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A Generalization of the Seifert-Threlfall Proof for the Lusternik-Schnirelman Category Inequality

E. H. ROTHE

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48104

Submitted by C. L. Dolph

1. INTRODUCTION

Let V be an open bounded set on a Hilbert Fredholm Riemannian manifold M, and let f be a real valued function defined on the closure $\bar{V}$ of V. Let cat $\bar{V}$ denote the Lusternik–Schnirelman category of $\bar{V}$ (with respect to itself) and let r denote the number of stationary point of f in $\bar{V}$. In an earlier paper [8] it was shown (among other things) that the Lusternik–Schnirelman inequality

$$\operatorname{cat} \bar{V} \leqslant r \tag{1.1}$$

holds under assumptions specified in Theorem 4.1 of that paper. The proof was based on results of F. Browder in [1]. It was pointed out in the introduction to [8] that under additional assumptions a more constructive proof for (1.1) could be given in which r closed sets of category 1 covering $\bar{V}$ are exhibited. It is the purpose of the present paper to carry out such a proof based on a generalization of a method used by Seifert and Threlfall [9, p. 91] in the case of a finite dimensional manifold without boundary. (For notations and concepts not explained in the present paper the reader is referred to [8]).

Since (1.1) is trivial if $r = \infty$ we assume that r is finite. Moreover the assumptions of Theorem 4.1 of [8] are supposed to be satisfied. In the present paper we make the following additional assumption: if $x_1, x_2, ..., x_r$ are the stationary points of f then x_ρ is nondegenerate of some (finite) order $p_\rho \geqslant 2$, i.e., all differentials of f of order $p_\rho - 1$ vanish at x_ρ while the differential of order p_ρ is a nondegenerate homogeneous form of order p_ρ in the "increments." (The exact definition of this kind of nondegeneracy is recalled in Section 3.)

The proof of (1.1) is based on the notion of a "cylindrical neighborhood" of a critical point introduced by Seifert and Threlfall in the finite dimensional case in [9; Section 9]. In Section 2 the definition for the Hilbert *space* case (as given in [7, Section 5]) will be recalled and generalised for the case of a Hilbert *manifold*.

The proof of (1.1) then consists in the following three steps:

I. It is shown that the cylindrical neighborhood c_ρ of the stationary point x_ρ (if small enough) is contractible to x_ρ (Section 3). Thus c_ρ is of category 1.

II. By the use of segments of "gradient lines": (see (2.1)) each c_ρ is extended to a certain set γ_ρ (Section 4). γ_ρ' can be deformed into c_ρ. Thus, by I, γ_ρ and its closure $\bar{\gamma}_\rho$ are of category 1 (Lemma 4.1).

III. The $\bar{\gamma}_\rho$ cover $\bar{V}$ (Lemma 4.2).

It is clear from the definition of the category that II and III together imply (1.1).

2. Definition and Basic Properties of Cylindrical Neighborhoods of a Stationary Point

Before treating the case of a Hilbert *manifold* we recall the relevant facts for the case of a Hilbert *space* E referring to [7, Section 5] for details and proofs.

Let then $h(u)$ be a real valued function whose domain lies in E. Without restriction of generality we assume the isolated stationary point of h considered to be the zero point θ of E. Let R be such a positive number that θ is the only stationary point in the ball $B(\theta, 2R)$ of center θ and radius $2R$ and that h is bounded in that ball. Let u_0 be a point of $B(\theta, 2R) - \{\theta\}$. Throughout this paper the "gradient line through u_0" is meant to be the solution $u(\tau, u_0)$ of the problem

$$\frac{du}{d\tau} = \frac{\gamma(u)}{\|\gamma(u)\|^2}, \qquad u(\tau_0, u_0) = u_0, \tag{2.1}$$

where the parameter τ is normalised by

$$\tau_0 = h(u_0), \tag{2.2}$$

and where $\gamma = \text{grad}\, h$. (This terminology differs from the one used in [8]. See [8, Theorem 2.2].) Then

$$\tau = h(u(\tau, u_0)), \tag{2.3}$$

for all τ for which the gradient line is defined.

Definition 2.1. Let

$$h_0 = h(\theta). \tag{2.4}$$

Then the gradient line $u(\tau, u_0)$ is said to end at θ if $\lim u(\tau, u_0) = \theta$ as $\tau \downarrow h_0^+$, and it is said to start from θ if $\lim u(\tau, u_0) = \theta$ as $\tau \uparrow h_0^-$. If ϵ is a positive number then $C^+(\epsilon)$ is the set of those u_0 for which the gradient line through u_0 ends at θ and for which $\epsilon > h(u_0) - h_0 > 0$, and $C^-(\epsilon)$ is the set of those u_0 for which the gradient line through u_0 starts from θ and for which $\epsilon > h_0 - h(u_0) > 0$.

DEFINITION 2.2. Let ϵ and R_1 be positive numbers with $R_1 < R$. Then the sets $Z(R_1)$ and $Z(R_1, \epsilon)$ are defined as follows: $Z(R_1) = \{u \mid h(u) = h_0$ and $\| u \| < R_1\}$ while $Z(R_1, \epsilon)$ is the set of those u which lie on gradient lines through points of $Z(R_1)$ and for which $h_0 - \epsilon < h(u) < h_0 + \epsilon$.

It is not hard to see that $Z(R_1)$, and therefore $Z(R_1, \epsilon)$, is empty if and only if $h(\theta)$ is a relative maximum or minimum.

DEFINITION 2.3. The cylindrical (R_1, ϵ) neighborhood of the isolated stationary point θ of h is the union of the sets $C^+(\epsilon)$, $C^-(\epsilon)$, $Z(R_1, \epsilon)$ and $\{\theta\}$.

LEMMA 2.1. *To a positive $R_1 < R$ corresponds a positive ϵ_0 such that $C(R_1, \epsilon) \subset B(\theta, R)$ for $0 < \epsilon < \epsilon_0$. For such R_1, ϵ the set $C(R_1, \epsilon)$ is open* (*See* [7, *Lemmas* 5.1 *and* 5.4]).

LEMMA 2.2. *The boundary $\dot{C}(R_1, \epsilon)$ of $C(R_1, \epsilon)$ is the union of the following three sets*

β_1: *the set of those points u_0 on $\{h = h_0 + \epsilon\}$ for which the segment $h_0 < \tau < h_0 + \epsilon$ of the gradient line through u_0 belongs to $C(R_1, \epsilon)$*

β_2: *the set of those points u_0 on $\{h = h_0 - \epsilon\}$ for which the segment $h_0 - \epsilon < \tau < h_0$ of the gradient line through u_0 belongs to $C(R_1, \epsilon)$*

β_3: *the points on the segment $h_0 - \epsilon \leqslant \tau \leqslant h_0 + \epsilon$ of the gradient lines through points of the intersection $\{h = h_0\} \cap \| u \| = R_1\}$.*

This lemma seems rather obvious. The proof however—at least the one this author is able to present—is complicated. It is given in the appendix.

We now turn to the definition of a cylindrical neighborhood of an isolated stationary point $x_0 \in \bar{V}$ of f where $\bar{V}$ and f are as in the introduction. We recall that we assume the assumptions of [8, Theorem 4.1] to be satisfied. Let then (ϕ, U) be a chart at x_0 with the Hilbert space E as target space. We assume that

$$\phi(x_0) = \theta \tag{2.5}$$

and set for $u \in \phi(U)$

$$h(u) = f(\phi^{-1}(u)). \tag{2.6}$$

Since by [8, Assumption 2.4] the boundary $\dot{V}$ of V contains no stationary points of f we may and will assume that $U \subset V$. In addition we assume that x_0 is the only stationary point in U. By [8, Eq. (2.19)] (with $u_0 = \theta$) it then follows from (2.5) that θ is an isolated stationary point of h. Therefore there exists a positive R such that not only $B(\theta, 2R) \subset \phi(U)$ but that also θ is the only stationary point of h in $B(\theta, 2R)$. Moreover, we require that h is bounded and satisfies a Lipschitz condition in that ball (see [8, Assumption 2.2]). Let now R_1 be a positive number less than R. Then by Lemma 2.1 there exists a cylindrical neighborhood $C(R_1, \epsilon)$ of θ (for h) which is contained in $B(\theta, R)$.

Definition 2.4. If $C(R_1, \epsilon)$ is defined as above then the set

$$c(R_1, \epsilon) = \phi^{-1}(C(R_1, \epsilon), \tag{2.7}$$

is called a cylindrical (R_1, ϵ) neighborhood of the stationary point x_0 of f, and every cylindrical neighborhood of x_0 is obtained in the manner described.

