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Schur Multipliers of Some Sporadic Simple Groups

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We determine the Schur multipliers of several of the sporadic simple groups, and in one case get an upper bound. The groups treated are those of Held, Suzuki, Fischer, and Conway.

1. INTRODUCTION

This paper is a continuation of the author's work in "Schur multipliers of finite simple groups of Lie type," [19]. Here, we determine the multipliers of several sporadic simple groups (those which are not known to belong to infinite families). We state our results as follows:

MAIN THEOREM. *The sporadic simple groups below have multipliers as stated*:

Group		Order	Multiplier
Held		$2^{10}3^3 5^2 7^3 17$	1
Suzuki		$2^{13}3^7 5^2 7.11.13$	$\mathbb{Z}_6$
Fischer's	$M(22)$	$2^{17}3^9 5^2 7.11.13$	$\mathbb{Z}_6$
	$M(23)$	$2^{18}3^{13}5^2 7.11.13.17.23$	1
	$M(24)'$	$2^{21}3^{16}5^2 7^3 11.13.17.23.29$	1 or $\mathbb{Z}_3$
Conway's	0.3	$2^{10}3^7 5^3 7.11.23$	1
	0.2	$2^{18}3^6 5^3 7.11.23$	1
	0.1	$2^{21}3^9 5^4 7^2 11.13.23$	$\mathbb{Z}_2$

Furthermore, $M(24)$, *which contains* $M(24)'$ *with index* 2, *has trivial multiplier.*

We remark that the multiplier of $M(24)'$ is very likely to be $\mathbb{Z}_3$ (the author had believed [19] until recently that the multiplier was trivial). The reason for this is the strong evidence that a simple group F (popularly called "the monster") of order $2^{46}3^{20}5^{9}7^{6}11^{2}13^{3}17.19.23.29.31.41.47.59.71$ exists [15, 18, 37], and its existence would imply that an element $h \in F$ of order 3 has the properties: $C_F(h) = C_F(h)'$, $C_F(h)/\langle h\rangle \cong M(24)'$ and $N_F(\langle h\rangle/\langle h\rangle \cong M(24)$. Proving directly that $M(24)'$ has multiplier of order 3 would be very difficult. If F is shown to exist, the ambiguity would be settled, of course.

Concerning other known simple groups, the multiplier situation is almost complete. For a general account of work on the Schur multipliers of finite simple groups, the reader is referred to the author's announcement [20]. The gaps in the tables of [20] are now filled, modulo the ambiguity concerning $M(24)'$. Since [20] was written, several new simple groups have appeared, and we present what is known about their multipliers.

Sporadic group	Order	Multiplier
Rudvalis [10, 29]	$2^{14}3^{3}5^{3}7.13.29$	$\mathbb{Z}_2$
O'Nan [27]	$2^{9}3^{4}5.7^{3}11.19.31$	$\mathbb{Z}_3$
F_1 [15, 18, 37]	$2^{46}3^{20}5^{9}7^{6}11^{2}13^{3}.17.19.23.29.31.41.47.59.71$	1
F_2 [13]	$2^{41}3^{13}5^{6}7^{2}11.13.17.19.23.31.47$	1 or $\mathbb{Z}_2$
F_3 [37]	$2^{15}3^{10}5^{3}7^{2}13.19.31$	?
F_5 [21]	$2^{14}3^{6}5^{6}7.11.19$	?

The multiplier of the Rudvalis group was settled by the combined work of A. Rudvalis, W. Feit, and R. Lyons; they also showed that the outer automorphism group is trivial. See [27] for the multiplier of the O'Nan group and see [18] for the above bounds on the multipliers of F_1 and F_2 (the groups F_i are defined to be the central factor groups of the centralizer in $F = F_1$ of certain elements of order $i = 1, 2, 3$, and 5). The existence question for the F_i has been settled affirmatively only for F_3 and F_5 as of this writing; both were handled with computer techniques by P. E. Smith of Cambridge. Also, the existence of F_1 would prove that a simple group satisfying all the properties listed for F_2 in [13] has multiplier of even order. A direct proof (independent of the existence question for F_1) that the multiplier of F_2 is $\mathbb{Z}_2$ has not yet been given.

Our main technique may be described as follows. Let G be one of the above groups, and let $\tilde{G}$ be a central extension of G. We study the possible extensions $\tilde{H}$ induced on various subgroups H of G to pinpoint information about $\tilde{G}$. Knowledge of the multipliers of other simple groups is very helpful in this regard, although it seems necessary to study nonsimple H (local subgroups, for example).

Most group theoretic notation used here is fairly standard; see [17] or [19]. Notation for groups of Lie type used here is that of [17, p. 491] and [5].

Other notation for classical groups is that found in [24]. Some notation special to this paper is the following:

$M(G)$	the multiplier of the group G,
$m(G)$	the order of $M(G)$,
$M_p(G)$	the Sylow p-subgroup of $M(G)$, p a prime,
$m_p(G)$	the order of $M_p(G)$,
$m_{p'}(G)$	$m(G)/m_p(G)$,
E_{p^n}	an elementary abelian group of order p^n, p a prime, $n \geqslant 1$.

Also, for group elements x, y, we define $x^y = y^{-1}xy$ and $[x, y] = x^{-1}y^{-1}xy$. The Assumed Results of [19] are referred to by (1) through (14) and are listed at the end of this article.

2. Held's Group

Let G be the simple group discussed by Held in [22]; $|G| = 2^{10}3^3 5^2 7^3 17$. We shall prove $m(G) = 1$. Trivially, $m_{17}(G) = 1$. There is an element $t \in G$ of order 3, central in a Sylow 3-subgroup such that $C_G(t) = C_G(t)'$, $C_G(t)/\langle t\rangle \cong A_7$ [37, page 275]. Since $m(A_7) = 6$, and $|C_G(t)|_3 = |G|_3$, Gaschütz' theorem implies $m_3(G) = 1$. Now, $G \supset S \cong \mathrm{Sp}(4, 4)$ (unpublished). Since $|S|_5 = |G|_5$ and $m(S) = 1$, it follows as above that $m_5(G) = 1$.

A Sylow 7-subgroup P of G is nonabelian of order 7^3, exponent 7. It is easy to see that a covering group Q of P must have $A \subseteq Q \cap Z(Q)$, $Q/A \cong P$, $A \cong Z_7 \times Z_7$, and that A is generated by $[a, b]$, $[a, c]$ where (denoting images of $Q \to P$ by $^-$) $\langle \bar{a} \rangle = Z(P)$ and $\{\bar{b}, \bar{c}\}$ is any set of generators for P. Checking centralizer orders for 7-elements in G, we find that $C_G(\bar{a}) \cong P \cdot Z_3$ and $C_G(s) \cong Z_7 \times L_2(7)$ or $Z_7 \times D_{14}$ for $|s| = 7$, s not conjugate in G to an element of $Z(P)$. Assume $\tilde{G}$ is an extension of G by $B \cong Z_7$, $B \subseteq \tilde{G}'$. Then $B \subseteq \tilde{P}'$ by a transfer lemma. By (10), $\tilde{P}$ is a quotient of some Q as above. Denote images of $Q \to \tilde{P}$ by $^\sim$. Say $[\tilde{a}, \tilde{b}] \neq 1$ in $\tilde{P}$. If $C_G(\bar{b}) \cong Z_7 \times L_2(7)$, choose $y, g \in L_2(7)$ with $|y| = 7$, $y^g = y^2$. Since $\bar{a} \in C_G(\bar{b})$, we may assume $y = \bar{a}\bar{b}^j$. Let $\tilde{y}$ be a preimage of y in $\tilde{P}$. Then, as $[\tilde{a}, \tilde{b}]$ is central, $[\tilde{a}, \tilde{b}] = [\tilde{a}, \tilde{b}]^g = [\tilde{y}, \tilde{b}]^g = [\tilde{y}^2, \tilde{b}] = [\tilde{y}, \tilde{b}]^2 = [\tilde{a}, \tilde{b}]^2$, which implies $[\tilde{a}, \tilde{b}] = 1$. Similarly, if $C_G(\bar{b}) \cong Z_7 \times D_{14}$, we get $[\tilde{a}, \tilde{b}] = [\tilde{a}, \tilde{b}]^{-1}$, implying $[\tilde{a}, \tilde{b}] = 1$. So the image of $A \subseteq Q$ in $\tilde{P}$ is trivial. Thus, $\tilde{G}$ does not exist and so $m_7(G) = 1$.

