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On the Generalization of the Volterra Principle of Inversion*

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In this article a linear operator, K, defined on a Hilbert space equipped with a chain of orthoprojectors is considered. It is proved that if K enjoys a particular property with respect to the chain of orthoprojectors, then the series $\sum_{n=0}^{\infty} K^n$ converges in the uniform operator norm. The proof uses purely algebraic techniques and does *not* require compactness of K. As such, it is a significant generalization of the well-known Volterra principle of inversion.

1. INTRODUCTION

A bounded linear operator P on a Hilbert space H is called an *orthoprojector* if for all pairs $x, y \in H^2$ one has $\langle Px, y\rangle = \langle x, Py\rangle$, and $P^2x = Px$. If P_1 and P_2 are two orthoprojectors, the symbol $P_1 < P_2$ is used to indicate that P_1H (the range of P_1) is contained in P_2H. A set $\mathfrak{P}$ of orthoprojectors is called a ***chain*** if for every pair $P_1, P_2 \in \mathfrak{P}$ one has either $P_1 < P_2$ or $P_2 < P_1$; the chain $\mathfrak{P}$ is called ***bordered*** if it contains the null operator 0, and the identity operator I; $\mathfrak{P}$ is *closed* if it has the property that whenever a sequence of orthoprojectors $\{P_i\} \subset \mathfrak{P}$ has a limit, $\lim\{P_ix\} = Px$ for all $x \in H$, then $P \in \mathfrak{P}$. A chain z is a ***partition*** of $\mathfrak{P}$ if z is composed of a finite number of orthoprojectors in $\mathfrak{P}$. If z_1 and z_2 are two partitions of $\mathfrak{P}$, the symbol $z_2 \supset z_1$ will indicate that if $P \in z_1$, then $P \in z_2$.

Suppose that $\Phi(\cdot)$ is an operator-valued function which associates to each partition z of $\mathfrak{P}$ a bounded operator $\Phi(z)$: $H \to H$. The operator T is said

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to be a uniform limit point of the function $\Phi(\cdot)$ if given any $\epsilon > 0$, there exists a partition z such that for any other partition z with the property $z \supset z_\epsilon$, one has $| \Phi(z) - T | < \epsilon$.

Suppose that Y is a bounded operator in H, where H is equipped with a bordered and closed chain $\mathfrak{P}$, and consider the operator valued function $S(\cdot)$ defined as follows; if $z = \{0 = P_0 < P_1 \cdots < P_{n-1} < P_n = I\}$ is a partition of the bordered and closed chain $\mathfrak{P}$, then $S(z) = \sum_{i=1}^{n} \Delta P_i Y P_{i-1}$, where $\Delta P_i = P_i - P_{i-1}$. If $S(\cdot)$ has a unique uniform limit point, T, then we shall denote this limit point by $(m) \int_{\mathfrak{P}} dPYP$ which is said to converge to T. Similarly if P_{i-1} is replaced by ΔP_i in the definition of $S(\cdot)$ and a limit point exists, it is denoted by $\int_{\mathfrak{P}} dPYdP$.[1]

The principal result of this paper is embodied in the theorem.

THEOREM 1. *Suppose that a linear, bounded operator K on a Hilbert space H satisfies $K = m \int_{\mathfrak{P}} dPKP$, where $\mathfrak{P}$ is a closed bordered chain in H. Then the series $\sum_{n=0}^{\infty} K^n$ converges in the uniform operator norm.*

It is noted that less general forms of this theorem have been offered by other authors. In particular, an early version of Theorem 1 was established by Volterra [5] whose principle of inversion establishes the validity of Theorem 1 in the special case where H is given by the space of square integrable functions on the real line, L_2, and K is a Hilbert–Schmidt operator on L_2. More recently, this latter result has been extended to the case where H is any abstract Hilbert space [1]. Finally, Gohberg and Krein [4], have proved Theorem 1 in the case where K is a compact operator.

To illustrate the content of the theorem and associated notation, we consider the following simple example.