Remark. The boundary of $c(R_1, \epsilon)$ is obtained from lemma 2.2 by application of ϕ^{-1}.

3. The Contractibility of a Cylindrical Neighborhood

We first recall the following definition of nondegeneracy of order p [7, Definitions 4.1 and 4.2]. (cf. the related definition of "property $Q(r)$ in [3, p. 200].

Definition 3.1. Let h be a real valued function defined in some neighborhood of the zero point θ of the Hilbert space E and suppose that $h \subset C^{p+1}(\theta)$, i.e., that h possesses continuous (Fréchet-) differentials up to and including order $p + 1$ at every point of some neighborhood of θ, where p is an integer $\geqslant 2$. Then θ is called a stationary nondegenerate point of order p of h if

$$dh(\theta; \eta) = \cdots = d^{p-1}h(\theta; \eta) = 0 \tag{3.1}$$

while $d^p h(\theta; \eta)$ (as homogeneous form of degree p) is nondegenerate, i.e., if the norm of the gradient of this function of η divided by $\| \eta \|^{p-1}$ is bounded away from zero.

Lemma 3.1. *Let $v = \psi(u)$ be a $(p + 1)$-diffeomorphism of a neighborhood* *of* $\theta \in E$ *into* E *with* $\psi(\theta) = \theta$, *and let* $h_1(v) = h(\psi^{-1}(v))$. *It is asserted*: (i) *if* θ *is a stationary point of* h *for which* (3.1) *is true then*

$$d^p h_1(\theta; \eta_1) = d^p h(\theta; \eta), \tag{3.2}$$

where

$$\eta_1 = d\psi(\theta; \eta); \tag{3.3}$$

(ii) *if* θ *is a nondegenerate stationary point of order* p *for* h *then* θ *has the same property for* h_1.

Proof. (i) Inspection of the chain rule for differentials of higher order shows that (3.1) implies (3.2). (A statement of this chain rule may be found in [5; p. 164]).

(ii) follows from (i) together with the fact that the right member of (3.3) is not singular.

In what follows we use the definitions assumptions and notations of the preceding two sections. Lemma 3.1 allows us to state the following

DEFINITION 3.2. Let $x_0 \in V$ be a stationary point of f, let (ϕ, U) be a chart at x_0, with $\phi(x_0) = \theta$, and let $h(u) = f(\phi^{-1}(u)$. Then x_0 is called a nondegenerate stationary point of order p of f if θ is a nondegenerate stationary point of order p of h.

The object of the present section is to prove Theorem 3.1.

THEOREM 3.1. *A cylindrical* (R_1, ϵ) *neighborhood* $c(R_1, \epsilon)$ *of a nondegenerate stationary point* x_0 *of order* p *of* f *is contractible to* x_0 *if* R_1 *is small enough.*

It follows from (2.7) that it will be sufficient to show that $C(R_1, \epsilon)$ is, for R_1 small enough, contractible to θ, and for the proof of this latter fact it will obviously be sufficient to establish the following two theorems

THEOREM 3.2. $C(R_1, \epsilon)$ *can be deformed into* $Z(R_1) \cup \{\theta\}$.

THEOREM 3.3. $Z(R_1) \cup \{\theta\}$ *is contractible to* θ.

Proof of Theorem 3.2. Let $0 \leqslant \alpha \leqslant 1$ and let $u_0 \in C(R_1, \epsilon)$. Let $u(\tau, u_0)$ be the gradient line through u_0 (cf. (2.1)–(2.3)). We set

$$\delta(u_0, \alpha) = \begin{cases} u_0, & \text{for } u_0 \in Z(R_1) \cup \{\theta\}, \quad 0 \leqslant \alpha \leqslant 1 \\ u(\tau_0(1-\alpha), u_0), & \text{for } u_0 \in Z(R_1, \epsilon), \quad 0 \leqslant \alpha \leqslant 1 \\ u(\tau_0(1-\alpha), u_0), & \text{for } u_0 \in C^+(\epsilon) \cup C^-(\epsilon), \quad 0 \leqslant \alpha < 1 \\ \theta, & \text{for } u_0 \in C^+(\epsilon) \cup C^-(\epsilon), \quad \alpha = 1. \end{cases} \tag{3.4}$$

Then $\delta(u_0, 0) = u_0$, and $\delta(u_0, 1) \in Z(R_1) \cup \{\theta\}$, and Theorem 3.2 will be proved once the joint continuity of δ in its two arguments is established. This was done by Seifert and Threlfall in the case of a finite dimensional space E. ([9, p. 96]). Their proof however uses the local compactness of such a space. A continuity proof valid in case E is a Hilbert space is given in the appendix.

Proof of Theorem 3.3. In order to construct a deformation $\delta(u_0, \alpha)$ for which

$$\delta(u_0, 0) = u_0, \qquad \delta(u_0, 1) = \theta, \qquad u_0 \in Z(R_1) \cup \theta, \tag{3.5}$$

we employ the solution $u = u(t, u_0)$ of the problem

$$\frac{du}{dt} = \frac{\gamma(u)\,\langle u, \gamma(u)\rangle - u\,\|\,\gamma(y)\|^2}{\|\,\gamma(u)\|^2}, \qquad u_0 \in Z(R_1), \quad \gamma = \operatorname{grad} h. \tag{3.6}$$

(For the purpose of critical point theory this differential equation was used by A. B. Brown [2] in the finite dimensional case. For the Hilbert space case see [6; p. 449]).

We will need some properties of $u(t, u_0)$.

LEMMA 3.2. *h is constant along the trajectories of* (3.6)

Proof. Direct computation shows that

$$\frac{dh(u(t, u_0)}{dt} = \left\langle \gamma, \frac{du}{dt} \right\rangle = 0$$

(cf. [6; p. 449]).

LEMMA 3.3. *For R_1 small enough $\|\,u(t, u_0)\|$ is monotone decreasing in t. Moreover*

$$\|\,u_0\,\|^2\, e^{-2t} \leqslant \|\,u(t)\|^2 \leqslant \|\,u_0\,\|^2\, e^{-t}, \qquad \textit{for} \quad 0 \leqslant t < \infty. \tag{3.7}$$

Proof. Scalar multiplication of (3.6) by $2u$ shows that

$$\frac{d\,\|\,u\,\|^2}{dt} = 2\left\langle u, \frac{du}{dt} \right\rangle = -2\,\|\,u\,\|^2 \left\{1 - \left\langle \frac{u}{\|\,u\,\|}, \frac{\gamma}{\|\,\gamma\,\|} \right\rangle^2 \right\} \tag{3.8}$$

(cf. [6; (4.36)]). To estimate the right member of (3.8) we will show that

$$\left| \left\langle \frac{u}{\|\,u\,\|}, \frac{\gamma(u)}{\|\,\gamma(u)\|} \right\rangle \right| < \frac{1}{\sqrt{2}}, \qquad u = u(t, u_0), \quad u_0 \in Z(R_1), \quad t \geqslant 0, \tag{3.9}$$

Since θ is a nondegenerate critical point of h of some order $p \geqslant 2$ the differentials of h of order $\leqslant p-1$ vanish at θ. Thus Taylor's theorem shows that

$$h(u) - h(u_0) = \frac{1}{p!} d^p h(\theta; u) + R_p{}^1, \tag{3.10}$$

where

$$R_p{}^1 = \frac{1}{p!} \int_0^1 d^{p+1} h(tu; u) (1-t)^p \, dt. \tag{3.11}$$

But $\gamma = \operatorname{grad} h$ and its differentials of order $\leqslant p-2$ also vanish at θ (see [7; (4.13)]). Therefore

$$\gamma(u) = \frac{1}{(p-1)!} d^{p-1} \gamma(\theta; u) + R_{p-1}^2, \tag{3.12}$$

where

$$R_{p-1}^2 = \frac{1}{(p-1)!} \int_0^1 d^p \gamma(tu; u) (1-t)^{p-1} \, dt. \tag{3.13}$$

We also recall ([7; (4.15)]) that

$$d^{p-1} \gamma(\theta; u) = p^{-1} \operatorname{grad}_2 d^p h(\theta; u) \tag{3.14}$$

where the subscript 2 indicates that the gradient operation refers to the second argument of $d^p h$.