Proving $m_2(G) = 1$ will finish the proof. The following information comes from Held's paper [22]. G has two conjugacy classes of involutions, represented by z and i; z is central in a Sylow 2-subgroup, i is not.

$$C_G(z) \cong (D_8 \circ D_8 \circ D_8) \cdot L_2(7),$$
$$[C_G(i) : C_G(i)'] = 2,\ C_G(i)' = C_G(i)'',\ Z(C_G(i)') \cong Z_2 \times Z_2,$$
$$C_G(i)'/Z(C_G(i)') \cong L_3(4),\ Z(C_G(i)) = \langle i \rangle.$$

G has a subgroup $S \simeq Sp(4, 4) = C_2(4)$. Let X be a one-parameter subgroup corresponding to a root of maximal height. Then X is a four-group lying in the center of a Sylow 2-subgroup of S and $C = C_S(X) \simeq O_2(C) \cdot L_2(4)$, $N_S(X) = C \cdot Z_3$, $C = C'$, $O_2(C)$ is elementary abelian of order 2^6 and $L_2(4)$ acts "naturally" on $O_2(C)/X$. Since $5 \,\|\, |C|$, $5 \nmid |C_G(z)|$, $X^\#$ consists of G-conjugates of i. We claim $C_G(X) = C_G(x)'$, for $x \in X^\#$. $|C_S(X)|_2 = 2^8 = |C_G(x)'|_2$ implies that the kernel of the map $C_G(x)' \to L_3(4)$ is contained in $C_S(X)$. Since X is the only normal four-group in $C_S(x)$, X must be this kernel, i.e., $X = Z(C_G(x)')$. The structure of $C_G(x)$ now establishes the claim. Since $[N_S(X) : C_S(X)] = 3$, we must have $N_G(X)/N_G(X)'' \simeq \Sigma_3$, $N_G(X)'' = C_G(X) = C_G(x)'$. Under conjugation, $N_G(X)$ is transitive on $X^\#$.

Let $\tilde{G}$ denote a central extension of G by $\langle \alpha \rangle \simeq A = Z_2$. Since $Z_4 \times Z_4 \simeq M_2(L_3(4))$ and any outer automorphism θ of order 3 acts fix point freely on $M_2(L_3(4))$ (and on any quotient by a θ-invariant subgroup), the action of $N_G(X)$ on X implies $A \not\subseteq \widetilde{(C_G(X))'}$. Thus $\widetilde{C_G(X)} = C_0 \times A$ where $C_0 \simeq C_G(X)$. Each factor is invariant under $N_G(X)$ as $C_0 = (C_0 \times A)'$ and $A = Z(\widetilde{(N_G(X))'})$. Also, if $\tilde{x} \in x \in X^\#$, $\tilde{x}^2 = 1$ (i.e., $\tilde{\imath}^2 = 1$ for $\tilde{\imath} \in i$).

For $\tilde{z} \in z$, we show that $\tilde{z}^2 = 1$. Since $\langle z \rangle = Z(Q)$, Q a quaternion subgroup of $O_2(C_G(z)) = D_8 \circ D_8 \circ D_8$ [22, p. 204], $m(Q) = 1$ implies $\tilde{z}^2 = 1$. In particular α has no square root in $\tilde{G}$.

Now, $\widetilde{(N_G(X))}$ splits over A since $[N_G(X)' : C_G(X)] = 3$. Any $t \in N_G(X) \backslash N_G(X)'$ effects an outer automorphism of $C_G(X)/X \simeq L_3(4)$. Since $\mathrm{Out}(L_3(4)) \simeq Z_2 \times \Sigma_3$ [5], and is generated by the automorphism classes of the diagonal automorphism, the graph automorphism, and the field automorphism, there is in $\mathrm{Aut}(L_3(4))$ a complement to the group of inner automorphisms. Hence, we may assume t has period two on $C_G(X)/X$. Then t^2 induces a central automorphism of $C_G(X)$, a perfect group. Thus t^2 is trivial on $C_G(X)$, forcing $t^2 = 1$ or $t^2 = x \in X^\#$. Pick $\tilde{t} \in t$. If $t^2 = 1$, then $\tilde{t}^2 = 1$ because α has no square root in G. In this case $\widetilde{N_G(X)}$ splits, because $\tilde{t}$ with a complement to A in $\widetilde{(N_G(X)')}$ generates a complement to A. The other case is $t^2 = x$. Choose $s \in X - \langle x \rangle$. Then $s^t = sx$. So, $(ts)^2 = t^2 s^t s = xsxs = 1$. Replacing t by $ts \in N_G(X)$, the same argument yields $\widetilde{N_G(X)}$ split over A.

Checking centralizer orders, $2 \mid C_G(i)|_2 = |C_G(z)|_2 = |G|_2$. Since $\widetilde{N_G(X)}$ splits, the induced extension of some maximal subgroup of a Sylow 2-subgroup splits over A.

We now look at the induced extension of $C_G(z)$, which contains a Sylow 2-subgroup of G, and try to prove $\widetilde{C_G(z)}$ splits over A. Choose $L \subset C_G(z)$,

$L \simeq L_2(7)$. Set $R = O_2(C_G(z))$. As $\tilde{z}^2 = 1$, $\widetilde{(R')}$ is elementary abelian. $\widetilde{(R')} \supseteq \widetilde{Z(C_G(z))}$ obviously, but they are equal since $C_G(z)$ is perfect. So, L acts on $R_0 = \tilde{R}/\langle\tilde{z}\rangle$, where we choose $\tilde{z} \in (\tilde{R})' \cap z$ (conceivably, there are two choices for $\tilde{z}$). Let $A_0 = \langle A, \tilde{z}\rangle/\langle\tilde{z}\rangle$. Since A is central in $\tilde{G}$, L preserves the bilinear form $R_0/A_0 \times R_0/A_0 \to A_0$ given by commutation. We claim this form is identically trivial, i.e., that R_0 is abelian. Since the induced extension of a maximal subgroup of a Sylow 2-subgroup of G splits over A, the same is true for a maximal subgroup of R. This means that R_0/A_0 has a subspace of codimension 1 isotropic under the form. Hence, the radical of the form has codimension no more than 2. Since R_0/A_0 has the same irreducible constituents under L as R/R' does (dimensions 3 and 3 for R/R'), the radical must be the whole space R_0/A_0, as L leaves the radical invariant. This means R_0 is abelian. Furthermore, R_0 is elementary abelian. If not, L leaves invariant the kernel of the squaring endomorphism of R_0, which contains A_0. This would force R/R' to have a 5-dimensional L-invariant subspace, contradiction. Hence, $\tilde{R}$ splits over A because A_0 is a direct summand of R_0.

Let K be a subgroup of L isomorphic to Σ_4. K contains a Sylow 2-subgroup of L and RK contains one of G. We aim to show $\widetilde{RK}$ splits over A.

Let X_1, X_2 be the indecomposable constituents of L on R/R' as described in [22, Sect. 1]. Set $X_i^*/R' = X_i$. X_i^* is elementary abelian and so is $\widetilde{X_i^*}$, $i = 1, 2$. As before, R_0 denotes $\tilde{R}/\langle\tilde{z}\rangle$ where $\langle\tilde{z}\rangle = (\tilde{R})'$. Set $Y_i = \widetilde{X_i^*}/\langle\tilde{z}\rangle$. Y_i is elementary abelian, $\langle Y_1, Y_2\rangle = R_0$, $Y_1 \cap Y_2 = B = A\langle\tilde{z}\rangle/\langle\tilde{z}\rangle \simeq A$, $|Y_i| = 2^4$, $i = 1, 2$.

We claim it suffices to show that $Y_i = Y_{i0} \oplus B$ as $\tilde{K}$-modules. For then, $Y/\langle\tilde{z}\rangle = \langle Y_{10}, Y_{20}\rangle$ is invariant under K and disjoint from B and $\widetilde{KR}/Y$ is a central extension of K by $B_0 \simeq AY/Y = A$. Now, $B_0 \not\subseteq (\widetilde{KR}/Y)'$, or else some involution in $K = \Sigma_4$ is represented in KR/Y by an element of order four (for example, the induced extension of $K' \simeq A_4$ would be isomorphic to $SL(2, 3)$, whatever the covering of K), a contradiction. Taking preimages in $\widetilde{KR}$, this gives $A \not\subseteq (\widetilde{KR})'$. By a transfer lemma, $A \not\subseteq \tilde{G}'$, and so $\tilde{G}$ splits, implying $m_2(G) = 1$.

For convenience, we consider L (rather than $\tilde{L}$) as a group of operators on Y_i. There are two conjugacy classes of Σ_4 in $L \simeq SL(3, 2)$ and for a fixed Y_i we choose K to be the stabilizer of a nonzero vector of X_i. Let V be the normal four-group of K and set $U = [Y_i, V]$. U is K-invariant. If s is an element of order 3 in K, $Y_i = F_1 \oplus F_2$, $F_1 = C_{Y_i}(s)$, $F_2 = [Y_i, s]$, $|F_1| = |F_2| = 4$. By the structure of L and the point-stabilizer K, $|UB/B| = 2$.

We prove $|U| = 2$. Since s acts trivially on B and UB/B, $UB = F_1$. Suppose $|U| = 4$, i.e., $U \supset B$. We claim that $\beta = [y, v]$ where $\langle\beta\rangle = B$, $y \in Y_i$, $v \in V$. If not, $|U| = 4$ implies that there are $y_1, y_2 \in Y_i$, $v_1, v_2 \in V$

with $[y_1, v_1] = u$, $[y_2, v_2] = u\beta$, for $u \in U \backslash B$. Conjugating the second by a power of s, we may assume $v_1 = v_2$. Then $[y_1 y_2, v_1] = \beta$. Recall that $\widetilde{X_i}^*$ is elementary abelian. Choosing $\tilde{y} \in y_1 y_2$, $\tilde{v}_1 \in v_1$, the above implies that, in $\tilde{G}$, $(\tilde{y}\tilde{v}_1)^2 = [\tilde{y}, \tilde{v}_1] = \alpha\tilde{z}^k$, $k = 0, 1$. Now, in $C_G(z)$, L normalizes a complement W_i to $\langle z \rangle$ in X_i^* (see [22, p. 259]). Thus, $[\tilde{W}_i, L] \subseteq \tilde{W}_i$ which intersects $\widetilde{\langle z \rangle}$ in $\langle \alpha \rangle$. This forces $k = 0$ and the equation reads $(\tilde{y}\tilde{v}_1)^2 = \alpha$, a contradiction to α having no square root.