EXAMPLE 1. Let H be given by $L_2[0, \infty)$, the Hilbert space of real-valued Lebesgue square integrable functions defined on the interval $[0, \infty)$. Consider the bounded, linear operator K defined as follows. If $x, y \in L_2[0, \infty)$ and $y = Kx$, then

$$y(t) = \int_0^t h(t, s)\, x(s)\, ds + \mu(t) \sum_{n=0}^{\infty} g_n(t - \tau_n)\, x(\tau_n),$$

where: $h(\cdot, \cdot)$ is a Lebesgue square integrable Kernel; $\{g_n\}$ is an ℓ_1 sequence, i.e. $\sum_{n=0}^{\infty} | g_n | < \infty$, and the sequence $\{\tau_n\}$ is such that $\tau_0 > 0$ and $\tau_{n+1} > \tau_n$;

$$\mu(\cdot) \in L_\infty[0, \infty) \qquad \text{and} \qquad \lim_{t\to\infty} \operatorname{ess}[\sup_{s\in[t,\infty)} | \mu(s)|] = 0.$$

[1] All the concepts presented up to this point are taken directly from the book of Gohberg and Krein [4, Sections 1.3 and 1.4] and are included for completeness.

This operator K satisfies the condition $K = (m) \int_{\mathfrak{P}} dPKP$, where $\mathfrak{P}$ is the bordered and closed chain of orthoprojectors indexed by $t \in [0, \infty)$ and defined as follows. If $x, y \in L_2[0, \infty)$ and $x = P^t y$, then $x(s) = y(s)$ for $s \in [0, t)$, and $x(s) = 0$ for $s \in [t, \infty)$; when $t = \infty$, then $P^\infty x = x$. In short, K satisfies the hypotheses of the theorem, and we conclude that the series $\sum_{n=0}^{\infty} K^n$ converges in the uniform operator norm. This conclusion could not have been obtained from earlier versions of Theorem 1, because K is not necessarily Hilbert–Schmidt nor compact.

2. Preliminary Results

The proof of Theorem 1 will be developed in steps using a sequence of four lemmas. Throughout the following, H is a Hilbert space equipped with the bordered and closed chain of orthoprojectors $\mathfrak{P}$, and K is a bounded linear operator on H. Moreover, P and Q are orthoprojectors in $\mathfrak{P}$ with $Q > P$; the difference $Q - P$ will be represented by the symbol Δ.

Lemma 1. *If $K = (m) \int_{\mathfrak{P}} dPKP$, then K is such that $PK = PKP$ for* every $P \in \mathfrak{P}$, and $\int_{\mathfrak{P}} dPKdP = 0$.

Lemma 2. *If $| K | < 1$, then $I - K$ is invertible and $(I - K)^{-1} = \sum_{j=0}^{\infty} K^n$, where the series converges in the uniform operator topology; moreover, this series is absolutely convergent in the sense that*

$$\sum_{n=0}^{\infty} K^n \leqslant \sum_{n=0}^{\infty} | K |^n < \infty.$$

Lemma 3. *If $(I - PKP)$ and $(I - \Delta K \Delta)$ are invertible and if $PK = PKP$, then $I - QKQ$, is invertible and*

$$Q(I - QKQ)^{-1} = P(I - PKP)^{-1} + (I - \Delta K \Delta)^{-1} [\Delta + \Delta KP(I - PKP)^{-1}].$$

Lemma 4. *Suppose $I - PKP$ is invertible and its inverse is computed by the convergent series*

$$(I - PKP)^{-1} = \sum_{j=0}^{\infty} (PKP)^j. \tag{1}$$

Suppose that $\| \Delta K \Delta \| < 1$ and that $PK = PKP$. Then, $I - QKQ$ is invertible; moreover, the inverse is computed by the convergent series

$$(I - QKQ)^{-1} = \sum_{j=0}^{\infty} (QKQ)^j.$$

Lemma 1 is a consequence of the definition of $(m) \int_{\mathfrak{P}} dPKP$. The proof Gohberg and Krein [4, Theorem 6.1, p. 27] carries over to the new setting without change and will not be repeated here. Lemma 2 is a familiar consequence of the Banach contraction principle [2, p. 131].

Lemma 3 can be established by manipulation. Note first that $P\Delta = \Delta P = 0$ and $PK\Delta = 0$ and hence

$$Q(I - QKQ) = \{(P - PKP) + (\Delta - \Delta K\Delta) - \Delta KP\}.$$

It is then easy to verify that

$$\{P(I - PKP)^{-1} + (I - \Delta K\Delta)^{-1} \Delta[I + KP(I - PKP)^{-1}]\} Q(I - QKQ) = Q,$$
$$Q(I - QKQ) \{P(I - PKP)^{-1} + (I - \Delta K\Delta)^{-1} \Delta[I + KP(I - PKP)^{-1}]\} = Q.$$

Recognizing this as the right and left inverse, Lemma 3 follows.