Multiplying (3.12) scalar by u and using (3.14) we obtain

$$\langle u, \gamma(u) \rangle = \frac{1}{(p-1)!} \langle u, \operatorname{grad}_2 d^p h(\theta; u) \rangle + \langle u, R_{p-1}^2 \rangle. \tag{3.15}$$

Now $d^p h(\theta; u)$ is homogeneous of degree p in u. Therefore using Euler's theorem on homogeneous functions we conclude from (3.15) that

$$\langle u, \gamma(u) \rangle = \frac{1}{(p-1)!} d^p h(\theta; u) + (u, R_{p-1}^2). \tag{3.16}$$

Here and in (3.10) we set $u = u(t, u_0)$ with $u_0 \in Z(R_1)$. Then by Lemma 3.2 the left member of (3.10) equals zero. Thus we see from (3.16) and (3.10) that

$$\langle u, \gamma(u) \rangle = -pR_p{}^1 + (u, R_{p-1}^2), \qquad u = u(t, u_0), \quad u_0 \in Z(R), \quad t \geqslant 0. \tag{3.17}$$

Now using Schwarz' inequality we see from (3.8) that $\| u \| = \| u(t, u_0)\|$ is not increasing in t, and therefore

$$\| u \| = \| u(t, u_0)\| \leqslant \| u_0 \| \leqslant R_1 , \qquad t \geqslant 0. \tag{3.18}$$

But (3.17) and (3.18) together with the definitions (3.8) and (3.10) of $R_p{}^1$ and R^2_{p-1} and together with the continuity of the differentials involved implies (for small enough R_1) the existence of a constant C_1 such that

$$\langle (u, \gamma(u)\rangle < C \| u \|^{p+1}. \tag{3.19}$$

Now by [7, Lemma 4.3] the nondegeneracy of order p of the critical point θ implies for small enough $\| u \|$ the existence of a constant k such that $\| \gamma(u)\| > k \| u \|^{p-1}$. Combining this inequality with (3.19) we see that the left member of (3.9) is majorized by $\| u \| C/k$. Because of (3.18) this proves (3.9) for small enough R_1.

The first assertion of Lemma 3.3 is now an immediate consequence of (3.8) and (3.9). Another immediate consequence of (3.8) and (3.9) are the inequalities

$$\frac{d \| u \|^2}{dt} + \| u \|^2 < 0 < \frac{d \| u \|^2}{dt} + 2 \| u \|^2,$$

from which (3.7) follows routinely.

With $u(t, u_0)$ defined as above (see (3.6)) we set

$$\delta(u_0 , \alpha) = \begin{cases} u\left(\dfrac{1}{1-\alpha} - 1, u_0\right), & \text{for } u_0 \in Z(R_1), \quad 0 \leqslant \alpha < 1 \\ \theta, & \text{for } u_0 \in Z(R_1), \quad \alpha = 1 \\ \theta, & \text{for } u_0 = \theta, \qquad 0 \leqslant \alpha \leqslant 1. \end{cases} \tag{3.20}$$

(3.5) (cf. Lemma 3.2) is obviously satisfied. Moreover $\delta(u_0 , \alpha) \in Z(R_1) \cup \theta$ if $u_0 \in Z(R_1) \cup \theta$. We also note that on account of (3.7)

$$\lim_{\alpha \to 1} \delta(u_0 , \alpha) = \delta(u_0 , 1), \qquad \text{for } u_0 \in Z(R_1).$$

The proof that δ is jointly continuous in u_0 and α is given in the appendix.

4. The Extension γ_ρ of the Cylindrical Neighborhood c_ρ

Let $x_1 , x_2 ,..., x_r$ be the stationary points of f in $\bar{V}$. By Assumption 2.4 of [8] there are no stationary points on the boundary $\dot{V}$ of V. Let $c(R_1 , \epsilon, x_\rho)$ be an (R_1 , ϵ)-cylindrical neighborhood of x_ρ with R_1 and ϵ chosen, independent of ρ, in such a way that the closure of $c(R_1, \epsilon, x_\rho)$ are disjoint from each other and from the boundary of V (see Definition 2.4 and the paragraph preceding it). Moreover we may assume R_1 so small that the assertions of the theorems of Section 3 established with the proviso "for R_1 small enough" are true. For R_1 and ϵ fixed in such a way we use the notation

$$c_\rho = c(R_1, \epsilon, x_\rho), \qquad f_\rho = f(x_\rho). \tag{4.1}$$

DEFINITION 4.1 (cf. [9, p. 91]). The roof of c_ρ consists of those points of the boundary $\dot{c}_\rho$ of c_ρ which are at the level $f_\rho + \epsilon$.

Let now x_0^ρ be a point of the roof of c_ρ, and let $x(\tau, x_0^\rho)$ be the gradient line through x_0^ρ. It is then clear from our definitions and our choice of R_1 and ϵ that

$$x(\tau, x_0^\rho) \in V - \bigcup_1^r \bar{c}_j \qquad \text{for small enough positive } \tau - \tau_0^\rho. \tag{4.2}$$

For fixed x_0^ρ let $\bar{\tau}_0^\rho$ denote the (finite or infinite) least upper bound of all τ_1 such that (4.2) holds for $\tau_0^\rho < \tau \leqslant \tau_1$. Then $\bar{\tau}_0^\rho > \tau_0^\rho$.

DEFINITION 4.2. The extension γ_ρ of c_ρ is the union of c_ρ and the set of those points $x \in \bar{V}$ which lie on a segment $f_\rho + \epsilon = \tau_0^\rho < \tau < \bar{\tau}_0^\rho$ of the gradient line $x(\tau, x_0^\rho)$ through some point x_0^ρ on the roof of c_ρ.

THEOREM 4.1. *Let the Assumptions* 2.1–2.4 *of* [8] *be satisfied. Moreover f is supposed to be bounded below. Finally it is assumed that each of the stationary points $x_1, x_2, \ldots, x_r$ is non-degenerate of some finite order $p \geqslant 2$. Then* (1.1) *holds.*

Proof. The theorem follows from the Lemmas 4.1 and 4.2 below.

LEMMA 4.1. *γ_ρ, and therefore $\bar{\gamma}_\rho$, is of category* 1.

Proof. By Theorem 3.1 $\bar{c}_\rho$ is of category 1. It is therefore sufficient to show that $\bar{c}_\rho$ is a deformation retract of γ_ρ. If $x(\tau, x_0)$ is the gradient line through the point $x_0 \in \gamma_\rho - \bar{c}_\rho$ then $x(f_\rho + \epsilon, x_0)$ is a point of the roof of c_ρ and therefore $\in \bar{c}_\rho$. We now define a retracting deformation by setting

$$\delta(x_0, \alpha) = \begin{cases} x(\alpha(f_\rho + \epsilon) + (1 - \alpha) f(x_0), x_0), & \text{for } x_0 \in \gamma_\rho - \bar{c}_\rho \\ x_0, & \text{for } x_0 \in \bar{c}_\rho. \end{cases}$$

LEMMA 4.2. $\bar{V} \subset \bigcup_1^r \bar{\gamma}_\rho$.

For every $x_0 \in \bar{V}$ we have to prove

$$x_0 \in \bar{\gamma}_\rho \qquad \text{for some positive integer} \quad \rho \leqslant r. \tag{4.3}$$

This assertion is trivial if x_0 is contained in some $\bar{c}_\rho$. Therefore we have to consider the following cases (α) and (β).

$$(\alpha) \quad x_0 \in V - \bigcup_{\rho=1}^{r} \bar{c}_\rho , \qquad (\beta) \quad x_0 \in \dot{V}. \tag{4.4}$$

Case (α). We consider the gradient line $x(\tau, x_0)$ through x_0 for $\tau \leqslant \tau_0 = f(x_0)$. Let $\bar{\tau}_0$ be the greatest lower bound of those σ for which

$$x(\tau, x_0) \in V - \bigcup_{\rho=1}^{r} \bar{c}_\rho , \qquad \text{for } \sigma \leqslant \tau \leqslant \tau_0 . \tag{4.5}$$

Since $V - \bigcup_1^r \bar{c}_\rho$ is an open set and since f is bounded from below in V

$$-\infty < \bar{\tau}_0 < \tau_0 . \tag{4.6}$$

We now claim

$$\bar{x} = \lim_{\tau \to \bar{\tau}_0} x(\tau, x_0) \tag{4.7}$$

exists, and

$$\bar{x} = x(\bar{\tau}_0 , x_0) \in \gamma_\rho \qquad \text{for some } \rho. \tag{4.8}$$

