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On the Minimal Degrees of Projective Representations of the Finite Chevalley Groups

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1. INTRODUCTION

For $G = G(q)$, a Chevalley group defined over the field $\mathbb{F}_q$ of characteristic p, let $l(G, p)$ be the smallest integer $t > 1$ such that G has a projective irreducible representation of degree t over a field of characteristic other than p. In this paper we present lower bounds for the numbers $l(G, p)$. As a corollary we determine those Chevalley groups having an irreducible complex character of prime degree. Recently there have been a number of results making use of lower bounds on the degrees of representations of Chevalley groups. See for example Curtis, Kantor, and Seitz [4], Hering [9], and Patton [11]. Also in Fong and Seitz [7] such bounds played an important role, although there the representations considered were over fields of characteristic p.

For most types of Chevalley groups and for most primes p it is not difficult to obtain reasonable lower bounds for the complex irreducible characters of $G = G(q)$, using the existence of certain p-subgroups of G resembling extraspecial groups. Indeed this was carried out in Landazuri [10]. However to be complete we must take into certain problems that occur with fields of characteristic 2 and 3. Also, since we are considering projective irreducible representations the groups with exceptional Schur multipliers present some difficulties. There is also the problem of deciding whether or not a lower bound is "good." In some cases our bounds are actually attained and there is no problem in this regard. Otherwise let $\{G(q)\}$ be a family of Chevalley groups of given type and with q ranging over suitable prime powers. Then our bounds will be in the form of a polynomial in q. In Curtis, Iwahori, and Kilmoyer [3] there is a list of certain character degrees for the family $\{G(q)\}$

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which are also polynomials in q. For most cases the degree of the polynomial $l(G(q), p)$ will equal that of one of the polynomials in [3]. We were also guided by the needs of Hering [9] in obtaining our bounds, as he uses the results in this paper.

Throughout the paper we use the term Chevalley group to mean a group (of normal or twisted type), $G = G(q)$, generated by its root subgroups and having trivial center. Once we have a bound $l(G, q)$ we will also have the same bound for all groups of Chevalley type $\hat{G}$ such that $G \leqslant \hat{G}/Z(\hat{G}) \leqslant \mathrm{Aut}(G)$ as long as we only consider representations not having $\bar{G}'$ in the kernel, where $\bar{G}/Z(\hat{G}) = G$.

THEOREM. *If $G = G(q)$ is a Chevalley group then a lower bound for $l(G, p)$ is given in the following table.*

$G(q)$	Bound	Exceptions
$PSL(2, q)$	$(1/d)(q-1)$, $d = (2, q-1)$	$l(PSL(2, 4), 2) = 2$, $l(PSL(2, 9), 3) = 3$
$PSL(n, q)$, $n > 2$	$q^{n-1} - 1$	$l(PSL(3, 2), 2) = 2$, $l(PSL(3, 4), 2) = 4$
$PSp(2n, q)$, $n \geqslant 2$	$\frac{1}{2}(q^n - 1)$, q odd; $\frac{1}{2}q^{n-1}(q^{n-1} - 1)(q - 1)$, q even	$l(PSp(4, 2)', 2) = 2$, $l(PSp(6, 2), 2) = 7$
$PSU(n, q)$, $n \geqslant 3$	$q(q^{n-1} - 1)/(q + 1)$, n odd; $(q^n - 1)/(q + 1)$, n even	$l(PSU(4, 2), 2) = 4$, $l(\mathrm{PSU}(4, 3), 3) = 6$
$PSO^+(2n, q)'$, $n \geqslant 4$	$(q^{n-1} - 1)(q^{n-2} + 1)$, $q \neq 2, 3, 5$; $q^{n-2}(q^{n-1} - 1)$, $q = 2, 3$, or 5	$l(PSO^+(8, 2), 2) = 8$
$PSO^-(2n, q)'$, $n \geqslant 4$	$(q^{n-1} + 1)(q^{n-2} - 1)$	
$PSO(2n + 1, q)'$, $n \geqslant 3$ and q odd	$q^{2(n-1)}_{-1}$, $q > 5$; $q^{n-1}(q^{n-1} - 1)$, $q = 3$ or 5	$l(PSO(7, 3)', 3) \geqslant 27$
$E_6(q)$	$q^9(q^2 - 1)$	
$E_7(q)$	$q^{15}(q^2 - 1)$	
$E_8(q)$	$q^{27}(q^2 - 1)$	
$F_4(q)$	$q^4(q^6 - 1)$, q odd; $\frac{1}{2}q^7(q^3 - 1)(q - 1)$, q even	$l(F_4(2), 2) \geqslant 44$
${}^2E_6(q)$	$q^8(q^4 + 1)(q^3 - 1)$	$l({}^2E_6(2), 2) \geqslant 3 \cdot 2^9$
$G_2(q)$	$q(q^2 - 1)$	$l(G_2(3), 3) \geqslant 14$
${}^3D_4(q)$	$q^3(q^2 - 1)$	
${}^2F_4(q)$	$(q/2)^{1/2}\, q^4(q - 1)$	
$Sz(q)$	$(q/2)^{1/2}(q - 1)$	$l(Sz(8), 2) \geqslant 8$
${}^2G_2(q)$	$q(q - 1)$	

