On the Generalization of the Volterra Principle of Inversion*

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In this article a linear operator, $K$, defined on a Hilbert space equipped with a chain of orthoprojectors is considered. It is proved that if $K$ enjoys a particular property with respect to the chain of orthoprojectors, then the series $\sum_{n=0}^{\infty} K^n$ converges in the uniform operator norm. The proof uses purely algebraic techniques and does not require compactness of $K$. As such, it is a significant generalization of the well-known Volterra principle of inversion.

1. INTRODUCTION

A bounded linear operator $P$ on a Hilbert space $H$ is called an orthoprojector if for all pairs $x, y \in H^2$ one has $\langle Px, y \rangle = \langle x, Py \rangle$, and $P^2 x = Px$. If $P_1$ and $P_2$ are two orthoprojectors, the symbol $P_1 < P_2$ is used to indicate that $P_1 H$ (the range of $P_1$) is contained in $P_2 H$. A set $\mathcal{P}$ of orthoprojectors is called a chain if for every pair $P_1, P_2 \in \mathcal{P}$ one has either $P_1 < P_2$ or $P_2 < P_1$; the chain $\mathcal{P}$ is called bordered if it contains the null operator 0, and the identity operator $I$; $\mathcal{P}$ is closed if it has the property that whenever a sequence of orthoprojectors $\{P_i\} \subseteq \mathcal{P}$ has a limit, $\lim\{P_i x\} = Px$ for all $x \in H$, then $P \in \mathcal{P}$.

A chain $\mathcal{P}$ is a partition of $\mathcal{P}$ if $\mathcal{P}$ is composed of a finite number of orthoprojectors in $\mathcal{P}$. If $\mathcal{z}_1$ and $\mathcal{z}_2$ are two partitions of $\mathcal{P}$, the symbol $\mathcal{z}_2 \supset \mathcal{z}_1$ will indicate that if $P \in \mathcal{z}_1$, then $P \in \mathcal{z}_2$.

Suppose that $\Phi(\cdot)$ is an operator-valued function which associates to each partition $\mathcal{z}$ of $\mathcal{P}$ a bounded operator $\Phi(\mathcal{z}) : H \to H$. The operator $T$ is said

* This research was in part supported by the Canadian National Research Council, Grant No. A-8244, and by the US Air Force Office of Scientific Research, Grant No. 73-2427.

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to be a uniform limit point of the function $\Phi(\cdot)$ if given any $\epsilon > 0$, there exists a partition $z$ such that for any other partition $z$ with the property $z \supset z'$, one has $|\Phi(z) - T| < \epsilon$.

Suppose that $Y$ is a bounded operator in $H$, where $H$ is equipped with a bordered and closed chain $\mathcal{P}$, and consider the operator valued function $S(\cdot)$ defined as follows; if $z = \{0 = P_0 < P_1 \cdots < P_{n-1} < P_n = I\}$ is a partition of the bordered and closed chain $\mathcal{P}$, then $S(z) = \sum_{i=1}^{n} \Delta P_i Y P_{i-1}$, where $\Delta P_i = P_i - P_{i-1}$. If $S(\cdot)$ has a unique uniform limit point, $T$, then we shall denote this limit point by $(m) \int_{\mathcal{P}} dP Y P$ which is said to converge to $T$. Similarly if $P_{i-1}$ is replaced by $\Delta P_i$ in the definition of $S(\cdot)$ and a limit point exists, it is denoted by $\int_{\mathcal{P}} dP Y dP.1$

The principal result of this paper is embodied in the theorem.

**Theorem 1.** Suppose that a linear, bounded operator $K$ on a Hilbert space $H$ satisfies $K = m \int_{\mathcal{P}} dP K P$, where $\mathcal{P}$ is a closed bordered chain in $H$. Then the series $\sum_{n=0}^{\infty} K^n$ converges in the uniform operator norm.

It is noted that less general forms of this theorem have been offered by other authors. In particular, an early version of Theorem 1 was established by Volterra [5] whose principle of inversion establishes the validity of Theorem 1 in the special case where $H$ is given by the space of square integrable functions on the real line, $L_2$, and $K$ is a Hilbert–Schmidt operator on $L_2$. More recently, this latter result has been extended to the case where $H$ is any abstract Hilbert space [1]. Finally, Gohberg and Krein [4], have proved Theorem 1 in the case where $K$ is a compact operator.

To illustrate the content of the theorem and associated notation, we consider the following simple example.

