $(1,4,1,3)$ | $(\frac{1}{2},0,0,\frac{1}{2})$, $(\frac{1}{2},0,\frac{1}{2})$ |
| $D$ | $(1,4,4)^2$ | $(\frac{1}{2},0,0,\frac{1}{2})$, $(0,0,0,1)$ |
| $E_n$, $n \geqslant 5$ | $(\underbrace{2,2,\ldots,2}_{n-3\ 2\text{'s}},n)$ | $(0,1)$, $(0,\frac{1}{2},0,\ldots,\frac{1}{2})$ ← $n$th slot |
| $F$ | $(2,4)^2$ | $(0,\frac{1}{2},0,\frac{1}{2})$ |
| $G$ | $(\bar{1},4,1,2,2,2)$ | $(\frac{1}{2},0,0,\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2})$, $(0,1)$ |
| $H$ | $(1,2,3)^2$ | $(0,\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},0,\frac{1}{2})$ |
| $I$ | — | $(0,\frac{1}{2},0,\frac{1}{2})$, $(0,\frac{1}{2},\frac{1}{2})$ |
| $J$ | — | $(\frac{1}{2},0,0,\frac{1}{2})$, $(\frac{1}{2},0,0,0,\frac{1}{2})$. |

there are countably many edges of every type. Let $H_1$ denote 2 disjoint copies of $H'$. We may now proceed as in Lemma VII.7 to apply the Köning–Hall theorem to each $S_i$ and obtain a POOF for all $H_1$, for $L$ of index $\Sigma_i \lambda_i$, since in each $S_i$ there are $(2\alpha/\Sigma\lambda_i)\lambda_j$ first (or second) vertices whose link corresponds to $L_j$ ($2\alpha/\Sigma\lambda_i$ corresponds to the $\alpha$ in Lemma VII.7).

The point here is that the $R_k$'s behave almost as if there were a graph with half its vertices having a link corresponding to $L_i$ and the other half corresponding to $L_j$. The important thing to remember here, of course, is that the affine coordinate of $R_k$, $r_k$, is always positive since we can never quite wipe out the other arcs. Although their proportions of the whole can be made quite small.

**Corollary VII.12.** $C' = \{(s_1, s_2, \ldots)$: *there is a Z-regular graph G with common link L, and m-ad,* $m \geqslant 3$, *such that* $\lambda_i$ *of the arms of L are of length i, and* $s_i = \lambda_i/\Sigma\lambda_i$, $i = 1, 2, \ldots\}$ *is a (rational) convex set.*

The trouble is that $C'$ may not be "compact" as in the finite case.

Let us now compute the vertex sequence for some of the graphs of Section IV. From here on $L_i$, $i = 1, 2, \ldots$, will represent the arc of length $i$, and an endpoint will represent the preferred vertex. The one-factor will be understood to be along $\partial H$ in each case. Let us make a table, where the reader is referred to Section IV to verify that the appropriate

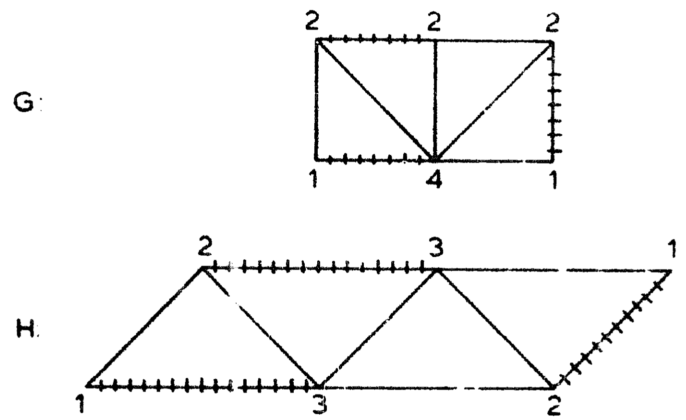



Fig. 9.