So, $|U| = 2$. Setting $W = [Y_i, s]$, we have $|\langle W, U \rangle| = 8$ and the definitions of U and W imply that $Y_{i0} = \langle W, U \rangle$ is K-invariant. This gives the required decomposition of Y_i. Y_{i0} is L-invariant since it is invariant under K (of odd index in L), by Gaschütz' theorem. So each Y_{i0}, $i = 1, 2$, is K-invariant for any $K \subset L$, $K \cong \Sigma_4$ (even though K is not a vector stabilizer for both $X_1^{\#}$ and $X_2^{\#}$).

The proof of $m(G) = 1$ is complete.

3. Suzuki's Group

Let G be the simple group constructed by Suzuki [41]. We shall prove $m(G) = 6$. By Lindsey [26], it is enough to prove $m(G) \mid 6$. Now, $|G| = 2^{13}3^7 5^2.7.11.13$ and $G \supset G_1 \cong G_2(4)$, simple of order $2^{12}3^3 5^2.7.13$. Since we know $m(G_1) = 2$ (see [19, 20]), we need only determine $m_p(G)$ for $p = 2, 3$ by Gaschütz' theorem and the cyclicity of a Sylow 11-subgroup of G.

Let U be the standard unipotent (Sylow 2-) subgroup of G_1, and let H be the standard Cartan subgroup. If $U \subseteq T$, an S_2 of G, then $[T : U] = 2$ and $U \lhd T$. If $N = N_G(U)$, then $N \supseteq \langle H, T \rangle$, $H \cong Z_3 \times Z_3$. Let $C \subseteq N$ be the subgroup inducing trivial automorphisms on U/U'. $|N/C|$ is n or $2n$, n odd. In either case, N/C has a normal 2-complement and H maps isomorphically into N/C by Theorem 1.4 of [17], since H is faithful on U and $(|H|, 2) = 1$.

Suppose $\tilde{G}$ is a central extension of G by $A \cong Z_2$ such that $\tilde{G}_1$ splits. Then $\tilde{U} \cong U \times A$. Furthermore, by the structure of a Borel subgroup in $G_2(4) \cong G_1$, $U/U' = [U/U', H]$. So, we may write $\tilde{U} = U_0 \times A$, where each factor is H-invariant, $U_0 \cong U$ as H-groups. Note $U_0' = \tilde{U}'$ is characteristic in $\tilde{U}$. In particular, $U_0/U_0' = [U_0/U_0', \tilde{H}]$. Take $\tilde{t} \in \tilde{T} \backslash \tilde{U}$. Its image in $\tilde{N}/\tilde{C} \cong N/C$ normalizes $[U_0/U_0', L] = X$ where $L = O_{2'}(N/C)$. But $U_0/U_0' \supseteq X \supseteq [U_0/U_0', H] = U_0/U_0'$. So equality holds, and $\tilde{t}$ normalizes U_0. Since $\tilde{T}$ is a Sylow 2-subgroup of $\tilde{G}$, and $U_0 \lhd \tilde{T}$, $[\tilde{T} : U_0] = 4$, we have $\tilde{T}/U_0$ abelian with $A \not\subseteq \tilde{T}'$. By a transfer lemma (2), $A \not\subseteq \tilde{G}'$, and so $\tilde{G}$ splits.

The conclusion is that if $\tilde{G}$ is a perfect central extension of G by a 2-group A, then $A \subseteq \tilde{G}_1'$. Now, $m(G_1) = 2$ implies $m_2(G) \mid 2$.

Lindsey [26] discusses a group G_0, $G_0 = G_0'$, $Z(G_0) \simeq Z_6$, $G_0/Z(G_0) \simeq G$. These results, together with [25], imply that there is a subgroup $S \subset G_0$, $S = S'$, $Z(S) \simeq Z_3 \times Z_3$, $S/Z(S) \simeq U_4(3) \simeq {}^2A_3(3)$. Since $m(U_4(3)) = 2^2 3^2$ (see [19, 20]) and S contains a Sylow 3-subgroup of G_0, $M_3(G_0) = 1$ by Gaschütz' theorem. Thus, $M_3(G) \mid 3$.

All this gives $m(G) \mid 6$. But, G_0 must be a covering group of G, exhibiting that $m(G) = 6$.

4. Fischer's Group $M(22)$

Let $G = M(22)$. Fischer [14] has shown that $2 \mid m(G)$, and [16] implies $3 \mid m(G)$, as Rudvalis has observed. We shall prove that $m(G) \mid 6$, and conclude $m(G) = 6$.

$|G| = 2^{17}3^9 5^2 7.11.13$. So, $m_p(G) = 1$ for $p = 7, 11, 13$. G contains a subgroup $S \simeq B_3(3)$. Since $m(B_3(3)) = 6$ and $|S|_3 = |G|_3 = 3^9$, we have $m_3(G) \mid 3$.

According to the character table of $M(22)$ [23], a Sylow 5-subgroup P is elementary abelian and all elements of order 5 in G are conjugate. So, $N(P)$ is transitive on $P^{\#}$. If $N(P)$ effects a nonspecial transformation on the vector space P, we have $m_5(G) = 1$, by (7). Assume otherwise. For $x \in P^{\#}$, $|C_G(x)| = 2^3.3.5^2$, and if $t \in C_G(x)$, $|t| = 2$, then t commutes with no element of order 5 in $C_G(x)$ outside $\langle x \rangle$ by [23]. If $C_G(x)$ does not have a normal 5-complement, we get a nonspecial transformation, and we are done as above. So, assume $C_G(x) = O_{5'}(C_G(x)) \cdot P$. But then P normalizes a Sylow 2-subgroup (of order 2^3) in $C_G(x)$, and must centralize it, contradiction. Therefore, $m_5(G) = 1$, and $m_{2'}(G) = 3$.

G has an involution i wih $C_G(i)/\langle i \rangle \simeq U_6(2) \simeq {}^2A_5(2)$ and $C_G(i) = C_G(i)'$ [14]. To show $m_2(G) \mid 2$ we shall prove that if $\tilde{G}$ is a central extension of G by $A \simeq Z_2$ and $\widetilde{C_G(i)}$ is split, then $\tilde{G}$ is split. The result then follows from $M_2(U_6(2)) \simeq Z_2 \times Z_2$.

$|C_G(i)| = 2\,|U_6(2)| = 2^{16}3^6 5.7.11$. $\widetilde{C_G(i)} = C_0 \times A$ by assumption. Let V be a Sylow 2-subgroup of $C_0 \times A$, and let T be a Sylow 2-subgroup of G containing V; $[T : V] = 2$, $V \lhd T$. If u is an involution central in a Sylow 2-subgroup of $K \simeq U_6(2)$, then $C = C_K(u) \simeq O_2(C) \cdot U_4(2)$, $C' = C$, $|O_2(C)| = 2^9$, and $O_2(C)$ is extra-special with center $\langle u \rangle$. If $\hat{K} = C_G(i) \simeq C_0$ is a perfect central extension of K by $\langle i \rangle$, then $\langle \hat{u}, i \rangle = Z(\hat{C})$, as C is perfect and it contains a Sylow 2-subgroup W of K; also $Z(\hat{W}) = Z(\hat{C})$ (see the discussion of $U_6(2)$ in [19]).

We may take $\hat{W} = C_0 \cap V$. $Z(V) = Z(\hat{W}) \times A$ is normal in T. Since i is not central in any Sylow 2-subgroup of G, $i^t \neq i$ for $t \in T \setminus V$. But t normalizes $C_G(Z(V)) = \hat{C} \times A$ and $\hat{C} = (\hat{C} \times A)'$. So we may assume t normalizes $\hat{W}$, by the Frattini argument. Now $T/\hat{W}$ has order four and is abelian. $A \cap \hat{W} = 1$ implies $A \not\subseteq T'$. So, $A \not\subseteq \tilde{G}'$ by the transfer lemma (2), and we are done.

5. Fischer's Group $M(23)$

Let $G = M(23)$, $|G| = 2^{18}3^{13}5^27.11.13.17.23$. G contains K, a perfect extension of $M(22)$ by Z_2. Since $m(M(22))$ is known and $|K|_2 = |G|_2$, $|K|_5 = |G|_5$, we get $m_p(G) = 1$, for $p = 2, 5$. Trivially $m(G)_p = 1$ for $p = 7, 11, 13, 17, 23$. By 18.3.4 of [14], G contains a subgroup S, $S/S'' = \Sigma_3$, $S'' = D_4(3)$, a simple group. Since $m_3(D_4(3)) = 1$, an extension $\tilde{G}$ of G by $A \simeq Z_3$ must yield $\widetilde{(S'')}$ split. As $|S/S''|_3 = 3$, $A \not\subseteq (\tilde{S})'$. As $|S|_3 = |G|_3$, this implies $A \not\subseteq \tilde{G}'$. So, $\tilde{G}$ splits and $m_3(G) = 1$. Thus, $m(G) = 1$.