To prove that the limit (4.7) exists we note that by Lemma 2.1 the set $V - \bigcup_1^r \bar{c}_\rho$ has a positive distance from the set $\{x_\rho\}$ ($\rho = 1,..., r$) of stationary points of f, and that therefore by the Palais–Smale condition ([8, Assumption 2.3]) there exists a constant m such that $\| g(x)\| > m > 0$ for $x \in V - \bigcup_1^r \bar{c}_\rho$. By (4.5) this inequality holds in particular for $x = x(\tau, x_0)$ if $\bar{\tau}_0 < \tau \leqslant \tau_0$. From this and the differential equation (2.2) for the gradient line we obtain easily the estimate

$$\| x(\tau'', x_0) - x(\tau', x_0)\| < (\tau'' - \tau')/m, \qquad \text{for } \bar{\tau}_0 < \tau' \leqslant \tau'' < \tau_0 .$$

The existence of the limit (4.7) follows now from Cauchy's convergence criterion, and we turn to the proof of (4.8). The equality in (4.8) is an obvious consequence of (4.7). The inclusion in (4.8) is by definition of γ_ρ equivalent to the assertion

(4.9) $\bar{x} = x(\bar{\tau}_0 , x_0)$ is contained in the roof of some c_ρ.

To prove this assertion we note first that by (4.7) and (4.8), $x(\bar{\tau}_0 , x_0)$ is in the closure of $V - \bigcup_1^r \bar{c}_\rho$. On the other hand it follows from the definition of $\bar{\tau}_0$ and the local existence theorem for differential equations that $\bar{x} = x(\bar{\tau}_0 , x_0)$ is not an element of the open set $V - \bigcup_1^r \bar{c}_\rho$. Thus $\bar{x}$ must be on the boundary of this set, i.e.,

$$\text{either} \quad \bar{x} \in \dot{c}_\rho \qquad \text{for some } \rho, \text{ or} \quad \bar{x} \in \dot{V}. \tag{4.10}$$

Now since $\tau = f(x(\tau, x_0))$ decreases strictly as τ varies from τ_0 to $\bar{\tau}_0$ it follows from Definition 4.1, Lemma 2.2 and Definition 2.4 that $\bar{x}$ lies on the roof of c_ρ, and therefore $x_0 \in \gamma_\rho$, if the first part of the alternative (4.10) takes place.

Thus the assertion (4.3) will be proved in our present case (4.4) α) if we can show that the second alternative in (4.10) can not occur. To show that this is so we note that the substitution

$$t = -\int_{\tau_0}^{\tau} \frac{d\theta}{\| g(x(\tau, x_0))\|^2}, \qquad x(\tau, x_0) = y(t, y_0), \quad \tau_0 \geqslant \tau \geqslant \bar{\tau}, \tag{4.11}$$

transforms the problem (2.1) into the problem

$$\frac{dy}{dt} = -g(y), \qquad y(0, x_0) = x_0 \in V, \tag{4.12}$$

and we have only to refer to Theorem 2.2 of [8] which states that $y(t, x_0)$ does not reach the boundary $\dot{V}$ of V for $t \geqslant 0$. (This theorem is essentially based on the boundary Assumption 2.4 of [8]).

This finishes the proof of Lemma 4.2 if (4.4α) holds. For the proof in case of (4.4β) we will need Lemma 4.3.

LEMMA 4.3. *Let E^2 be a linear hypersubspace of the Hilbert space E and let e' be an element of E which is of length* 1 *and which is orthogonal to E^2. If for $\rho > 0$, B_ρ denotes the ball with center θ and radius ρ we set*

$$B_\rho^- = \{x \in B_\rho \mid \langle e', x\rangle < 0\}. \tag{4.13}$$

Finally let the map γ: $B_\rho^- \cup \{\theta\} \to E$ be Lipschitz and satisfy

$$\langle e', \gamma(\theta)\rangle > 0. \tag{4.14}$$

Then there exists a positive α_1, such that the problem

$$du/d\alpha = -\gamma(u), \qquad u(0) = \theta, \tag{4.15}$$

has a unique solution $u = u(\alpha)$ for $0 \leqslant \alpha < \alpha_1$. Moreover

$$u(\alpha) \in B_\rho^- \qquad \text{for} \quad 0 < \alpha < \alpha_1. \tag{4.16}$$

The proof consists in a suitable modification of the classical Picard proof for the local existence of solutions of ordinary differential equations and will be given in the appendix.

We return to the proof of Lemma 4.2. Let (U, ϕ) be a chart at the point x_0

for which (4.4β) holds. We assume $\phi(x_0) = \theta$. Let E^2 and e' be as in Lemma 2.3 of [8]. Then by that lemma there exists a positive ρ such that

$$\phi^{-1}(B_\rho^-) \subset V, \tag{4.17}$$

with B_ρ^- as in (4.13). We may assume e' to be orthogonal to E^2. We claim that (4.14) holds if $\gamma = \operatorname{grad} f\phi^{-1}$. Indeed then by [8, Lemma 2.6 and Definition 2.3] the left member of (4.14) equals the scalar product of $g(x_0)$ with the exterior unit normal to $\dot{V}$ at x_0, and this product is positive by Assumption 2.4 of [8]. Thus we can apply Lemma 4.3 and we see that (4.16) holds. Therefore by (4.17)

$$\phi^{-1}(u(\alpha)) \in V \qquad \text{for} \quad 0 < \alpha < \alpha_1 . \tag{4.18}$$

But $u(\alpha)$ is the solution of (4.15). Consequently (see [8, Lemma 2.7]) $y(\alpha) = \phi^{-1}(u(\alpha))$ satisfies (4.12), and the $x(\tau, x_0)$ obtained from $y(\alpha)$ by the substitution (4.11) is the gradient line through x_0. We thus see from (4.18) that

$$x(\tau, x_0) \in V \qquad \text{for} \quad \tau_0 > \tau > \tau_1 , \tag{4.19}$$

where τ_0 and τ_1 are the τ-values corresponding by the substitution (4.11) to the α-values α_0 and α_1, respectively. Now the point $x_0 \in \dot{V}$ has a positive distance from $\bigcup c_\rho$. Therefore (4.19) implies that $\bar{x} = x(\bar{\tau}, x_0) \in V - \bigcup c_\rho$ if $\bar{\tau} - \tau_0$ is positive and small enough. Thus $\bar{x}$ satisfies the assumption made in (4.4α) for x_0, and from the result proved for this case we know that the gradient line through $\bar{x}$ reaches the roof of some c_ρ for some $\tau > \bar{\tau}$. This finishes the proof since the gradient line through $\bar{x}$ is the same as that through x_0 (up to a parameter translation).

5. Appendix

The purpose of this section is to give proofs of Lemmas 2.2 and 4.3, and of the continuity of the deformations defined in (3.1) and (3.20). Basic for these proofs is the following lemma

Lemma 5.1. *Let $u_1(t)$ and $u_2(t)$ be solutions of the differential equation $du/dt = \psi(u)$, which for $t_0 \leqslant t \leqslant t_1$ lie in the subset S of the domain of ψ the latter being open in the Hilbert space E. We assert*

(i) *if ψ satisfies in S a Lipschitz condition with Lipschitz constant λ then*

$$\| u_2(t) - u_1(t)\| \leqslant \| u_2(t_0) - u_1(t_0)\| \, e^{\lambda(t-t_0)} \qquad \textit{for } t_0 \leqslant t \leqslant t_1 .$$

(ii) *if there exists a constant* $\bar{m}$ *such that*

$$\|\psi(u)\| \leqslant \bar{m} \qquad \text{for} \quad u \in S \quad \textit{then} \quad \|u_1(t_1) - u_2(t_2)\| \leqslant \bar{m}(t_1 - t_0).$$

Assertion (i) of this lemma is classical in a finite dimensional space. For a proof valid in Banach spaces see [4, p. 56, Proposition 2]. Assertion (ii) follows trivially on writing the differential equation in integral form.

In the application of this lemma to the differential Equation (2.1) for the gradient lines one has to observe that the assumptions of a uniform Lipschitz condition and of boundedness in S are satisfied only if u is bounded away from the set of stationary points. To cope with this complication we need three more lemmas.