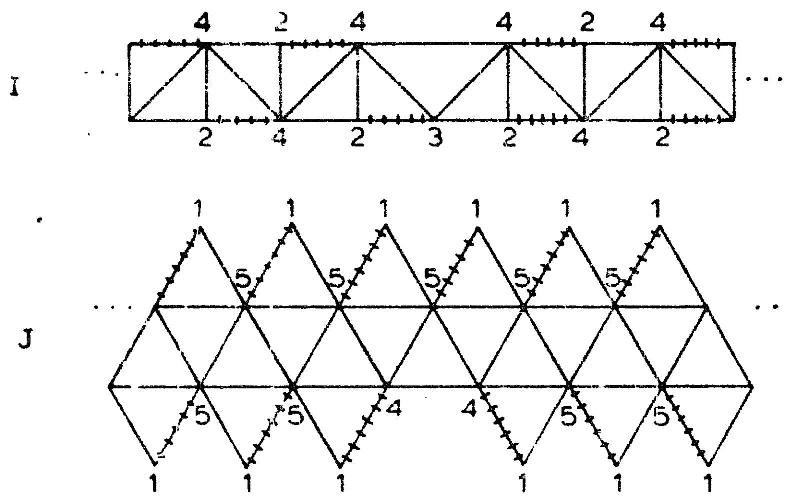



Fig. 10.

type exists (Table 1). The last four graphs are not discussed in Section IV so we demonstrate their existence here (see Figs. 9 and 10).

We have also indicated the appropriate one-factor in each graph.

We next apply Lemma VII.11 to Table 1 to obtain specific results about which $m$-ads can occur as common links in a $Z$-regular graph. Note that conditions (b) and (d) of Theorem VII.10 do not now hold, nevertheless condition (c), $\lambda_1 \leqslant \lambda_2 + \lambda_3 + \ldots$, must still hold since no two adjacent vertices of $\partial H$ can have links of length one, for there to be a one-factor.

**Lemma VII.13.** *Let $\lambda_1, \lambda_2, \ldots$ be a sequence of non-negative integers where all but a finite number are zero, and $\Sigma\lambda_i \geqslant 3$. Suppose $\lambda_1 \leqslant \lambda_2 + \lambda_3 + \ldots$ and*

$$\lambda_i \neq 0 \quad \textit{for some } i \geq 5,$$

*or*

$$\lambda_1 \leq \lambda_2 \ \textit{and} \ \begin{cases} \lambda_2 \leq \lambda_1 + \lambda_3 + \lambda_4 \\ \textit{or} \\ \lambda_1 \neq 0 \ \textit{and} \ \lambda_4 \neq 0 \end{cases} \textit{and} \ \begin{cases} \lambda_1 \neq 0 \ \textit{and} \ \lambda_2 < \lambda_1 + \lambda_3 \\ \textit{or} \\ \lambda_4 \neq 0 \end{cases}$$

*or*

$$\lambda_2 \leq \lambda_1 \ \textit{and} \ \begin{cases} \lambda_1 < \lambda_2 + \lambda_4 \\ \textit{or} \\ \lambda_3 \neq 0 \\ \textit{or} \\ \lambda_4 = 0. \end{cases}$$

*Let L be a $\Sigma\lambda_i$-ad, where $\lambda_i$ of the arms have length i. Then there is a POOF of index $\Sigma\lambda_i$ for L, and thus a Z-regular graph with L as common link.*

**Proof.** In what follows we assume that we have a vector $(r_1, r_2, \ldots)$, $\Sigma r_i = 1$, where $r_i = \lambda_i / \Sigma\lambda_i$. We wish to express this vector as an appropriate linear combination of vectors of the vector sequence in Table 1. We then apply Lemma VII.11. To facilitate notation, a vector $(-)_X$ will mean that that vector comes from graph $X$ in the table and we must be sure that all the coefficients of each "$X$" vector are either all zero (the graph is not used) or all non-zero.

*Case* I: $\lambda_1 \leq \lambda_2$.

IA: $\lambda_n \neq 0$ for some $n \geq 5$. Let $0 < \epsilon < \min[2r_n, 2(r_2 - r_1)]$, $\epsilon$ rational.

$$2r_1(\tfrac{1}{2},\tfrac{1}{2})_{B_2} + (r_2 - r_1 - \tfrac{1}{2}\epsilon)(0,1)_{E_n} + \epsilon(0,\tfrac{1}{2},0,\ldots,\tfrac{1}{2})_{E_n} + (r_n - \tfrac{1}{2}\epsilon)(0,\ldots,1)_{A_n} + \sum_{\substack{i \geq 3 \\ i \neq n}} r_i(0,\ldots,1)_{A_i}.$$

If $r_1 = r_2$, let $\epsilon = 0$.