6. Fischer's Group $M(24)'$

Let $G = M(24)$, $G_0 = G'$. $|G| = 2^{22}3^{16}5^27^311.13.17.23.29$, $[G : G_0] = 2$, G_0 simple. We let D denote the conjugacy class of 3-transpositions in G. Trivially, $m_p(G_0) = 1$, $p \geqslant 11$. Since $G_0 \supset M(23)$, $|G_0|_5 = |M(23)|_5$, $m(M(23)) = 1$, we have $m_5(G_0) = 1$. Fischer states (personal communication) G contains a subgroup X isomorphic to the holomorph of $Z_7 \times Z_7$. A Sylow 7-subgroup P of $G_0 \cap X$ is then normalized by $h \in C(P')$, $|h| = 3$, and h is fixed-point free on P/P'. We now get $m_7(G_0) = 1$ by an argument similar to that for $m_7(\text{Held}) = 1$.

Lemma 6.1. *Say $e = d_1d_2$, $d_1, d_2 \in D$, $d_1d_2 = d_2d_1$. Then if $e = d_3d_4$, $d_3, d_4 \in D$, we have $\{d_1, d_2\} = \{d_3, d_4\}$. Consequently, $C(d_1) \cap C(d_2)$ has index 2 in $C(e)$. Also $e \in (C(d_1) \cap C(d_2))'$ and $C(d_1) \cap C(d_2)/\langle e\rangle \simeq M(22)$.*

Proof. See Sections 17, 18, 19 of [14].

Set $F = \langle D_d\rangle = \langle d\rangle \times F'$, $F' \simeq M(23)$. Also set $K = C_G(e)$, where $e = dd'$, $d' \in D_d$. Note that $K \cap F'$ is a perfect central extension of $U_6(2)$ by a four-group, and that $e \in Z(K \cap F')$.

Lemma 6.2. *$m_2(G) = 1$ and $m_2(G_0) = 1$.*

Proof. Let $\tilde{G}$ be an extension of G by $\langle\alpha\rangle = A \simeq Z_2$. We show $A \not\subseteq \tilde{G}'$. In Fischer's notation, G contains a subgroup L, elementary abelian of order 2^{12}, and $N/L \simeq M_{24}$, the Mathieu group, where $N = N_G(L)$. Furthermore, L has a subgroup L_0 of index 2 consisting of elements of L which are products of an even number of elements from $G \cap L$. $L_0 \lhd N$, $d \notin L_0$ for $d \in D \cap L$, and $d \notin G'$. In the notation of the previous paragraph, $A \not\subseteq \tilde{F}'$ as $m(F) = 1$. For d_i, $d_j \in D \cap L$, $d_i \neq d_j$, select coset representatives $\widetilde{e_{ij}}$ for $e_{ij} = d_i d_j$ by the rule $\{\widetilde{e_{ij}}\} = e_{ij} \cap \tilde{F}'$. Let $K_{ij} = C_G(e_{ij})$. Since $\langle e_{ij} \rangle = Z(K'_{ij})$ and $m_2(K_{ij}) = 1$, we see that $\widetilde{e_{ij}}$ is not $\tilde{G}$-conjugate to $\widetilde{e_{ij}}\alpha$ and that $e_{ij} \cap \tilde{F}' = e_{ij} \cap \widetilde{K_{ij''}}$.

Since the set $\{d_i d_j \mid d_i \neq d_j \text{ in } D \cap L\}$ generates L_0, the associated $\widetilde{e_{ij}}$ in $\tilde{L}$ generate a subgroup L_1 of $\tilde{L}$ normalized by $\tilde{N}$. We claim $A \not\subseteq L_1$. If $\alpha \in L_1$, then α is some product of the $\widetilde{e_{ij}}$'s. But each $\widetilde{e_{ij}} \in \tilde{F}'$ and $A \not\subseteq \tilde{F}'$. So, $L_1 \cap A = 1$. Then $\tilde{N}/L_1$ is a central extension of $Z_2 \times M_{24}$ by A. Since $m(M_{24}) = 1$, $A \not\subseteq \tilde{N}'$. This implies $A \not\subseteq \tilde{G}'$ as $|N|_2 = |G|_2$. Thus, $m_2(G) = 1$. Since G is a split extension $G_0\langle d \rangle$, $m_2(G_0) = 1$ by (9).

The following arguments show that the 3-part of the multiplier of $M(24)'$ has order at most 3. We use Fischer's paper [14] and the following information about a subgroup of $M(24)$ (due to private correspondence with B. Fischer).

There is a D-subgroup H with $V = O_3(H)$ elementary of order 3^7, $H/V \cong PO(7, 3)$. Set $E = D \cap H$. Then, there is a nondegenerate, symmetric, bilinear form f on V, preserved by H/V, so that members of E act as reflections

$$x \mapsto x + \pi f(x, a)a$$

where $f(a, a) = \pi$ is either 1 or -1. By replacing f with $-f$, if necessary, we may assume that f has discriminant 1. Since $E \cap H' = \varnothing$, the structure of $PO(7, 3)$ implies that $\pi = -1$. Also, for $d \in E$, $\langle E_d \rangle \cong O^+(6, 3) = O^{+-}(6, 3)$, in Fischer's notation.

Lemma 6.3. *Let V be the above module for $PO(7, 3)$. Then,*

$$H^1(PO(7, 3), V) = 0 \qquad \text{and} \qquad H^1(\Omega(7, 3), V) = 0.$$

Proof. Choose $u \in V$ with $f(u, u) = 1$. Let J be the subgroup of $\Omega(7, 3)$ stabilizing $\langle u \rangle$. Then, J stabilizes $\langle u \rangle^\perp$ and the form f restricted to $\langle u \rangle^\perp$ has discriminant 1, whence J is isomorphic to a subgroup of index 2 in $GO^-(6, 3)$. Also, $Z(J) \subset J'$, $|Z(J)| = 2$, $|J : J'| = 2$, and elements of $J \backslash J'$ invert $\langle u \rangle$. We have $|J| = 2^9 3^6 5.7$, and so J contains a Sylow 2-subgroup of $\Omega(7, 3)$.

Let w generate $Z(J)$. Then, w has six eigenvalues -1, one 1. Let $w' = w^g$ be a conjugate of w in $\Omega(7, 3)$ which commutes with w and $w' \neq w$. Then $w' \in J$. Since w and w' have the same eigenvalues, w' inverts $C_V(w) = \langle u \rangle$.

Set $\mathscr{L} = \{\langle w, w' \rangle, J, J^g\}$. By above remarks, $H^0(L, V) = 0$, for each $L \in \mathscr{L}$. Since $\langle w, w' \rangle$ is a 2-group, $H^1(\langle w, w' \rangle, V) = 0$. Write $V = \langle a \rangle \oplus W$ as a J-module, where $W = [V, w]$. The argument of [24, p. 124] shows that $H^1(J, W) = 0$. Since $|J : J'| = 2$ is prime to 3, $H^1(J, \langle u \rangle)$ is isomorphic to a subgroup of $H^1(J', \langle u \rangle)$, by [3]. Since $\langle u \rangle$ is a trivial J'-module, the latter is $\mathrm{Hom}(J', \langle u\rangle) = 0$ (see [4]). So, $H^1(J, \langle u\rangle) = 0$. The decomposition of V as a direct sum now implies that $H^1(J, V) = 0$. Similarly, $H^1(J^g, V) = 0$.

Since the members of $\mathscr{L}$ generate $\Omega(7, 3)$ and $H^i(L, V) = 0$, for each $L \in \mathscr{L}$, $i = 0, 1$, the hypotheses of the "Vanishing Theorem" of Alperin and Gorenstein [1] are satisfied. We conclude that $H^1(\Omega(7, 3), V) = 0$. Since $|PO(7, 3) : \Omega(7, 3)| = 2$ is prime to 3, we also get $H^1(PO(7, 3), V) = 0$. The lemma is proven.

Now we are ready to show that $m_3(G) \leqslant 3$. Let $G_0 = M(24)'$ and let $\tilde{G}_0$ be a central extension of G_0 by an abelian 3-group A. For any subgroup G_1 of G_0, we let $\tilde{G}_1$ denote the extension of G_1 induced by $\tilde{G}_0$.

Set $K = H' = H \cap G_0$. Since K acts irreducibly on V and since $|V|$ is an odd power of 3, V must be elementary abelian. By a well-known isomorphism $\mathrm{Ext}_{Z_3K}(Z_3, V) \simeq H^1(K, V)$ (see [4]), Lemma 6.3 implies that we may write $V = V^* \oplus A$, as $\tilde{K}$-modules, with $V^* = [\tilde{V}, K]$. Now, $\tilde{K}/V^*$ is a central extension of K/V by $A/A \cap V^* \simeq A$. Since $m_3(\Omega(7, 3)) = 3$, this extension is possibly nonsplit. If it were split, then $\tilde{G}_0$ would split over A, since $(|G_0 : K|, 3) = 1$. But, if $\tilde{G}_0 = \tilde{G}_0'$, then $m_3(\Omega(7, 3)) = 3$ and a transfer lemma (2) implies $|A| \leqslant 3$, i.e., $m_3(G_0) \leqslant 3$, as required.