LEMMA 5.2. *Let h, θ and R be as described in the paragraph preceding* (2.1), *and let R_1, ϵ be as in Lemma* 2.1. *For simplicitys sake we assume that $h_0 = h(\theta) = 0$. We consider a set $Y \subset Z(R_1)$ (see Definition* 2.2) *having the property that there exists a constant η_0 such that*

$$0 < \eta_0 \leqslant \|y\| < R_1, \qquad \textit{for} \quad y \in Y. \tag{5.1}$$

Let $u(\tau, y)$ be the gradient line through the point y of Y such that

$$x(0, y) = y, \qquad h(y) = 0. \tag{5.2}$$

We assert the existence of a number $\zeta = (\eta_0, R_1)$ such that

$$\|u(\tau, y)\| \geqslant \zeta \qquad \textit{for} \quad y \in Y \quad \textit{and} \quad -\epsilon < \tau < \epsilon. \tag{5.3}$$

Proof. Let

$$\mu = \min\{\eta_0/2; R/2\}. \tag{5.4}$$

Then the set

$$X = \bigcup_{y \in Y} B(y, \mu), \tag{5.5}$$

has a positive distance from the set of stationary points of h, and by the Palais–Smale condition there exists a positive number $m = m(\mu)$ such that

$$\|\gamma(u)\| \geqslant m > 0, \qquad \text{for} \quad u \in X. \tag{5.6}$$

Because of $h(\theta) = 0$ there exists a $\zeta > 0$ such that

$$|h(x)| < \min\{\eta_0/2, \mu m\} \qquad \text{for} \quad \|u\| < \zeta. \tag{5.7}$$

In addition we subject ζ to the condition

$$0 < \zeta < \eta_0/2.$$

With this choice of ζ we claim (5.3) to be true.

Suppose that

$$\| u(\tau, y)\| < \zeta \tag{5.9}$$

for some $y \in Y$ and some τ in $(-\epsilon, \epsilon)$. For such y and τ we see from (5.7) that $|\tau| = |h(u(\tau, y))| < \mu m$. This shows that (5.9) is not true for $\tau \geqslant \mu m$, in other words for such τ our assertion (5.3) holds.

It remains to prove the assertion for $|\tau| < \mu m$. We see from (5.6) that in the ball $B(y, \mu)$ the right member of the differential equation (2.2) is majorized by m^{-1}. Consequently $u(\tau, y)$ stays in this ball for $|\tau| < \mu m$, i.e., $\| u(\tau, y) - y \| < \mu$ for these τ. Consequently use of (5.1), (5.4) and (5.8) shows that

$$\| u(\tau, y)\| \geqslant \| y \| - \| u(\tau, y) - y \| \geqslant \eta_0 - \mu > \zeta \qquad \text{for} \quad |\tau| < \mu m.$$

LEMMA 5.3. *Let* $0 < \eta_0 < R_1$, *and let* $\{y_n\}$ *be a sequence of points of the following properties*:

$$\eta_0 \leqslant \| y_n \| \leqslant R_1, \qquad h(y_n) = 0, \qquad \lim_{n\to\infty} y_n = y_0 .$$

Then $\lim_{n\to\infty} u(\tau, y_n) = u(\tau, y_0)$ *for* $-\epsilon \leqslant \tau \leqslant \epsilon$ *where as usual* $u(\tau, y)$ *denotes the gradient line through* y.

Proof. By Lemma 5.2 and by the choice of R_1 and ϵ the set S of the segments $-\epsilon \leqslant \tau \leqslant \epsilon$ of the gradient lines through y_n is bounded away from the stationary set of h. Consequently by the Palais–Smale condition the assumptions of Lemma 5.1 are satisfied in S with $\psi = \gamma/\| \gamma \|^2$. Thus our assertion is a consequence of the (i)-part of that lemma.

LEMMA 5.4. *Let* h, R, R_1, *be as in Lemma* 5.2. *Let* $u_0 \in Z(R_1, \epsilon)$ *(Definition* 2.2) *and let* $u(\tau, u_0)$ *denote the gradient line through* u_0. *Then to a given positive* σ *there corresponds a positive* ρ *such that*

$$\| u(\tau, \bar{u}) - u(\tau, u_0)\| < \sigma, \qquad \text{for} \quad -\epsilon < \tau < \epsilon, \tag{5.10}$$

if

$$\| u - u_0 \| < \rho. \tag{5.11}$$

Proof. From $u_0 \in Z(R_1, \epsilon)$ it follows that $u(\tau, u_0) \in C(R_1, \epsilon) \in B(R)$ (see Lemma 2.1) for $-\epsilon < \tau < \epsilon$. Thus for these values of τ

$$\| u(\tau, u_0)\| \leqslant R. \tag{5.12}$$

On the other hand it follows from $h(u(0, u_0) = 0$ and from $h(\theta) = 0$ together with the fact that $h(u(\tau, u_0)$ is strictly increasing with τ that $u(\tau, u_0) \neq \theta$ for $-\epsilon \leqslant \tau \leqslant \epsilon$. Since this segment of our gradient line is compact we infer the existence of a positive constant $\mu < R/3$ such that

$$\| u(\tau, u_0)\| > 4\mu, \qquad \text{for} \quad -\epsilon \leqslant \tau \leqslant \epsilon. \tag{5.13}$$

It follows from (5.12) and (5.13) that the set S defined by

$$S = \bigcup_{|\tau| \leqslant \epsilon} B(u(\tau, u_0), 3\mu), \tag{5.14}$$

has a distance $\geqslant \mu$ from the set of stationary points of h. Thus for some positive m the inequality (5.6) holds for $u \in S$. This implies the existence of a Lipschitz constant λ for the right member of the differential equation (2.1) in S.

Let now ρ be a positive number such that

$$| h(\bar{u}) - h(u_0)| < \mu m \qquad \text{if} \quad | \bar{u} - u_0 | < \rho. \tag{5.15}$$

In addition we require

$$\rho < \mu e^{-\lambda\epsilon}. \tag{5.16}$$

We show next that with such choice of ρ

$$u(\tau, \bar{u}) \in S \qquad \text{for} \quad -\epsilon \leqslant \tau \leqslant \epsilon \tag{5.17}$$

if (5.11) is satisfied. To better compare $u(\tau, \bar{u})$ with $u(\tau, u_0)$ we introduce the solution $\phi(\sigma, \xi)$ of (2.1) for which

$$\phi(\tau_0, \xi) = \xi. \tag{5.18}$$

Then

$$u(\tau, u_0) = \phi(\tau, u_0), \qquad u(\tau, \bar{u}) = \phi(\tau + \tau_0 - \bar{\tau}, \bar{u}). \tag{5.19}$$

Now the assertion (5.17) is identical with the following one: if $T = \{\tau \mid \phi(\tau + \tau_0 - \bar{\tau}_0 \bar{u}) \subset S$, then

$$T \supset [-\epsilon_1 + \epsilon]. \tag{5.20}$$

Now it is easily seen from (5.19), (5.11), (5.16) and (5.13) that T contains $\tau = \bar{\tau}$ and therefore some neighborhood of $\bar{\tau}$. Suppose now (5.20) were not true. Then there exists either a $\tau_1 < \bar{\tau}$ such that $\phi(\tau + \tau_0 - \bar{\tau}, \bar{u}) \in S$ for $\tau_1 < \tau \leqq \bar{\tau}$ while each neighborhood of τ_1 contains numbers τ for which $\phi(\tau + \tau_0 - \bar{\tau}, \bar{u}) \notin S$, or there exists a $\tau_1 > \bar{\tau}$ with the corresponding properties.

It will be sufficient to consider the first case. To arrive at a contradiction we observe that Lemma 5.1 can be applied to any closed τ-interval contained in $(\tau_1, \bar{\tau}]$. Taking account also of (5.15) and (5.1) we obtain the following inequality valid for all such τ and for u satisfying (5.11)

$$\begin{aligned} &\| u(\tau, \bar{u}) - u(\tau, u_0)\| \\ &\quad = (\| \phi(\tau + \tau_0 - \bar{\tau}, \bar{u}) - \phi(\tau, u_0)\| \\ &\quad \leqslant \| \phi(\tau + \tau_0 - \bar{\tau}, \bar{u}) - \phi(\tau, \bar{u})\| + \| \phi(\tau, \bar{u}) - \phi(\tau, u_0)\| \\ &\quad \leqslant | \tau_0 - \bar{\tau} |/m + \| \bar{u} - u_0 \| < e^{\lambda\epsilon} < 2\mu, \end{aligned} \tag{5.21}$$

and it is easy to see that (5.21) holds also for $\tau = \tau_1$. But this shows that the ball with center $\phi(\tau_1 + \tau_0 - \bar{\tau}, \bar{u})$ and radius μ is still contained in S, and so is $\phi(\tau + \tau_0 - \bar{\tau}, u)$ if $0 < \tau_1 - \tau < \mu m$. This contradicts the definition of τ_1.