IB: $\lambda_2 < \lambda_1 + \lambda_3$, $\lambda_1 \neq 0$, $\lambda_i = 0$ for all $i \geq 5$. Note $r_1 \neq 0$, $r_2 \neq 0$ and $r_3 \neq 0$. Let $0 < \epsilon < \min[2r_1, r_3 + r_1 - r_2]$ be rational.

$$\epsilon(\tfrac{1}{2},0,\tfrac{1}{2})_H + (2r_1 - \epsilon)(\tfrac{1}{2},\tfrac{1}{2})_H + (2r_2 - 2r_1 + \epsilon)(0,\tfrac{1}{2},\tfrac{1}{2})_H + (r_3 + r_1 - r_2 - \epsilon)(0,0,1)_{A_3}.$$

IC: $\lambda_2 \leqslant \lambda_1 + \lambda_3 + \lambda_4$, $\lambda_4 \neq 0$, $\lambda_i = 0$ for all $i \geqslant 5$. If $r_2 > r_1 + r_4$,

$$2r_1(\tfrac{1}{2},\tfrac{1}{2})_{B_2} + 2r_4(0,\tfrac{1}{2},0,\tfrac{1}{2})_I + 2(r_2-r_1-r_4)(0,\tfrac{1}{2},\tfrac{1}{2})_I + (r_1+r_3+r_4-r_2)(0,0,1)_{A_3}.$$

If $r_2 \leqslant r_1 + r_4$,

$$2r_1(\tfrac{1}{2},\tfrac{1}{2})_{B_2} + 2(r_2-r_1)(0,\tfrac{1}{2},0,\tfrac{1}{2})_F + (r_1+r_4-r_2)(0,0,0,1)_{A_4} + r_3(0,0,1)_{A_3}.$$

ID: $\lambda_1 \neq 0$, $\lambda_4 \neq 0$, $\lambda_i = 0$ for all $i \geqslant 5$. Let $0 < \epsilon < \min\{2r_1, 2r_4\}$ be rational.

$$(2r_1-\epsilon)(\tfrac{1}{2},\tfrac{1}{2})_G + (r_2-r_1+\tfrac{1}{2}\epsilon)(0,1)_G + \epsilon(\tfrac{1}{2},0,0,\tfrac{1}{2})_G + (r_4-\tfrac{1}{2}\epsilon)(0,0,0,1)_{A_4} + r_3(0,0,1)_{A_3}$$

*Case* II: $\lambda_2 \leqslant \lambda_1 \leqslant \Sigma_{k\neq 1,4}\lambda_k$. Choose integers $i$ and $j$ such that $i, j \neq 4$, $2 \leqslant i < \infty$ and

$$\sum_{k=2,3,5,\ldots,i} r_k \leqslant r_1 \leqslant \sum_{k=2,3,5,\ldots,j} r_k. \qquad j = i+1 \text{ if } i \neq 3, j = 5 \text{ if } i = 3.$$

$$\sum_{k=2,3,5,\ldots,i} 2r_k(\tfrac{1}{2},0,0,\ldots,\tfrac{1}{2})_{B_k} + 2\Big(r_1 - \sum_{k=2,3,5,\ldots,i} r_k\Big)(\tfrac{1}{2},0,\ldots,\tfrac{1}{2})_{B_j}$$
$$+ \Big(\sum_{k=2,3,5,\ldots,j} r_k - r_1\Big)(0,\ldots,1)_{A_j}$$
$$+ \sum_{k\neq 1,3,5,\ldots,j} r_k(0,0,\ldots,1)_{A_k}.$$

*Case* III: $\Sigma_{i\neq 1,4}\lambda_i < \lambda_1 \leqslant \Sigma_{i\geqslant 2}\lambda_i$. (Note $\lambda_4 \neq 0$.)
IIIA: $\lambda_n \neq 0$ for $n = 3$ or $n \geqslant 6$.

$$\sum_{i\neq 1,4,n} 2r_i(\tfrac{1}{2},0,\ldots,\tfrac{1}{2})_{B_i} + 2r_n(\tfrac{1}{2},0,\ldots,\tfrac{1}{2})_{C_n}$$
$$+2\Big(r_1 - \sum_{i\neq 1,4} r_i\Big)(\tfrac{1}{2},0,0,\tfrac{1}{2})_{C_n} + \Big(\sum_{i\geqslant 2} r_i - r_1\Big)(0,0,0,1)_{A_4}.$$