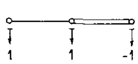
It remains only to show that $m_3(M(24)) = 1$. Let $G = M(24)$. By (11), $m(G) \mid 3$. Assume $m(G) = 3$ and let G be a perfect central extension of G by $A \simeq \mathbb{Z}_3$. Then $\tilde{H}$ (the induced extension on H) is nonsplit, since $(|G : H|, 3) = 1$. The analysis in the last paragraph shows that $\tilde{V} = V^* \oplus A$, as H-modules, and that $\tilde{K}/V^*$ is a perfect central extension of $K/V \simeq \Omega(7, 3)$ by $\tilde{V}/V^* \simeq \mathbb{Z}_3$. But $H/V \simeq PO(7, 3) \simeq \mathrm{Aut}(\Omega(7, 3))$. We shall have a contradiction if we show that an outer diagonal automorphism of $B_3(3)$ inverts $M_3(B_3(3))$.

We need some detailed information about B^*, a perfect central extension of $B_3(3)$ by $A^* \simeq Z_3$. Regard $B_3(3)$ as the quotient B^*/A^*. We may choose a system of representatives $y_r(t) \in B^*$ for $x_r(t) \in B_3(3)$, all $r \in \Sigma$, $t \in \mathbb{F}_3$, so that each $y_r(t)$ has order 3 and all Chevalley commutator relations holding between $x_r(t)$, $x_s(u)$, $r \neq \pm s$, hold between the corresponding $y_r(t)$, $y_s(u)$, except for the following one:

Whenever r, s are roots of unequal length, with r orthogonal to s, and t, u are nonzero, then $[x_r(t), x_s(u)] = 1$, while $1 \neq [y_r(t), y_s(u)] \in A^*$.

The multiplier of $B_3(3)$ has been determined by Steinberg in unpublished work.

We now leave to the reader the verification of the claim that the function



(Dynkin diagram for B_3)

from fundamental roots to $\mathbb{F}_3^\times$ extends to an outer diagonal automorphism which inverts $M_3(B_3(3))$. This completes the proof that $m(M(24)) = 1$ and $m(M(24)') = 1$ or 3.

7. Conway's Group .3

We shall prove $m(.3) = 1$. Our sources of information about .3 are [8] and [12].

It is easy to obtain $m_p(.3) = 1$ for $p \neq 3$. $|\,.3\,| = 2^{10}3^7 5^3 7.11.23$. Since the Sylow 7-, 11-, and 23-subgroups are cyclic, $m_p(.3) = 1$ for these primes. If z is an involution in .3 central in a Sylow 2-subgroup, $C(z)$ is a perfect extension of Sp(6, 2) by $\langle z \rangle$. Hence, any central extension of .3 by a 2-group splits off $C(z)$, as this group is a covering group for $C(z)/\langle z \rangle$. By Gaschütz' theorem, $m_2(.3) = 1$. We may show $m_5(.3) = 1$ by a direct argument, or by noting that .3 contains .332, which is isomorphic to the Higman–Sims group, and that $m_5(.332) = 1$ [40], $|\,.3\,|_5 = 5^3 = |\,.322\,|_5$ imply $m_5(.3) = 1$ as above.

Showing $m_3(.3) = 1$ will finish the proof. L. Finkelstein points out that .3 contains a subgroup X containing .333 with $M = O_3(X)$ elementary abelian of order 3^5, $X/M \simeq Z_2 \times M_{11}$. If t is an involution in X mapping onto a generator for Z_2, we claim t inverts M. By [12], t does not centralize M, since $3^5 \nmid |C(t)|$. Write $M = M_1 \times M_2$, where M_1 are the elements inverted by t, M_2 are the elements centralized by t (use Fitting's theorem). M_1 and M_2 are normal in X since the action of t commutes with the automorphisms of M induced by X. If $M_2 \neq 1$, M_1 and M_2 are centralized by the action of M_{11} (order 8.9.10.11) because M_{11} is simple and $11 \nmid 3^k - 1$ for $0 < k < 5$. This leads to a contradiction which proves the claim. Since $C_X(t) \cap M = 1$, the extension splits. Write $C_X(t) = \langle t \rangle \times M_{11}$.

Let $.\tilde{3}$ be a central extension of .3 by $\langle \alpha \rangle \simeq Z_3$. Since $|M|$ is an odd power of 3 and $C_X(t)$ acts irreducibly on M, $\tilde{M}$ is elementary abelian. Let $M_0 = [\tilde{M}, t]$. Then $\tilde{M} = M_0 \times \langle \alpha \rangle$, by Fitting's theorem for the action of $\langle t \rangle$ on M. The factors are M_{11}-invariant, hence normal in $\tilde{X}$. Then $\tilde{X}/M_0$ is a central extension of $\langle t \rangle \times M_{11}$ by $\langle \alpha \rangle$. Since the former group has trivial multiplier and M_{11} is perfect, $\tilde{X}$ splits. Since $|X|_3 = |\,.3\,|_3$, Gaschütz' theorem implies .3 splits, and we are done.

8. Conway's Group .2

Set $G = .2$; $|G| = 2^{18}3^6 5^3 7.11.23$. We will prove that $m(G) = 1$. Easily, $m_p(G) = 1$, for $p = 7, 11, 13$. Since G contains .322, which is isomorphic to McLaughlin's simple group, M^cL, we get $m_5(G) = 1$ because $m_5(M^cL) = 1$ and .322 contains a Sylow 5-subgroup of G.

Showing that $m_3(G) = 1$ and $m_2(G) = 1$ is more difficult.

LEMMA 8.1. *Let Z be the center of a Sylow 3-subgroup of G. Then $N_G(Z) = O_3(N_G(Z)) \cdot H$, where $O_3(N_G(Z))$ is extra special of order 3^5, exponent 3; where $O_2(H) \cong Q_8 \circ D_8$, $H/O_2(H) \cong \Sigma_5$.*

Proof. We use the containments $U_4(3) \subset M^cL \subset G$. Let Z be a Sylow 3-center in $U_4(3)$. Its centralizer in $U_4(3)$ is a semidirect product XS, where X is a normal extra special group of order 3^5 exponent 3, and the unique involution of $S \cong SL(2, 3)$ inverts X/X'.

Now, let C be the centralizer of Z in M^cL. Using the character table of M^cL (J. Thompson, unpublished), we see that $|C| = 5|Z|$, and it is an easy exercise, using centralizer orders in M^cL, to get that X is normal in C and $C/X \cong SL(2, 5)$ and $N_G(Z)/X$ is a covering group of Σ_5.

Finally, since $|M^cL|_3 = |G|_3$, Z is a Sylow 3-center in G, and the centralizer orders from the character table of G [9] force the conclusion of the Lemma, with $X = O_3(C_G(Z))$.

LEMMA 8.2. $m_3(N_G(Z)) = 1$.

Proof. Set $D = N_G(Z)$ and let $\tilde{D}$ be a central extension of D by $A \cong \mathbb{Z}_3$. Let $X = O_3(D)$. We first show $\tilde{X}$ splits over A.

Since X has class 2, $\tilde{X}$ has class 2 or 3. Let $L(\tilde{X}) = L_1 \oplus L_2 \oplus L_3$ be the Lie ring associated with $\tilde{X}$. Suppose $L_3 \neq 0$, i.e., that $A = L_3$. Then there is $x \in L_1$, $y \in L_2$ whose Lie product $[x, y]$ generates L_3. Let $\langle j \rangle = Z(H)$. Then j acts on $L(\tilde{X})$ inverting L_1, centralizing L_2. So, $[x, y]^j = [x^{-1}, y] = [x, y]^{-1}$. But this contradicts $A \subseteq Z(\tilde{D})$. So $L_3 = 0$.

Next, we suppose $A \subseteq \tilde{X}'$, i.e., $A \subseteq L_2$. Since H/H' inverts $X' = Z$, we have $L_2 = A \times B$, a decomposition as an H-module, with $B = [\tilde{X}', H]$. The group $\tilde{X}/B$ is nonabelian, and commutation induces a nontrivial alternating form from $\tilde{X}/\tilde{X}'$ into $AB/B \cong A$. This form is nondegenerate, since H acts irreducibly on $\tilde{X}/\tilde{X}'$. Since $A \subseteq Z(\tilde{D})$, the action of $\tilde{H}$ preserves this form. However, A is the kernel of this action of $\tilde{H}$, and so $\tilde{H}/\tilde{A} \cong H$ is faithful on $\tilde{X}/\tilde{X}'$. However, H is not isomorphic to a subgroup of Sp(4, 3), a contradiction. Thus, $A \not\subseteq \tilde{X}'$.

Since H is irreducible on X/X', it follows that $\tilde{X}/\tilde{X}'$ is elementary. Since j inverts X/X' and centralizes A, we have a decomposition $\tilde{X} = Y \times A$ of H-groups, where $Y = [\tilde{X}, \langle j \rangle]$. Now, $\tilde{D}/Y$ is a central extension of H by A. Since H has cyclic Sylow 3-subgroups, $A \not\subseteq \tilde{H}'$, and so $A \not\subseteq \tilde{D}'$. Since $\tilde{D}$ is an arbitrary central extension of D by A, $m_3(D) = 1$, and the Lemma follows.

Since D contains a Sylow 3-subgroup of G, Gaschütz' Theorem and Lemma 2.2 imply the following.

COROLLARY 8.3. $m_3(G) = 1$.