This proves (5.20) and, therefore, (5.17). It follows that (5.21) holds for all τ in $[-\epsilon, \epsilon]$ with $\bar{u}$ satisfying (5.11).

We now recall that μ was defined as a positive number satisfying (5.13) and the inequality $\mu < R$. If then σ is a given positive number we subject μ to the additional restriction that $2\mu < \sigma$. Then (5.10) follows from (5.21).

Proof of Lemma 2.2. Since by Lemma 2.1, $C(R_1, \epsilon)$ is open, Lemma 2.2 is equivalent to Lemma 5.5.

LEMMA 5.5. *Let $\beta_1, \beta_2, \beta_3$ be defined as in the statement of Lemma* 2.2, *and let $\beta = \bigcup \beta_i$. Then for R_1 and ϵ small enough*

$$\overline{C(R_1, \epsilon)} = C(R_1, \epsilon) \cup \beta. \tag{5.22}$$

Proof. We again assume $h(\theta) = 0$. As in Section 2 let $R > 0$ be such that θ is the only critical point in $B(2R)$. It is possible to choose positive numbers $R_1, \epsilon_1, \epsilon$ in such a way that

$$\overline{C(R_1, \epsilon)} \subset C(R, \epsilon_1) \subset B(2R), \qquad \epsilon < \epsilon_1. \tag{5.23}$$

Indeed by Lemma 2.1 there exists a positive ϵ_1 such that $C(R, \epsilon_1) \subset B(2R)$, and a positive R_1 such that $B(2R_1) \subset C(R, \epsilon_1)$. Moreover by the same lemma we can choose ϵ such that $C(R_1, \epsilon) \subset B(2R_1)$, and in addition $\epsilon < \epsilon_1$. With this choice of R_1 and ϵ, (5.23) is satisfied.

To prove (5.22) we will first show that

$$\overline{C(R_1, \epsilon)} \subset C(R_1, \epsilon) \cup \beta, \tag{5.24}$$

i.e., that

$$u_0 \in C(R_1, \epsilon) \cup \beta, \tag{5.25}$$

if

$$u_0 \in \overline{C(R_1, \epsilon)}. \tag{5.26}$$

Now if (5.26) is satisfied there exists a sequence of points $u_1, u_2, \ldots,$ such that

$$u_n \in C(R_1, \epsilon), \qquad \text{and} \qquad \lim_{n\to\infty} u_n = u_0 . \tag{5.27}$$

Then

$$-\epsilon < h(u_n) < \epsilon_n , \qquad -\epsilon \leqslant h(u_0) \leqslant \epsilon, \tag{5.28}$$

and by (5.23)

$$u_0 \subset C(R, \epsilon_1). \tag{5.29}$$

Therefore by definition of $C(R, \epsilon_1)$

$$\text{either} \quad u_0 \in C^+(\epsilon_1) \cup C^-(\epsilon_1), \quad \text{or} \quad u_0 \in Z(R, \epsilon_1) \tag{5.30}$$

where we omitted the case $u_0 = \theta$ in which our assertion (5.25) is trivially satisfied.

In discussing the first of the two cases in (5.30) it will be sufficient to suppose that $u_0 \in C^+(\epsilon_1)$. Then u_0 is on the segment $0 < \tau \leqslant \epsilon_1$ of a gradient line ending at θ. But from (5.28) we see that u_0 is actually on the smaller segment $0 < \tau \leqslant \epsilon$ of such gradient line. This proves (5.25).

We turn to the second part of the alternative in (5.30). In that case

$$0 < \| u(0, u_0)\| < R. \tag{5.31}$$

From the second part of (5.27) in conjunction with Lemma 5.4 we see that

$$\lim_{n\to\infty} u(0, u_n) = u(0, u_0), \tag{5.32}$$

and from the first part of (5.27) we conclude that $\| u(0, u_0)\| < R_1$, an inequality which together with (5.31), (5.32) implies that $0 < \| u(0, u_0)\| \leqslant R_1$. This together with the second part of (5.28) proves (5.25).

We now want to prove the inclusion opposite to (5.24), i.e., we want to show that (5.25) implies (5.26). Since this is obvious if $u_0 \in C(R_1, \epsilon)$ we assume

$$u_0 \in \beta = \bigcup_1^3 \beta_i , \tag{5.33}$$

and will establish the existence of a sequence u_n satisfying (5.27).

Suppose first $u_0 \in \beta_1$. Then the segment $0 < \tau < \epsilon$ of the gradient line $u(\tau, u_0)$ through u_0 is in $C(R_1, \epsilon)$ and the points $u_n = u(\tau_n, u_0)$ with $\tau_n \uparrow \epsilon$ will satisfy (5.27).

The corresponding argument holds if $u_0 \in \beta_2$.

It remains to consider the case that $u_0 \in \beta_3$. Then with the notation $y_0 = u(0, u_0)$

$$\| y_0 \| = R_1 . \tag{5.34}$$

Let now $y(t, y_0)$ be the solution of the differential equation (3.6) with the initial condition $y(0, y_0) = y_0$, and let $y_n = y(n^{-1}, y_0)$. It follows from Lemmas 3.2 and 3.3 that

$$h(y_n) = 0, \qquad \lim_{n\to\infty} y_n = y_0 , \qquad 0 < \eta_0 \leqslant \| y_n \| < \| y_0 \| = R_1 , \tag{5.35}$$

for some positive η_0. Then by Lemma 5.3, $\lim_{n\to\infty} u(\tau, y_n) = u(\tau, y_0)$ which shows that $u_n = u(\tau, y_n)$ satisfies both conditions (5.27) if $-\epsilon < \tau_0 < \epsilon$. But $u_0 = u(\epsilon, y_0)$ also satisfies (5.26) since $u(\epsilon, y_0) = \lim_{\tau\uparrow\epsilon} u(\tau, y_0)$.

Proof of the Continuity of the Deformation (3.4)

We have to prove the continuity of $\delta(u, \alpha)$ at every point (u_0, α_0) where $u_0 \in C(R_1, \epsilon)$ and $0 \leqslant \alpha_0 \leqslant 1$. We assume again $h(\theta) = 0$ and consider different cases.

Case A. $u_0 \in C^+(\epsilon) \cup C^-(\epsilon)$, $\alpha_0 = 1$. It will be sufficient to consider the case that $u_0 \in C^+(\epsilon)$. Then by (3.4)

$$\delta(u_0, 1) = \theta. \tag{5.36}$$

We have to prove

$$\delta(u_n, \alpha_n) \to \theta, \tag{5.37}$$

if

$$u_n \to u_0 \qquad \text{and} \qquad \alpha_n \to 1. \tag{5.38}$$

We will first consider the special case that all α_n equal 1, i.e., we want to prove that

$$\delta(u_n, 1) \to \theta. \tag{5.39}$$

Now $h(u_0) > 0$ by definition of $C^+(\epsilon)$. Therefore we may, on account of (5.38) assume $h(u_n) > 0$ for all n. Therefore for each n either (case i) $u_n \in C^+(\epsilon)$ or (case ii) $u_n \in Z(R_1, \epsilon)$. Now in case (i), $\delta(u_n, 1) = \theta$ by (3.4). This proves