IIIB: $\lambda_1 < \Sigma_{i\geqslant 2}\lambda_i$, $\lambda_i = 0$ for $i = 3$ and for all $i \geqslant 6$.

$$2r_2(\tfrac{1}{2},\tfrac{1}{2})_{B_2} + 2r_5(\tfrac{1}{2},0,0,0,\tfrac{1}{2})_{B_5} + 2(r_1-r_2-r_5)(\tfrac{1}{2},0,0,\tfrac{1}{2})_D$$
$$+(r_2+r_4+r_5-r_1)(0,0,0,1)_D.$$

IIIC: $\lambda_1 = \Sigma_{i \geqslant 2} \lambda_i$, $\lambda_5 \neq 0$, $\lambda_i = 0$ for $i = 3$ and for all $i \geqslant 6$.

$$2r_2(\tfrac{1}{2},\tfrac{1}{2})_{B_2} + 2r_4(\tfrac{1}{2},0,0,\tfrac{1}{2})_J + 2r_5(\tfrac{1}{2},0,0,0,\tfrac{1}{2})_J .$$

It is easy to check that these cases exhaust all the possibilities of the theorem.

## VIII. The non-existence of certain graphs

We wish to show that for the graphs $L$ (of the appropriate type) left out of Corollary VII.3 and Lemma VII.13 there are *no* graphs (infinite or not) that have $L$ as a common link. In particular we show:

**Theorem VIII.1.** *Let L be a finite disjoint union of arcs, where* $\lambda_i$, $i = 1, 2, \ldots$, *of the arcs have length i. Then there is an (infinite) Z-regular graph with L as the common link if and only if* $\lambda_2(1-\lambda_1-\lambda_4-\lambda_5 \ldots) \leqslant 0$.

**Theorem VIII.2.** *Let L be an m-ad* ($m \geqslant 3$), *where* $\lambda_i$, $i = 1,2,\ldots,$ *of its arms have length i. Then there is an (infinite) Z-regular graph with L as its common link if and only if*

$$\lambda_1 \leqslant \lambda_2 + \lambda_3 + \lambda_4 + \ldots \quad \textit{and}$$

$$\lambda_i \neq 0 \quad \textit{for some } i \geqslant 5$$

*or*

$$\text{(C)} \qquad \lambda_1 \leqslant \lambda_2 \ \textit{and} \begin{cases} \lambda_2 \leqslant \lambda_1+\lambda_3+\lambda_4 \ \textit{and} \begin{cases} \lambda_1 \neq 0 \ \textit{and} \ \lambda_2 < \lambda_1+\lambda_3 \\ \textit{or} \\ \lambda_4 \neq 0 \end{cases} \\ \textit{or} \\ \lambda_1 \neq 0 \ \textit{and} \ \lambda_4 \neq 0 \end{cases}$$

*or*

$$\lambda_2 \leqslant \lambda_1 \ \textit{and} \begin{cases} \lambda_1 < \lambda_2 + \lambda_4 \\ \textit{or} \\ \lambda_3 \neq 0 \\ \textit{or} \\ \lambda_4 = 0. \end{cases}$$

Note that the if part of these theorems is Corollary VII.3 and Lemma VII.13, respectively.

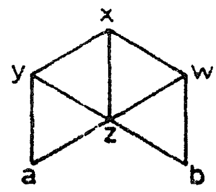



Fig. 11.

We first concentrate on Theorem VIII.1. We apply Corollary VI.6 to observe that if $G$ is $Z$-regular with common link $L$, a disjoint union of arcs, then we can "cut" $G$ along vertices to obtain a graph $H$ that has the property that each one of the arcs of $L$ has length $i$ if and only if one of the vertices of $H$ has a link an arc of length $i$. Thus Theorem VIII.1 follows from the following.

**Lemma VIII.3.** *There does not exist a graph $H$ (possibly infinite) such that each link$(x, H)$ is an arc of length 2 or 3 in which at least one vertex has a link of length 2.*

**Proof.** Let $x \in H$ be a vertex such that link$(x, H)$ is an arc $\langle y, z\rangle$, $\langle z, w\rangle$. Then since the link$(y, H)$ has length greater than 1, there is a vertex $a \in H$ adjacent to $y$ and $z$. Similarly there is a vertex $b \in H$ adjacent to $w$ and $z$. Then link$(z, H)$ has length greater than 3, a contradiction (see Fig. 11).