Our next task is to show $m_2(G) = 1$. Let KM_{24} be the subgroup N of Conway's paper [8], $K = O_2(N)$ elementary abelian of order 2^{12}. Let $\{v_1, \ldots, v_{24}\}$ be an orthonormal basis for $\mathbb{R}^{24}$ and let Λ be the Leech lattice. Take $G = .2$ as the subgroup of $.0$ fixing the vector $b = \sum a_i v_i$, where $\{i \mid a_i \neq 0\} = C$ is an octad and $a_i = 2$, all $i \in C$.

LEMMA 8.4. *.442 is a split extension of a normal extra special group of order* 2^9 by A_7.

Proof. .4 is isomorphic to the split extension $E_{2^{11}}M_{23}$, the subgroup of N fixing the vector $-8v_i$ of type 4. Our lattice vector $v = \sum a_k v_k$ of type 2, along with the origin and $-8v_i$ forms a triangle of type 442.

Now, $C = \{k \mid a_k \neq 0\}$ is an octad. We argue that $L = .442 \cap O_2(.4)$ is elementary abelian of order 32. The transformations in L are the ϵ_S, where S is a $\mathscr{C}$-set disjoint from C. There are 32 such S: 30 octads, the 16-set complementing C, and the empty set. Thus, $L \simeq E_{2^5}$.

By our choice of triangle, we see that $.442/L$ induces permutations of $\{\langle v_1\rangle, \ldots, \langle v_{24}\rangle\}$ fixing v_i and those $\langle v_k\rangle$ with $k \in C$. By inspecting the stabilizer of an octad in M_{24}, we get that $.442/L$ is the split extension $E_{2^4}A_7$. By inspecting the structure of N, we see that .442 is a split extension LH, where $H \subset M_{23}$. Since the subgroup of M_{24} stabilizing each of the 30 octads disjoint from C is trivial, H acts faithfully on L. Since H induces A_7 on $L/\langle\epsilon_{\Omega+C}\rangle$, the action of $O_2(H)$ on L makes $L \cdot O_2(H)$ extra-special with $\langle\epsilon_{\Omega+C}\rangle = \mathbb{Z}(LO_2(H))$. This proves the lemma.

Set $z = \epsilon_{\Omega+C}$. According to [8], $F = C_G(z)$ has order $2^{18}3^4 5\cdot 7$. Let B be the copy of .442 mentioned in the proof of Lemma 8.4, and set $Y = LO_2(H) = O_2(B)$. Also, set $X_1 = L$, $X_2 = O_2(H)\langle z\rangle$. We have $X_k \lhd B$, $|X_k| = 2^5$, $k = 1, 2$. Finally, let D be a subgroup of H isomorphic to A_7; D complements Y in B. Since D is contained in the "standard" copy of M_{23} in N, the involutions of D have eigenvalues {sixteen $+1$, eight -1} in the 24-dimensional representation. Since $X_2 \cap H$ is also contained in M_{23}, a group with one conjugacy class of involutions, the involutions of $X_2 \cap H$ have the same set of eigenvalues as those of D. Now a look at the class list [9] shows that the involutions of D and those of $X_2 \cap H$ fuse in G.

LEMMA 8.5. *The sublattice Λ_1 of Λ consisting of vectors fixed by z is fixed pointwise by Y. Also $Y \lhd F$ and $F/Y \simeq$* Sp(6, 2).

Proof. Λ_1 consists of all $\sum b_k v_k \in \Lambda$ for which $b_k \neq 0$ implies $k \in C$. The first statement is now clear from the way N acts on Λ. There are 240 vectors of type 2 in Λ_1, and it is straightforward to select eight of them whose inner product matrix identifies Λ_1 as a lattice of type E_8. Note that D acts faithfully on Λ_1. Let F_1 be the subgroup of F which acts trivially on Λ_1 and set $\bar{F} = F/F_1$; $\bar{F}$ contains a copy of A_7, with index dividing $2^6 3^2$. The Frattini argument would produce an element of order 21 in G, and a contradiction, if any 3-element were to act trivially on Λ_1. So F_1 is a 2-group. Similarly, $\bar{F}$ must be simple. Let $|\bar{F}| = |A_7| 2^n 3^2$. Since an element of order 7 in G has centralizer order $2^3.7$, D acts on $N_{F_1}(Y)/Y$ in such a way that an element of order 7 is fixed point free. So, $n = 0$ or 6. But $n = 0$ is out by simplicity of $\bar{F}$. Thus, $|\bar{F}| = 2^9 3^4 5.7 = |\mathrm{Sp}(6, 2)|$. Since the action of F on Λ_1 stabilizes v, $\bar{F}$ is identified as a subgroup of Sp(6, 2), the commutator subgroup of the Weyl group of E_7. The lemma is proven.

LEMMA 8.6. *$1 \to \langle z \rangle \to X_i \to X_i/\langle z \rangle \to 1$ is a split extension of $\mathbb{F}_2 D$-modules, $i = 1, 2$.*

Proof. We consider extensions

$$(*) \quad 1 \to Y_i \to E \to T \to 1$$

of the module Y_i dual to $X_i/\langle z \rangle$, where T is the one-dimensional trivial $\mathbb{F}_2 D$-module, and where E is dual to X_i. By taking annihilators, we get a one-to-one correspondence between subgroups of Y and those of E which inverts the lattice of submodules. To prove the lemma, it sufficies to show that (*) is split.

Any Sylow 3-subgroup P of D must act fixed-point-freely on Y_i, hence P fixes a unique element v of $E \backslash Y_i$. Choose x, y in P so that $\langle x \rangle \cap \langle y \rangle = 1$ and x, y each act fixed point freely on Y_i. The only element of $E \backslash Y_i$ fixed by either x or y is v, and the same is true for any Sylow 3-subgroup of D containing either x or y. Since D is generated by all such Sylow 3-subgroups, it follows that D fixes $\langle v \rangle$, and so the extension splits.

Remark. This "generation" argument closely resembles ideas in [1]. This Lemma actually shows that $H^1(A_7, Y_i) = \mathrm{Ext}_{\mathbb{F}_2 A_7}(\mathbb{F}_2, Y_i) = 0$.

Notation: write $X_i = X_{i0} \oplus \langle z \rangle$ as A_7-modules, $i = 1, 2$. In previous notation, we have $X_2 \cap H = X_{20}$.

LEMMA 8.7. *Let V be the irreducible $\mathbb{F}_2\bar{F}$-module Y/Y' of order 2^8. Then any extension $1 \to V \to W \to T \to 1$ of V by a trivial $\mathbb{F}_2\bar{F}$-module T is split. Also, any extension $1 \to T \to W_0 \to V \to 1$ is split.*

Proof. The group D is isomorphically contained in $\bar{F}$. Hence, $\bar{F}$ contains an element of order 3 acting fixed point freely on V, because V is isomorphic to $X_1/\langle z \rangle \oplus X_2/\langle z \rangle$ as $\mathbb{F}_2 D$-modules. By looking at the extension associated with the module dual to W, a generation argument as in the last lemma will imply the splitting. We omit details. The second assertion follows from the first, since V is a self-dual module.

LEMMA 8.8. *In the (unique) nonsplit central extension $\hat{S}$ of $S = \mathrm{Sp}(6, 2)$ by $\mathbb{Z}_2$, there is an involution in S represented by an element of order four in $\hat{S}$. Any subgroup of S isomorphic to A_7 contains such an involution.*

Proof. Consider the 7-dimensional rational representation of S which comes from identifying S as the commutator subgroup of the Weyl group E_7. The embeds S in the real orthogonal group $SO(7, \mathbb{R})$. The latter group has the property that an involution having -1 as an eigenvalue k times is represented in Spin $(7, \mathbb{R})$ by an element of order four if and only if $k \equiv 2 \pmod 4$ [7].

Any subgroup of S isomorphic to A_7 must have an irreducible constituents of dimension 1 and 6, and the trace of an involution in A_7 on the 7-dimensional space is $3 = 2(-1) + 5(1)$. Thus, the preimages of A_7 and S in Spin$(7, \mathbb{R})$ are not split extensions of those groups by the center of Spin$(7, \mathbb{R})$. Since $m(S) = 2$, this extension is unique up to isomorphism, and the parts of the Lemma follow.

LEMMA 8.9. *F/Y acts transitively on the nontrivial cosets of Y' in Y which contain involutions.*

Proof. Since $Y \cong D_8 \circ D_8 \circ D_8 \circ D_8$, there are 135 such cosets. The stabilizer of one of these in $\mathrm{Out}(Y) \cong O^+(8, 2)$ is isomorphic to $E_{2^6} \cdot O^+(6, 2) \cong E_{2^6} \cdot \Sigma_8$. We have $A_7 \cong D \subset F$, and D acts on $X_i \subset Y$, $i = 1, 2$. The stabilizer in D of xY', for $x \in X_i^{\#}$, is isomorphic to $GL(3, 2)$. Therefore, the stabilizer L of xY' in F/Y has index dividing $|\mathrm{Sp}(6, 2)|/|GL(3, 2)| = (2^9 3^4 5.7/2^3 3.7) = 2^6 3^3 5$. Now, all subgroups of order 5 in F/Y are conjugate. If we take an element h of order 5 in DY/Y, we see that h acts fixed point freely on X_1 and X_2. Thus, 5 divides $|F/Y : L|$. If $|L|$ were divisible by 3^2, then, since $L/O_2(L)$ contains a subgroup isomorphic to $GL(3, 2)$ (a maximal subgroup of A_7) and since $L/O_2(L)$ is isomorphic to a subgroup of Σ_8, we would have a contradiction. Thus, 5 and 3^3 divide $|F/Y : L|$, and so $|F/Y : L| = 3^3 5 = 135$.