(5.39) if case (ii) occurs only for a finite number of n. Let us assume now that case (ii) occurs infinitely often. From the argument given in case (i) it is clear that without loss of generality we may assume that case to occur for all n. Then by (3.4) and Definition 2.2

$$\delta(u_n, 1) = u(0, u_n) \qquad \text{(all } n\text{)} \tag{5.40}$$

where as usual $u(\tau, u_n)$ denotes the gradient line through u_n. Setting $u(0, u_n) = y_n$ we have to prove

$$\lim y_n = \theta. \tag{5.41}$$

Suppose (5.41) not to be true. Then there exists a positive η_0 such that

$$0 < \eta_0 \leqslant \| y_n \| < R_1, \tag{5.42}$$

for infinitely many n. Changing our notation we may assume (5.42) to be true for all n. Since the gradient line through u_n coincides with the one through y_n we conclude from Lemma 5.2 that the set S consisting of the segments $0 \leqslant \tau \leqslant \tau_n$ of the gradient lines through u_n has a positive distance from the stationary point of h, and (for suitable m and λ) the inequality

$$\| u(\tau, u_n) - u(\tau, u_m)\| \leqslant | \tau_n - \tau_m |/m + \| u_n - u_m \| e^{\lambda\epsilon} \tag{5.43}$$

holds which is derived in the same way as the corresponding part of inequality (5.21). (Note that the two agree for $\bar{u} = u_n$, $u_0 = u_m$.) Since $\tau_n = h(u_n)$ and since the u_n converge (see (5.38)) it follows from (5.43) that $y_n = u(0, u_n)$ converge. Let y_0 be the limit. Then $h(y_0) = 0$ since $h(y_n) = 0$; moreover $R_1 \geqslant \| y_0 \| \geqslant \eta_0$ by (5.42). It follows that $u(\tau_0, y_0) \in Z(R_1', \epsilon)$ for any $R_1' > R_1$. We will now arrive at a contradiction by showing that $u(\tau_0, y_0) = u_0$ which is in $C^+(\epsilon)$ by assumption: by Lemma 5.2,

$$\begin{aligned} u(\tau_0, y_0) &= \lim_{n\to\infty} u(\tau_0, y_n) \\ &= \lim u(\tau_n, y_n) + \lim(u(\tau_0, y_n) - u(\tau_n, y_n)). \end{aligned}$$

Here the second limit is zero since $\tau_n = h(u_n) \to \tau_0 = h(u_0)$. Moreover $u(\tau_n, y_n) = u_n$. Thus $u(\tau_0, y_0) = \lim u_n = u_0$ (see (5.38)). This contradiction proves (4.1).

This finishes the proof of (5.39). We now turn to the more general assertion that (5.38) implies (5.37). Let then σ be a given positive number. We have to exhibit a positive integer. n_0 such that

$$\| \delta(u_n, \alpha_n)\| < \sigma, \qquad \text{for} \quad n > n_0. \tag{5.44}$$

We choose n_0 in such a way that for $n > n_0$ the inequalities (5.45), (5.46), (5.47) below are satisfied.

$$\delta(u_n, 1) < \sigma/2. \tag{5.45}$$

This choice is possible because of the limit relation (5.39) just proved.

$$\| h(u_n) - h(u_0)\| < h(u_0)/2. \tag{5.46}$$

This choice is possible on account of (5.38) and on account of the fact (already mentioned in the line below (5.39)) that $h(u_0) > 0$.

$$1 > \alpha_n > 1 - 2\epsilon_2/3\tau_0 \qquad (\tau_0 = h(u_0)), \tag{5.47}$$

where ϵ_2 is a positive number such that

$$C(\sigma/2, \epsilon_2) \subset B(\sigma). \tag{5.48}$$

(Such ϵ_2 exists by Lemma 2.1).

In addition we require that for $n > n_0$

$$u_n \in C(R_1, \epsilon). \tag{5.49}$$

This is possible since u_0 is contained in the set $C(R_1, \epsilon)$ which is open (Lemma 2.1).

To show that with the above choice of n_0 the inequality (5.44) holds for $n > n_0$ we note first that by (5.46)

$$0 < h(u_0)/2 < h(u_n) < 3h(u_0)/2. \tag{5.50}$$

From (5.49) and (5.50) we see that

$$\text{either (i)} \quad u_n \in C^+(\epsilon), \qquad \text{or} \qquad \text{(ii)} \quad u_n \in Z(R_1, \epsilon). \tag{5.51}$$

In both case by (3.4)

$$\delta(u_n, \alpha_n) = u(\tau_n(1 - \alpha_n), u_n) \qquad (\tau_n = h(u_n)). \tag{5.52}$$

Now by (5.47) and (5.50)

$$\tau_n(1 - \alpha_n) < \tau_n\, 2\epsilon_2/3\tau_0 < \epsilon_2 .$$

It follows that (5.44) will be proved once it is shown that

$$\| u(\tau, u_n)\| < \sigma, \qquad \text{for} \quad 0 < \tau < \epsilon_2 . \tag{5.53}$$

Now in case i of (5.51): $u(\tau, u_n) \in C^+(\epsilon_2) \subset C(\sigma/2, \epsilon_2)$ for $0 < \tau < \epsilon_2$. By (5.48) this proves (5.53).

In case ii of (5.51) we see from (5.45) that $u(0, u_n) = (u_n, 1) \in B(\sigma/2)$ i.e., $\| u(0, u_n)\| < \sigma/2$. But this implies that $u(\tau, u_n) \in C(\sigma/2, \epsilon_2)$ for $0 < \tau < \epsilon_2$, and (5.53) follows now from (5.48).

Case B. $u_0 \in C^+(\epsilon) \cup C^-(\epsilon)$, $0 \leqslant \alpha_0 < 1$. Again it will be sufficient to consider the case that $u_0 \in C^+(\epsilon)$. It is clear from definition (3.4) of our deformation that we have to deal with gradient lines through u_0 and "neighboring points." Let k be a number for which

$$\alpha_0 < k < 1. \tag{5.54}$$

We consider the segment of the gradient line $u(\tau, u_0)$ through u_0 determined by the interval

$$\tau_0(k - \alpha_0) \leqslant \tau \leqslant (\tau_0 + \epsilon)/2. \tag{5.55}$$

(The motivation for the choice of this interval will become clear later on.) This compact segment does not contain the point θ. Consequently for τ in the interval (5.55) the inequality (5.13) holds for some positive $\mu < R$. We now define S_1 as the set obtained from (5.14) by taking the union over the τ in the interval (5.55). Let now m_1, λ_1, be two constants playing the same role for S_1 as m, λ for S. If then in the arguments following (5.14) we replace S by S_1, m by m_1, λ by λ_1 and the interval $|\tau| \leqslant \epsilon$ by the interval (5.55) we arrive at the following conclusion: if ρ is a constant such that (5.15) and (5.16) hold then (cf. (5.21))

$$\| u(\tau, \bar{u}) - u(\tau_0, u_0)\| \leqslant \tau - \tau_0/m_1 + \| u - u_0 \| e^{\lambda_1 \epsilon}, \tag{5.56}$$

for $\| u - u_0 \| < \rho$ and for τ in the interval (5.55).

In order to use (5.56) for the proof of the continuity of δ we subject $\bar{u}$ to the following additional restriction:

$$| \bar{\tau} - \tau_0 | = | h(\bar{u}) - h(u_0)| \leqslant \min \left\{ \frac{\tau_0(1-k)}{3(1-\alpha_0)}, \frac{\epsilon - \tau_0}{2} \right\},$$

and restrict α to the interval $| \alpha - \alpha_0 | < (1 - k)/3$. With these restrictions $\tau = \bar{\tau}(1 - \alpha)$ lies in the interval (5.55) as may be verified by an elementary computation.

From (3.1), from (5.56) (with $\tau = \bar{\tau}(1 - \alpha)$, and from the second part of Lemma 5.1 we obtain the inequality

$$\begin{aligned} &\| \delta(\bar{u}, \alpha) - \delta(u_0, \alpha_0)\| \\ &\quad = \| u(\bar{\tau}(1-\alpha), \bar{u}) - u(\tau_0(1-\alpha_0), u_0)\| \\ &\quad \leqslant \| u(\bar{\tau}(1-\alpha), \bar{u}) - u(\tau_0(1-\alpha_0), \bar{u})\| \\ &\qquad + \| u(\tau_0(1-\alpha_0), \bar{u}) - u(\tau_0(1-\alpha_0), u_0)\| \\ &\quad \leqslant | \bar{\tau}(1-\alpha) - \tau_0(1-\alpha_0 |/m_1 + | \bar{\tau} - \tau_0 |/m_1 + \| \bar{u} - u_0 \| e^{\lambda_1 \epsilon} \end{aligned} \tag{5.57}$$

which obviously implies the continuity of $\delta(\bar{u}, \alpha)$ at (u_0, α_0) since $\bar{\tau} = h(\bar{u})$ and $\tau_0 = h(u_0)$.

Case C. $u_0 \in Z(R_1, \epsilon)$, $0 \leqslant \alpha_0 \leqslant 1$. Here the assumptions of Lemma 5.4 are satisfied. If we choose ρ as indicated in the proof of that lemma (see (5.15), (5.16)) it follows immediately from (3.4), (5.21) and Lemma 5.1 that for $\| \bar{u} - u_0 \| < \rho$ the inequality holds which is obtained from (5.57) by replacing m_1, λ_1, by the numbers m, λ, respectively, defined in the proof of Lemma 5.4.