As for Lemma 4, a more formal proof is appropriate. First note that the condition $\| \Delta K\Delta \| < 1$ implies the existence of $(I - \Delta K\Delta)^{-1}$; moreover, the series expansion of Lemma 2 holds and this series is absolutely convergent. Using Eq. (1) and the series expansion of $(I - \Delta K\Delta)^{-1}$ in the result of Lemma 3, we have

$$Q(I - QKQ)^{-1} \\ = P \sum_{j=0}^{\infty} (PKP)^j + \Delta \sum_{j=0}^{\infty} (\Delta K\Delta)^j + \Delta \sum_{j=0}^{\infty} (\Delta K\Delta)^j KP \sum_{i=0}^{\infty} (PKP)^i. \tag{2}$$

Because the series in $\Delta K\Delta$ is absolutely convergent, a natural generalization of the Cauchy product of two series theorem can be invoked, (see [6, p. 65]). The rearrangement we need is embodied in the indentity

$$\sum_{j=0}^{\infty} (\Delta K\Delta)^j KP \sum_{i=0}^{\infty} (PKP)^i = \sum_{j=1}^{\infty} \sum_{q=0}^{j-1} (\Delta K\Delta)^q KP(PKP)^{j-1-q}.$$

Using this, Eq. (2) becomes

$$Q(I - QKQ)^{-1} \\ = P + \Delta + \sum_{j=1}^{\infty} \left[P(PKP)^j + \Delta(\Delta K\Delta)^j + \Delta \sum_{q=0}^{j-1} (\Delta K\Delta)^q KP(PKP)^{j-1-q}\right]$$

The desired result is now a consequence of the operator equality

$$(QKQ)^j = (PKP)^j + (\Delta K\Delta)^j + \Delta \sum_{q=0}^{j-1} (\Delta K\Delta)^q KP(PKP)^{j-1-q}, \tag{3}$$

where $j = 1, 2,\ldots$

The validity of Eq. (3) can be easily verified by an inductive process. It is true for $j = 1$, that is, $PKP + \Delta K\Delta + \Delta KP = QKQ$. Now, using this latter equality, we see that

$$\Big[(PKP)^j + (\Delta K\Delta)^j + \Delta \sum_{q=0}^{j-1} (\Delta K\Delta)^q KP(PKP)^{j-1-q}\Big] QKQ$$

$$= \Big[(PKP)^{j+1} + (\Delta K\Delta)^{j+1} + \Delta \sum_{q=0}^{j} (\Delta K\Delta)^q KP(PKP)^{j-q}\Big],$$

$$\text{for } j = 1, 2, \ldots.$$

3. Proof of Main Result

From the hypothesis that $K = (m) \int_{\mathfrak{P}} dPKP$, it follows that $\int dPKdP = 0$ (Lemma 1). This implies that there exists a partition

$$z = \{P_0 = 0, P_1, \ldots, P_N = I\} \in \mathfrak{P}$$

such that

$$\Big| \sum_{i=1}^{N} \Delta_i K \Delta_i \Big| < 1,$$

where $\Delta_i = P_i - P_{i-1}$. This, in turn, implies that

$$| \Delta_i K \Delta_i | < 1, \qquad i = 1, 2, \ldots, N.$$

Applying Lemma 2, $I - P_1KP_1$, and $I - \Delta_i K \Delta_i$, $i = 1, 2, \ldots, N$, are invertible. Moreover, from Lemma 3, one has

$$(I - P_1KP_1)^{-1} = \sum_{n=0}^{\infty} (P_1KP_1)^n \tag{4}$$

and

$$(I - \Delta_i K \Delta_i)^{-1} = \sum_{n=0}^{\infty} (\Delta_i K \Delta_i)^n. \tag{5}$$

From Eqs. (4) and (5), we can apply iteratively, Lemma 4 and obtain that the series $\sum_{n=0}^{\infty} (P_iKP_i)^n$ is uniformly convergent for each $i = 2, 3, \ldots, N$. For $i = N$, this implies the convergence of $\sum_{n=0}^{\infty} K^n$.

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