Now, we consider a central extension $\tilde{G}$ of G by $A \cong \mathbb{Z}_2$ and consider the extensions induced on subgroups. To get $m_2(G) = 1$, it suffices to show $\tilde{F}$ splits.

LEMMA 8.10. *$\tilde{Y}$ splits over A.*

Proof. Since F is perfect, the preimage of $\mathbb{Z}(F) = Y'$ in F is $\mathbb{Z}(\tilde{F})$. Thus, $\tilde{Y}$ has nilpotence class 2. Since $\tilde{Y}/\mathbb{Z}(\tilde{Y})$ is elementary, so is $(\tilde{Y})'$. Also, $\tilde{X}_{i0}$ is elementary abelian, $i = 1, 2$ because D acts transitively on $X_{i0}^{\#}$.

Let α generate A. Suppose $A \subseteq (\tilde{Y})'$. Then $(\tilde{Y})'$ is a four-group. Also, α must be a product of commutators of the form $[\tilde{x}_1, \tilde{x}_2]$, $x_i \in X_{i0}$, $i = 1, 2$. Since $[\tilde{x}_1, \tilde{x}_2]$ is central in $\tilde{F}$ and D is transitive on $X_{i0}^{\#}$, we may assume that all the x_1 occurring in our expression for α are all congruent modulo A. Bilinearity of commutation then implies that we may assume $\alpha = [\tilde{x}_1, \tilde{x}_2] = (\tilde{x}_1\tilde{x}_2)^2$. That is, α has a square root in $\tilde{Y}\backslash(\tilde{Y})'$. By the previous lemma, we may conjugate x_1x_2Y' to x_1Y' in F. Then, $\tilde{x}_1^2 = \alpha$, which contradicts the splitting of $\tilde{X}_{i0}$ over A. Thus $A \cap (\tilde{Y})' = 1$. Since $\tilde{Y} = \langle \tilde{X}_1, \tilde{X}_2 \rangle$, $\tilde{Y}/(\tilde{Y})'$ is elementary abelian, which implies the Lemma.

LEMMA 8.11. *$\tilde{F}$ splits over A.*

Proof. Since $\tilde{Y}$ splits over A, $\tilde{Y}/(\tilde{Y})'$ is an extension of the $\bar{F} = F/Y$-modules Y/Y' and A. By Lemma 2.11, this extension of modules is split. We write $\tilde{Y}/(\tilde{Y})' = Y_1/(\tilde{Y})' \oplus A(\tilde{Y})'/(\tilde{Y})'$, direct sum of $\bar{F}$ modules. Let $D \cong A_7$ be our complement to Y.

By the argument in the proof of the last Lemma, every involution of X_2 is represented by an involution in $\tilde{G}$. Since the involutions of X_{20} and D are G-conjugate, every involution of D is represented by an involution in $\tilde{D} \subset \tilde{G}$. Since the unique perfect extension of A_7 by $\mathbb{Z}_2$ has quaternion Sylow 2-subgroups, we get that $\tilde{D}$ splits over A. Since $Y_1 \lhd \tilde{F}$ and $A \cap Y_1 = 1$. Lemma 8.12 tells us that $\tilde{F}/Y_1$ splits over AY_1/Y_1. Thus, $A \not\subseteq (\tilde{F})'$, whence F is split over A, as required.

9. CONWAY'S GROUP .1

Set $G = .0$; $|G| = 2^{22}3^9 5^4 7^2 11.13.23$. We first show that $m_2(G) = 1$. This will imply $m_2(.1) = 2$.

As in the last section, we consider the subgroup $M = KM_{24}$ of Conway's paper. The only proper M_{24}-submodule of K is $\langle \epsilon_\Omega \rangle = Z(G)$. Let V be the $\mathbb{F}_2M_{24}$-module dual to K, and let V_0 be the unique maximal submodule of V; $|V : V_0| = 2$.

LEMMA 9.1. *Let W be any $\mathbb{F}_2M_{24}$-module containing V_0 as a submodule of index 2 with the property that $1 \to V_0 \to W \to \mathbb{F}_2 \to 1$ is a nonsplit extension. Then there is a vector $v \in W\backslash W_0$ whose stabilizer is (up to conjugacy) M_{23}.*

Proof. Choose an element x in M_{24} of order 23. Since x necessarily acts fixed point freely on the submodule V_0 of order 2^{11}, there is a unique nonzero vector $v \in W\backslash V_0$ fixed by x. The normalizer of $\langle x \rangle$ in M_{24} is a Frobenius group of order 11.23 [39], so we choose $y \in M_{24}$, an element of order 11 normalizing $\langle x \rangle$. Since $\langle x, y \rangle$ is a Frobenius group, $|C_{V_0}(y)| = 2$, and so $|C_W(y)| = 2^2$. Next, the normalizer of $\langle y \rangle$ in M_{24} is a Frobenius group of order 55. Let u be an element of order 5 normalizing $\langle y \rangle$. Then, u leaves invariant $C_W(y)$, and must centralize it. Now, let S be the stabilizer of v in M_{24}. We know that $5 \cdot 11 \cdot 23$ divides $|S|$.

Let Ω be the usual 24 points on which M_{24} acts. Now, $\langle x, y\rangle$ fixes exactly one point of Ω. Let M be the subgroup of M_{24} fixing that point; $M \simeq M_{23}$. It follows that $u \in M$, since u fixes each of the two points of Ω fixed by y.

Recall that $|M| = 2^7 3^2 5.7.11.23$. So, $|S \cap M| = 2.11.23.a$, where a divides $2^7 3^2 7$. Since an element of order 11 is selfcentralizing and not conjugate to its inverse in M, the Frattini argument implies that $S \cap M$ is simple. Sylow's theorem implies that $5a \equiv 1 \pmod{23}$ and $a \equiv 1 \pmod{11}$, whence $a = 221 + (11 \cdot 23)\, k = 221 + 253k$, for some $k \geqslant 0$. If $k = 0$, 1 or 2, a does not divide $|M|$. But $k \geqslant 3$ implies $|M : S \cap M| < 8$, (and $k = 31$). This gives $M = S \cap M$, or $S \supseteq M$, a maximal subgroup of M_{24}. Since our extension is nonsplit, all of M_{24} may not stabilize v. Therefore, $S = M$. This proves the lemma.

LEMMA 9.2. *Let* $1 \to V_0 \to X \to T \to 1$ *be an extension of* $\mathbb{F}_2 M_{24}$-*modules, with* T *a trivial module, having the property that the extension, when restricted to any proper submodule of* T *is nonsplit. Then* $|T| \leqslant 2$.

Proof. Assume there is such an extension with $|T| > 2$. Choose T_1, T_2, distinct submodules of T, each of dimension 1. Let X_1, X_2 be the extensions of V_0 by T_1, respectively T_2, induced by X. By preceeding arguments and the fact that there is one conjugacy class of subgroups in M_{24} of index 24, there is a subgroup $M \simeq M_{23}$ of M_{24} fixing $t_1 \in X_1 \backslash V_0$ and $t_2 \in X_2 \backslash V_0$. Let $t_3 = t_1 t_2$ and set $X_3 = \langle V_0, t_3\rangle$. Choose $y \in M$, $|y| = 11$, and $s \in M_{24}$, $|s| = 2$, s normalizing $\langle y\rangle$. Then s inverts y and so $[V_0, y]$. of dimension 10, is a free $\mathbb{F}_2\langle s\rangle$-module. Also, s stabilizes $C_{X_i}(y)$, which has dimension 2, for $i = 1, 2, 3$. Let $\langle w\rangle = C_{V_0}(y)$. If s were to fix t_i, $i = 1$ or 2, then $M_{24} = \langle M, s\rangle$ fixes t_i and then X_i would split, a contradiction. So, we must have $t_1{}^s = t_1 w$ and $t_2{}^s = t_2 w$. But then, $(t_1 t_2)^s = t_1 w t_2 w = t_1 t_2$, which implies the splitting of X_3, another contradiction. We conclude that $|T| \leqslant 2$.

LEMMA 9.3. *Let* (*) $1 \to A \to L \to K \to 1$ *be an extension of* $\mathbb{F}_2 M_{24}$-*modules, with* A *a trivial module. Then,* $L \simeq K \oplus A$ *as* $\mathbb{F}_2 M_{24}$-*modules, i.e., the extension splits.*

Proof. Let R denote the $\mathbb{F}_2 M_{24}$-module dual to L. Then we have an exact sequence $1 \to V \to R \to A \to 1$ of $\mathbb{F}_2 M_{24}$-modules. By considering the submodule V_0 of V, we have another exact sequence (**) $1 \to V_0 \to X \to T \to 1$, with T a trivial module of order $2\,|A|$. If (*) were nonsplit, there would be a maximal subgroup A_0 of A (A_0 is also a submodule) so that the exact sequence with A and L replaced by, respectively, A/A_0 and L/A_0 is nonsplit. So, we may assume A has order 2. Then (**) satisfies the hypotheses of the last lemma, which tells us $|T| \leqslant 2$. This contradiction proves the lemma.