We now concentrate on Theorem VIII.2 which is more complicated. If we review the discussion just before Corollary VI.8, we see that, if we have a $Z$-regular graph with $L$, an $m$-ad ($m \geqslant 3$) as common link, then $G$ has a very natural 1-factor $F$ ($F$ consists of the edges $\langle v, w\rangle$ where $w$ is the center vertex of link$(v, G)$). Then by "cutting" along the arms as in Lemma VI.5 and Corollary VI.8, we obtain a graph $H$ and simplicial map $\pi : H \to G$ such that $\pi^{-1}(\langle v, w\rangle)$, $\langle v, w\rangle \in F$ is a POOF of index $m$ for $L$. Thus as before we shall consider such $H$'s and decide when such POOFs cannot exist. Thus we see that the following lemma together with Lemma VII.13 implies Theorem VIII.2.

**Lemma VIII.4.** *Let $L$ be an $m$-ad with $\lambda_i$ arms of length $i$, $i = 1, 2, \ldots$ . Let $H$ be a (possibly infinite) graph with the link of each vertex an arc (of varying length). Then the following condition implies that there does not exists a POOF, in $H$, of index $m$ for $L$.*

$$
\begin{array}{l}
\lambda_1 > \lambda_2 + \lambda_3 + \ldots \\
\textit{or} \\
\lambda_i = 0 \textit{ for all } i \geqslant 5 \textit{ and}
\left[
\begin{array}{l}
\lambda_1 > \lambda_2 \\
\textit{or} \\
\left\{
\begin{array}{l}
\lambda_2 > \lambda_1 + \lambda_3 + \lambda_4 \\
\textit{or} \\
\lambda_1 = 0 \textit{ and } \lambda_4 = 0 \\
\textit{or} \\
\lambda_2 \geqslant \lambda_1 + \lambda_3 \textit{ and } \lambda_4 = 0
\end{array}
\right\}
\textit{ and }
\left\{
\begin{array}{l}
\lambda_1 = 0 \\
\textit{or} \\
\lambda_4 = 0
\end{array}
\right\}
\end{array}
\right] \\
(\sim C) \\
\qquad\qquad \textit{and}
\left\{
\begin{array}{l}
\lambda_1 < \lambda_2 \\
\textit{or} \\
\lambda_1 \geqslant \lambda_2 + \lambda_4 \textit{ and } \lambda_3 = 0 \textit{ and } \lambda_4 \neq 0.
\end{array}
\right.
\end{array}
$$

(*Notice that condition* (~C) *is the negative of condition* (C).)

**Proof.** *Case* I: $\lambda_1 > \lambda_2 + \lambda_3 + \lambda_4 + \ldots$. Let $\{(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)\}$ be any element of the POOF, so that each link$(x_i, H)$ corresponds to some different arm of $L$, and similarly for link$(y_i, H)$. Then we see that for some $i$ the link$(x_i, H)$ and link$(y_i, H)$ both have length 1. But this implies that $H$ has a triangle as a component and thus has no one-factor, a contradiction.

*Case* II: $\lambda_1 < \lambda_2$, $\lambda_i = 0$ for $i \geqslant 5$. (Note $\lambda_2 \neq 0$ here.)

IIA: $\lambda_4 = 0$. $\lambda_1 = 0$. This is implied by Lemma VIII.3.

IIB: $\lambda_4 = 0$, $\lambda_2 \geqslant \lambda_1 + \lambda_3$. (Note $\lambda_3 \neq 0$.) In the one-factor if there are two adjacent vertices $v$, $w$ each of which has a link of length 2, $\langle v, w\rangle \in \partial H$, then if $z$ is the vertex adjacent to $v$ and $w$, then the link$(z, H)$ has length greater than 2. Thus the link$(z, H)$ has length 3. Let $a$ be the vertex shown in Fig. 12 adjacent to $v$ and $z$, and $b$ the vertex adjacent to $w$ and $z$. Then link $(a, H) = \langle v, z\rangle$ and link$(b, H) = \langle w, z\rangle$ and $v, w, a, b, z$ determines a component of $H$. However, its boundary is of odd length and thus cannot contain a one-factor. Thus if $\lambda_2 > \lambda_1 + \lambda_3$ in an element of a partition of the one-factor some vertex whose link has length 2 will be paired with another vertex whose link has length 2 and by the above we have a contradiction. Also we see that if $\lambda_2 = \lambda_1 + \lambda_3$, no

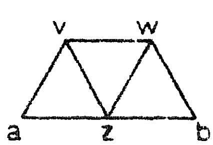



Fig. 12.