**Example 1.** Let $H$ be given by $L_2[0, \infty)$, the Hilbert space of real-valued Lebesgue square integrable functions defined on the interval $[0, \infty)$. Consider the bounded, linear operator $K$ defined as follows. If $x, y \in L_2[0, \infty)$ and $y = Kx$, then

$$y(t) = \int_{0}^{t} h(t, s) x(s) ds + \mu(t) \sum_{n=0}^{\infty} g_n (t - \tau_n) x(\tau_n),$$

where: $h(\cdot, \cdot)$ is a Lebesgue square integrable Kernel; $\{g_n\}$ is an $\ell_1$ sequence, i.e. $\sum_{n=0}^{\infty} |g_n| < \infty$, and the sequence $\{\tau_n\}$ is such that $\tau_0 > 0$ and $\tau_{n+1} > \tau_n$;

$$\mu(\cdot) \in L_\infty[0, \infty) \quad \text{and} \quad \lim \sup_{t \to \infty} \left( \sup_{x \in [t, \infty)} |\mu(x)| \right) = 0.$$

$^1$ All the concepts presented up to this point are taken directly from the book of Gohberg and Krein [4, Sections 1.3 and 1.4] and are included for completeness.
This operator $K$ satisfies the condition $K = (m) \int_{\Psi} dPKP$, where $\Psi$ is the bordered and closed chain of orthoprojectors indexed by $t \in [0, \infty)$ and defined as follows. If $x, y \in L_2(0, \infty)$ and $x = P\gamma$, then $x(s) = y(s)$ for $s \in [0, t)$, and $x(s) = 0$ for $s \in [t, \infty)$; when $t = \infty$, then $P\infty x = x$. In short, $K$ satisfies the hypotheses of the theorem, and we conclude that the series $\sum_{n=0}^{\infty} K^n$ converges in the uniform operator norm. This conclusion could not have been obtained from earlier versions of Theorem 1, because $K$ is not necessarily Hilbert–Schmidt nor compact.

2. PRELIMINARY RESULTS

The proof of Theorem 1 will be developed in steps using a sequence of four lemmas. Throughout the following, $H$ is a Hilbert space equipped with the bordered and closed chain of orthoprojectors $\Psi$, and $K$ is a bounded linear operator on $H$. Moreover, $P$ and $Q$ are orthoprojectors in $\Psi$ with $Q > P$; the difference $Q - P$ will be represented by the symbol $\Delta$.

**Lemma 1.** If $K = (m) \int_{\Psi} dPKP$, then $K$ is such that $PK = PKP$ for every $P \in \Psi$, and $\int_{\Psi} dPKdP = 0$.

**Lemma 2.** If $|K| < 1$, then $I - K$ is invertible and $(I - K)^{-1} = \sum_{n=0}^{\infty} K^n$, where the series converges in the uniform operator topology; moreover, this series is absolutely convergent in the sense that

$$\sum_{n=0}^{\infty} K^n \leq \sum_{n=0}^{\infty} |K|^n < \infty.$$

**Lemma 3.** If $(I - PKP)$ and $(I - \Delta K\Delta)$ are invertible and if $PK = PKP$, then $I - QKQ$, is invertible and

$$Q(I - QKQ)^{-1} = P(I - PKP)^{-1} + (I - \Delta K\Delta)^{-1} [\Delta + \Delta KP(I - PKP)^{-1}].$$

**Lemma 4.** Suppose $I - PKP$ is invertible and its inverse is computed by the convergent series

$$(I - PKP)^{-1} = \sum_{j=0}^{\infty} (PKP)^j.$$

Suppose that $\|\Delta K\Delta\| < 1$ and that $PK = PKP$. Then, $I - QKQ$ is invertible; moreover, the inverse is computed by the convergent series

$$(I - QKQ)^{-1} = \sum_{j=0}^{\infty} (QKQ)^j.$$
Lemma 1 is a consequence of the definition of \((m) \int_\Phi dPKP\). The proof Gohberg and Krein [4, Theorem 6.1, p. 27] carries over to the new setting without change and will not be repeated here. Lemma 2 is a familiar consequence of the Banach contraction principle [2, p. 131].