LEMMA 9.4. *Let $\tilde{N}$ be an extension of N by $A \simeq \mathbb{Z}_2$. Then $\tilde{N}$ splits.*

Proof. Consider the induced extension $\tilde{K}$ of K. Since there is an element of order 23 acting on K, there is also one acting on $\tilde{K}$. This easily implies that $\tilde{K}$ is elementary abelian. By the last lemma, $\tilde{K} = K_0 \oplus A$, with $K_0 \simeq K$, as M_{24}-modules. So, K/K_0 is isomorphic to a central extension of M_{24} by $A \simeq AK_0/K_0$, and we get that $A \cap (\tilde{N})' = 1$, since $m(M_{24}) = 1$. Since N is perfect, $\tilde{N}$ splits.

COROLLARY 3.5. $m_2(G) = 1$.

Proof. Use Lemma 9.4, $|N|_2 = |G|_2$, and Gaschütz' theorem.

It remains to show $m_p(G) = 1$, for odd primes p. Since

$$|G| = 2^{22}3^9 5^4 7^2 11.13.23,$$

this is trivial for $p = 11, 13, 23$.

LEMMA 9.6. $m_7(G) = 1$.

Proof. From the list of centralizer orders, there are two conjugacy classes of elements of order 7, represented by, say, x and y with $|C_G(x)| = 2^4 3^2 5.7^2$ and $|C_G(y)| = 2^4 3.7^2$. So, the Sylow 7-subgroup is elementary abelian. We may assume x and y commute. If $\overline{C(x)} = C(x)/\langle x \rangle$ had a normal 7-complement, the Frattini argument and the Schur-Zassenhaus Theorem would imply that the image $\bar{y}$ of y normalizes, hence centralizes, a Sylow 3-subgroup of $\overline{C(x)}$. But 3^2 does not divide the order of $C(y)$, contradiction. Therefore, there is a $7'$-element u of $C(x)$ with $\bar{u}$ normalizing but not centralizing $\langle \bar{y} \rangle$. By an easy exercise, $\langle x, y, u \rangle$ has trivial multiplier. Since this subgroup contains a Sylow 7-subgroup of G, $m_7(G) = 1$ follows.

LEMMA 9.7. $m_5(G) = 1$.

Proof. Consider the subgroup $.533 \simeq U_3(5)$ of G. Its Sylow 5-subgroup has the form PQ, where $P = O_5(PQ)$ is a nonabelian group of order 5^3, exponent 5, and Q is cyclic of order 8 with the involution of Q inverting P/P'. Now, $m_5(PQ) = 1$ [19]. Let S be a Sylow 5-subgroup of G containing P; $|S| = 5^4$. Set $H = \langle PQ, S \rangle$. Let $\tilde{H}$ be a central extension of H by $A \simeq \mathbb{Z}_5$. By the above, $A \cap (\tilde{P})' = 1$. Since the involution t of Q inverts P/P', we get $\tilde{P} = R \times A$, where $R = [\tilde{P}, t]$. Since the actions of t and S on P/P' must commute, $\tilde{S}$ normalizes R. Thus, $\tilde{H}/R$ is isomorphic to a central extension of H/P by A. Since H/P has cyclic Sylow 5-subgroups, we get $A \cap (\tilde{H})' = 1$. This implies $m_5(H) = 1$, and so $m_5(G) = 1$.

LEMMA 9.8. $m_3(G) = 1$.

Proof. Let Z be a Sylow 3-center of G. By the table of centralizer orders, $|N_G(Z)| = 2^9 3^9 5$. Using the containment $.2 \subset G$, it is not difficult (see Lemma 8.1) to see that $N_G(Z) = PK \times B$, where $B = \mathbb{Z}(G)$, $P = O_3(C_G(Z))$ is extra special of order 3^5, exponent 5, and $K \cap C_G(Z) = K' \cong \mathrm{Sp}(4, 3)$, and K complements P in PK. But the argument of Lemma 8.2 goes through here without change to give $m_3(N_G(Z)) = 1$. Thus $m_3(G) = 1$.

This completes the proof that $m(G) = 1$ and $m(.1) = 2$.

Assumed Results

Most of these may be found in [17] or [24]; if not, a source is noted. Fundamental results about multipliers and covering groups are nicely presented in [24].

(1) (Gaschütz' Theorem). If G is a finite group, H a subgroup, M a finite G-module and $(|M|, |G:H|) = 1$, then an extension of G by M splits if the restriction to H is a split extension.

(2) ("Transfer Lemma"). If P is a Sylow p-subgroup of G, x a p-element in $Z(G)$, then $x \notin P'$ implies $x \notin G'$.

(3) (Fitting's Lemma). If A is a group of automorphisms of the finite abelian group M and $(|M|, |A|) = 1$, then $M = C_M(A) \times [M, A]$.

(4) An automorphism of order prime to p on P, a p-group, is nontrivial if and only if the induced automorphism of $P/\Phi(P)$ is nontrivial.

(5) The terms G_i of the lower central series of G satisfy $[G_i, G_j] \leqslant G_{i+j}$.

(6) (Cartan–Eilenberg). For $H \leqslant G$, the restriction map $H^n(G, M) \to H^n(H, M)$, where $p \nmid |G:H|$, induces a monomorphism of the p-primary parts of the cohomology groups. The image is the set of elements of $H^n(H, M)$ stable with respect to G [4, Chap. XII].

(7) If a Sylow p-subgroup P of G is elementary abelian of order p^2, then $p \nmid m(G)$ if the normalizer of P effects a transformation on P of determinant not 1. (This follows from [24, p. 644].)

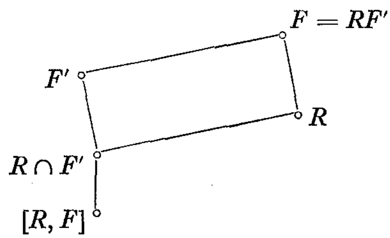
(8) All covering groups of a perfect group are isomorphic [24, 30].

(9) An automorphism α of a perfect group can be lifted to an automorphism of the covering group.

Proof. (Alperin) Let $1 \to R \to F \xrightarrow{\pi} G \to 1$ be a free presentation of the perfect group G. Say the free generators x_i of F map to g_i, $i = 1,\ldots, n$, a set of generators for G. Suppose $g_i^\alpha = h_i$. Write h_i as a word $w_i(g_1, \ldots, g_n)$ in the g_j. Define an endomorphism $\beta: F \to F$

by $x_i^\beta = w_i(x_1, ..., x_n)$. Then $\beta\pi = \pi\alpha$. Now, β maps R into itself because if a word $v = v(x_1, ..., x_n)$ lies in R, $v^\beta = v(x_1^\beta, ..., x_n^\beta)$ goes under π to $v(x_1^{\beta\pi}, ..., x_n^{\beta\pi}) = v(x_1^{\pi\alpha}, ..., x_n^{\pi\alpha}) = v(x_1^\pi, ..., x_n^\pi)^\alpha = 1$ because $v \in R$ means $v^\pi = 1$.

Since $G = G'$, $RF' = F$. β leaves invariant each vertex of the diagram below.



A covering group of G is obtained by taking F/S, where $S/[R, F]$ is a complement to $R \cap F'/[R, F]$ in $R/[R, F]$. In our case, taking incidence implies $F/S \cong F'/[R, F]$ and β induces α on $F'/R \cap F' \cong F/R$. We claim the endomorphism β^* induced by β is an automorphism of $E = F'/[R, F]$. Clearly the product of the image of β^* on E with $R \cap F'/[R, F]$ is E. But since $R \cap F'/[R, F]$ is central, it lies in the Frattini subgroup of E. Hence β^* is onto and so an isomorphism.

(10) If $K/A \cong G$, $A \leqslant Z(K) \cap K'$, then there is a covering group H of G with quotient isomorphic to K [24, 30].

(11) If $H \lhd G$, $H = H'$, and $m_p(G/H) = 1$, then $m_p(H) = 1$ implies $m_p(G) = 1$.

Proof. Take $\tilde{G}$, a central extension of G by a p-group A. Then $\tilde{H} = \tilde{H}' \times A \cong H \times A$ and each factor is normal in G. Let $G^* = \tilde{G}/\tilde{H}'$, a central extension of G/H. If $A^* = A\tilde{H}'/\tilde{H}'$, $m_p(G/H) = 1$ implies $A \cap G^{*\prime} = 1$. As $\tilde{G}$ is arbitrary, we get $m_p(G) = 1$ by (10).

(12) If $K/A = G$, $A \leqslant Z(K) \cap K'$ and the ordinary representations of K over an algebraically closed field k of characteristic $p > 0$ lift the projective representations of G over k, then $A \cong M(G)/M_p(G)$ (e.g., 3.2 of [35]).

(13) $[xy, z] = [x, z]^y[y, z]$, $[x, yz] = [x, z][x, y]^z$.

(14) Let A, B be subgroups of G. Suppose $[A, B]$ centralizes A and B. Then $[aa', b] = [a, b][a', b]$ and $[a, bb'] = [a, b'][a, b]$, $a, a' \in A$, $b, b' \in B$. (We say here that [,] is "biadditive" or "bimultiplicative.")

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