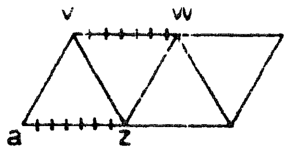



Fig. 13.

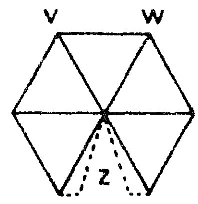



Fig. 14.

two vertices whose link has length 2 will be paired, and if a vertex has a link of length 1 or 3, then it must be paired with a vertex whose link has length 2. Since $\lambda_3 \neq 0$, there is at least one pair of vertices $v$, $w$, $\langle v, w\rangle \in \partial H$ such that link$(v, H)$ is of length 2 and link$(w, H)$ has length 3. Let $z$ be the vertex adjacent to $v$ and $w$, and $a$ the vertex $\neq w$ adjacent to $v$ and $z$. Note that the link$(z, H)$ has length 3 and thus the link$(a, H)$ $= \langle v, z\rangle$. Thus $\langle a, z\rangle$ must be in the one-factor of $H$ and a vertex whose link has length 1 is paired with a vertex whose link has length 3, a contradiction (see Fig. 13).

IIC: $\lambda_2 > \lambda_1 + \lambda_3 + \lambda_4$, $\lambda_1 = 0$. As before there is an edge $\langle v, w\rangle \in \partial H$ such that link$(v, H)$ and link$(w, H)$ are of length 2. Let $z$ be the vertex in $H$ adjacent to $v$ and $w$. Then since there are no vertices whose link has length 1, the length of link$(z, H)$ is at least 5, a contradiction (see Fig. 14).

*Case* III: $(\lambda_1 > \lambda_2)$, $\lambda_i = 0$ for all $i \geq 5$ and $\lambda_1 \geq \lambda_2 + \lambda_4$, $\lambda_3 = 0$, $\lambda_4 \neq 0$. Due to Case I we need only consider when $\lambda_1 = \lambda_2 + \lambda_4$ $(\lambda_4 \neq 0, \lambda_3 = 0)$. Consider some component of $\partial H$ which has a vertex whose link has length 4, and consider the sequence of integers obtained {length of link$(v_i, H)$}, where ... $v_{-1}, v_0, v_1, \ldots$ are the vertices of this component of $\partial H$ in order (a priori this may be finite). We know first that the triple 1,2,1 cannot appear in the sequence since if so, then 1,2,1,2 would be the whole component of $\partial H$ and not have a 4 in it. From the previous discussions we know that this component of $\partial H$ has a one-factor $F$ in which each edge of $F$ has exactly one vertex whose link has length 1. Thus by labeling correctly we know that $\langle v_{2i}, v_{2i+1}\rangle$ is that part of the one-factor in this component of $\partial H$. Thus if ... $n_{-1}, n_0, n_1, \ldots$ is our sequence, exactly one of $n_{2i}, n_{2i+1}$ is a 1 and the other is not. Also we know that no two 1's are adjacent. Thus the only possibilities we have for the sequence are:

(1) ... 1,4,1,4,2,1,4,1,4,1,4,1, ...,

(2) ... 1,4,1,4,4,1,4,1, ...,

(3) ... 1,4,1,4,1,4, ... .

(1) and (2) must be infinite and (3) may be finite. For possibilities (1) and (2), consider the graph obtained by deleting all the vertices on this particular component of $\partial H$ whose link has length 1. In the new graph the vertices whose link had length 4 and were adjacent along $\partial H$ to 2 of the deleted vertices now have a link of length 2, and we may find as many of these vertices in a row along the new boundary as we please. If we choose a string of 4 of the vertices, we find that they are adjacent to another vertex whose link is at least 5. Thus, possibilities (1) and (2) are eliminated. In possibility (3) if we assume that every boundary component is of type (1,4) (ignoring the (1,2) type), then by deleting all the vertices whose link has length 1 we obtain a graph all of whose vertices has a link of length 2, which we know is impossible.

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