Case D. $u_0 = \theta$, $0 \leqslant \alpha_0 \leqslant 1$. Then $\delta(\theta, \alpha_0) = \theta$ for $0 \leqslant \alpha_0 \leqslant 1$. Therefore to given positive σ we have to exhibit a neighborhood N of θ such that $\| \delta(\bar{u}, \alpha)\| < \sigma$ for $\bar{u} \in N$ and $0 \leqslant \alpha \leqslant 1$. Let $0 < \sigma_1 < \sigma$, and let ϵ_1 be such that $C(\sigma_1, \epsilon_1) \subset B(\sigma)$. (Such ϵ_1 exists by Lemma 2.1). Using (3.4) it is then easily verified that $N = C(\sigma_1, \epsilon_1)$ satisfies our requirement.

Proof of the Continuity of the Deformation (3.20)

Case A. $u_0 \in Z(R_1)$, $0 \leqslant \alpha_0 < 1$. Let ν be a number such that

$$\alpha_0 < \nu < 1. \tag{5.58}$$

With the notation

$$t_i = \alpha_i/(1 - \alpha_i), \qquad i = 0,1 \tag{5.59}$$

we see from (3.20) that for $u_1 \in Z(R_1)$

$$\delta(u_1, \alpha_1) - \delta(u_0, \alpha_0) = u(t_1, u_1) - u(t_0, u_0), \tag{5.60}$$

and have to show that the left member tends to θ as $(u_1, \alpha_1) \to (u_0, \alpha_0)$.

We restrict α_1 and u_1 to neighborhoods of α_0 and u_0, respectively, given by

$$| \alpha_1 - \alpha_0 | < \nu - \alpha_0, \qquad \| u_1 - u_0 \| < \| u_0 \|/2. \tag{5.61}$$

For later use we note that then by (5.58) and (5.59)

$$0 < t_1 < \nu(1 - \nu)^{-1}, \qquad | t_1 - t_0 | < | \alpha_1 - \alpha_0 | (1 - \nu)^{-2}. \tag{5.62}$$

Since $\| u(t, u_1)\|$ is (by Lemma 3.2) decreasing with increasing t we see from (3.4), (5.61) and (5.62) that

$$\| u(t, u_1)\| \geqslant \| u(t_1, u_1)\| \geqslant \| u_1 \| e^{-t_1} \geqslant \frac{\| u_0 \|}{2} e^{-\nu_1}, \qquad \text{for } 0 \leqslant t \leqslant t_1,$$

where $\nu_1 = \nu(1-\nu)^{-1}$. This shows that the set S of all points $u(t, u_1)$ where t varies over the interval just indicated and where u_1 satisfies (5.61) is bounded away from the zeros of γ. Therefore there exists for S an upper bound M and a Lipschits constant for the right member of the differential equation (3.20). Application of Lemma 5.1 then yields the inequality

$$\begin{aligned}\| u(t_1, u_1) - u(t_0, u_0)\| &\leqslant \| u(t_1, u_1) - u(t_1, u_0)\| + \| u(t_1, u_0) - u(t_0, u_0)\| \\ &\leqslant \| u_1 - u_0 \| e^{\lambda|t_1 - t_0|} + M \mid t_1 - t_0 \mid .\end{aligned}$$

On account of (5.62) and (5.60) this inequality shows that $\delta(u_1, \alpha_1) \to \delta(u_0, \alpha_0)$ as $(u_1, \alpha_1) \to (u_0, \alpha_0)$.

Case B. $u_0 \in Z(R_1)$, $\alpha_0 = 1$

Case C. $u_0 = \theta$, $0 \leqslant \alpha_0 \leqslant 1$.

In both cases $\delta(u_0, \alpha_0) = \theta$ by (3.20). Therefore for both cases it will be sufficient to prove

$$\lim_{(u_1,\alpha_1)\to(u_0,\alpha_0)} \delta(u_1, \alpha_1) = \theta. \tag{5.63}$$

Now by (3.20) for $u_1 \neq \theta$

$$\delta(u_1, \alpha_1) = \begin{cases} u(t_1, u_1), & \text{for } 0 \leqslant \alpha_1 < 1, \\ 0, & \text{for } \quad \alpha_1 = 1. \end{cases}$$

Therefore by (3.7) for $u_1 \neq \theta$ and $0 \leqslant \alpha_1 \leqslant 1$

$$\| \delta(u_1, \alpha_1)\| \leqslant \| u_1 \| e^{-t_1/2}. \tag{5.64}$$

Now in case B we note that $t_1 = \alpha_1(1-\alpha_1)^{-1} \to \infty$ as $\alpha_1 \to \alpha_0 = 1$. Thus (5.63) follows from (5.64) since $\| u_1 \| < R_1$.

In case C, $u_1 \to u_0 = \theta$, and (5.63) follows again from (5.64) since $t_1 \geqslant 0$.

Proof of Lemma 4.3. Let L be a Lipschitz constant for γ in $B_\rho^- \cup \theta$. We will show that the assertion of the lemma is true with an α_1 satisfying

$$0 < \alpha_1 < \min\{L^{-1}, \rho e'(\gamma(\theta))^{-1}, \langle\gamma(\theta), e'\rangle \tfrac{1}{2}(eL\gamma(\theta))^{-1}\}. \tag{5.65}$$

Where e is the base of the natural logarithm and e' as defined in the lemma. Let $u_0 = \theta$. We want to set

$$u_n(\alpha) = -\int_0^\alpha \gamma(u_{n-1}(\beta))\, d\beta, \qquad n = 1, 2,\ldots; \quad 0 \leqslant \alpha < \alpha_1 . \tag{5.66}$$

To legitimize this we will prove that

$$u_n(\alpha) \in B_\rho^- \qquad \text{for} \quad n = 1, 2,\ldots \quad \text{and} \quad 0 < \alpha < \alpha_1 . \tag{5.67}$$

Moreover we will prove that for the n and α indicated in (5.66)

$$\| u_n(\alpha) - u_{n-1}(\alpha)\| < L^{n-1} \frac{\alpha^n}{n!} \| \gamma(\theta)\| . \tag{5.68}$$

We claim first that (5.68) implies (5.67). Indeed $u_n(\alpha) = \sum_{\nu=1}^{n} (u_\nu(\alpha) - u_{\nu-1}(\alpha)$ since $u_0 = \theta$. From this, (5.68) and (5.65) we see that

$$\| u_n(\alpha)\| < \alpha e^{\alpha L} \| \gamma(\theta)\| < \alpha_1 e \| \gamma(\theta)\| < \rho. \tag{5.69}$$

This implies that $u_n(\alpha) \in B_\rho$. To establish (5.67) it remains to prove that

$$\langle u_n(\alpha), e'\rangle < 0 \qquad \text{for} \quad n = 1, 2,... \quad \text{and} \quad 0 < \alpha < \alpha_1 . \tag{5.70}$$

Now from (5.66) (with $n = 1$) we see that

$$\begin{aligned} \langle u_n(\alpha), e^1\rangle &= \langle u_1(\alpha), e'\rangle + \langle u_n(\alpha) - u_1(\alpha), e^1\rangle \\ &= -\alpha\langle \gamma(\theta), e^1\rangle + \langle u_n(\alpha) - u_1(\alpha), e^1\rangle. \end{aligned} \tag{5.71}$$

But from (5.68) and (5.65) one concludes easily that

$$\| u_n(\alpha) - u_1(\alpha)\| < L\alpha^2 e^{L\alpha} \| \gamma(\theta)\| < L\alpha\alpha_1 e \| \gamma(\theta)\|,$$

and combining this with (5.71) and using (5.65) we see that

$$\langle u_n(\alpha), e^1\rangle = -\alpha\{\langle \gamma(\theta), e'\rangle - L\alpha_1 e \| \gamma(\theta)\| < -\alpha \tfrac{1}{2}\langle \gamma(\theta), e^1\rangle.$$

Here the right member is, by assumption (4.14) of our lemma, negative for positive α. Thus (5.70) is true, and (5.68) implies (5.67) as claimed.

Thus to establish (5.67) it remains to prove (5.68). That this inequality is true for $n = 1$ is obvious from (5.66) and the fact that $u_0 = \theta$. The validity for $n \geqslant 2$ follows by a familiar induction procedure. Likewise the convergence of the u_n to a solution of our problem (4.15) and the uniqueness of that solution follow by a well known procedure from (5.68).

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