Lemma 3 can be established by manipulation. Note first that \(P \Delta = \Delta P = 0\) and \(PK\Delta = 0\) and hence

\[
Q(I - QKQ) = ((P - PKP) + (\Delta - \Delta K\Delta) - \Delta KP).
\]

It is then easy to verify that

\[
\{P(I - PKP)^{-1} + (I - \Delta K\Delta)^{-1} \Delta[I + KP(I - PKP)^{-1}]\} Q(I - QKQ) = Q,
\]

\[
Q(I - QKQ) \{P(I - PKP)^{-1} + (I - \Delta K\Delta)^{-1} \Delta[I + KP(I - PKP)^{-1}]\} = Q.
\]

Recognizing this as the right and left inverse, Lemma 3 follows.

As for Lemma 4, a more formal proof is appropriate. First note that the condition \(\|\Delta K\Delta\| < 1\) implies the existence of \((I - \Delta K\Delta)^{-1}\); moreover, the series expansion of Lemma 2 holds and this series is absolutely convergent. Using Eq. (1) and the series expansion of \((I - \Delta K\Delta)^{-1}\) in the result of Lemma 3, we have

\[
Q(I - QKQ)^{-1} = P \sum_{j=0}^{\infty} (PKP)^j + \Delta \sum_{j=0}^{\infty} (\Delta K\Delta)^j + \Delta \sum_{j=0}^{\infty} (\Delta K\Delta)^j KP \sum_{i=0}^{\infty} (PKP)^i.
\]

Because the series in \(\Delta K\Delta\) is absolutely convergent, a natural generalization of the Cauchy product of two series theorem can be invoked, (see [6, p. 65]). The rearrangement we need is embodied in the identity

\[
\sum_{j=0}^{\infty} (\Delta K\Delta)^j KP \sum_{i=0}^{\infty} (PKP)^i = \sum_{j=1}^{\infty} \sum_{q=0}^{j-1} (\Delta K\Delta)^q KP(PKP)^{j-1-q}.
\]

Using this, Eq. (2) becomes

\[
Q(I - QKQ)^{-1} = P + \Delta + \sum_{j=1}^{\infty} \left[ P(PKP)^j + \Delta(\Delta K\Delta)^j + \Delta \sum_{q=0}^{j-1} (\Delta K\Delta)^q KP(PKP)^{j-1-q} \right]
\]

The desired result is now a consequence of the operator equality

\[
(QKQ)^j = (PKP)^j + (\Delta K\Delta)^j + \Delta \sum_{q=0}^{j-1} (\Delta K\Delta)^q KP(PKP)^{j-1-q},
\]

where \(j = 1, 2, \ldots\)
The validity of Eq. (3) can be easily verified by an inductive process. It is true for $j = 1$, that is, $PKP + \Delta K\Delta + \Delta KP = QKQ$. Now, using this latter equality, we see that

$$\left[ (PKP)^j + (\Delta K\Delta)^j + \Delta \sum_{q=0}^{j-1} (\Delta K\Delta)^q KP(PKP)^{j-1-q} \right] QKQ$$

$$= \left[ (PKP)^{j+1} + (\Delta K\Delta)^{j+1} + \Delta \sum_{q=0}^{j} (\Delta K\Delta)^q KP(PKP)^{j-q} \right],$$

for $j = 1, 2, \ldots$.

3. Proof of Main Result

From the hypothesis that $K = (m) \int \delta dPK$, it follows that $\int dPK dP = 0$ (Lemma 1). This implies that there exists a partition

$$z = \{P_0 = 0, P_1, \ldots, P_N = I\} \in \Psi$$

such that

$$\left| \sum_{i=1}^{N} \Delta_i K\Delta_i \right| < 1,$$

where $\Delta_i = P_i - P_{i-1}$. This, in turn, implies that

$$|\Delta_i K\Delta_i| < 1, \quad i = 1, 2, \ldots, N.$$

Applying Lemma 2, $I - P_1 K\Delta_1$, and $I - \Delta_i K\Delta_i$, $i = 1, 2, \ldots, N$, are invertible. Moreover, from Lemma 3, one has

$$(I - P_1 K\Delta_1)^{-1} = \sum_{n=0}^{\infty} (P_1 K\Delta_1)^n \quad (4)$$

and

$$(I - \Delta_i K\Delta_i)^{-1} = \sum_{n=0}^{\infty} (\Delta_i K\Delta_i)^n. \quad (5)$$

From Eqs. (4) and (5), we can apply iteratively, Lemma 4 and obtain that the series $\sum_{n=0}^{\infty} (P_1 K\Delta_1)^n$ is uniformly convergent for each $i = 2, 3, \ldots, N$. For $i = N$, this implies the convergence of $\sum_{n=0}^{\infty} K^n$. 
References