COROLLARY. *Let $G(q)$ be a Chevalley group and suppose that $G(q)$ has a complex irreducible character χ such that $\chi(1)$ is prime. Then one of the following holds*:

(a) $G(q) = PSL(2, q)$ *and* $\chi(1) = q, \frac{1}{2}(q \pm 1)$, *or* $q \pm 1$,

(b) $G(q) = PSL(n, 2)$ *and* $\chi(1) = 2^{n-1} - 1$,

(c) $G(q) = PSL(n, q)$ *and* $\chi(1) = q^n - 1/q - 1$,

(d) $G(q) = PSp(2n, q)$, *q odd, and* $\chi(1) = \frac{1}{2}(q^n \pm 1)$,

(e) $G(q) = PSp(6, 2)$ *and* $\chi(1) = 7$,

(f) $G(q) = PSU(n, q)$, *n odd, and* $\chi(1) = q^n + 1/q + 1$,

(g) $G(q) = PSU(3, 2)$, $\chi(1) = 2$,

(h) $G(q) = PSU(4, 2)$, $\chi(1) = 5$.

Proof. We illustrate the idea as follows. Suppose $G(q) = PSp(2n, q)$ with q odd. Then $\frac{1}{2}(q^n - 1) \leqslant \chi(1) \mid |G(q)|$ and $|G(q)|$ divides

$$q^{n^2}(q^{2n} - 1)(q^{2(n-1)} - 1) \cdots (q^2 - 1)$$
$$= q^{n^2}(q^n - 1)(q^n + 1)(q^{n-1} - 1)(q^{n-1} + 1) \cdots (q - 1)(q + 1).$$

It follows that $\chi(1) \mid q^n - 1$ or $\chi(1) \mid q^n + 1$. Write $t\chi(1) = q^n - 1$ or $t\chi(1) = q^n + 1$ and obtain $t(\frac{1}{2}(q^n - 1)) \leqslant q^n + 1$. As $PSp(2, q) \cong PSL(2, q)$, we may assume $n > 2$. It follows that $t = 1, 2$ and since $\chi(1)$ is prime, $\chi(1) = \frac{1}{2}(q^n \pm 1)$. The other cases are similar. For the exceptional groups listed in the table it is handy to use the list of finite subgroups of $GL(k, C)$ for $1 \leqslant k \leqslant 7$ listed in [5].

We remark that for q odd $PSL(n, q)$ has an irreducible character of degree $q^n - 1/q - 1$ and that $PSp(2n, q)$ does have irreducible characters of degree $\frac{1}{2}(q^n \pm 1)$. Also $PSp(6, 2)$ has an irreducible character of degree 7. As in the Corollary the bounds presented in the theorem can be used to investigate characters of Chevalley groups having small degree relative to a fixed prime divisor r, of $|G(q)|$. For example, one could investigate characters of degree $r + 1$ or $2r$.

For most of the exceptions in the table $m_p(G(q)) \neq 1$ (Schur multiplier), and the lower bound given is a lower bound for the degree of a projective representation of $G(q)$ such that p divides the order of the center of the representation group. The lower bounds for $PSL(2, q)$, $PSp(2n, q)$ q odd, $PSU(3, q)$, $Sz(q)$, and ${}^2G_2(q)$ are known to be best possible, as are the bounds for the indicated exceptional groups.[1]

[1] *Note added in proof*: The second author has shown that the bounds for $PSU(m, q)$ are also best possible.

The outline of the paper is as follows. In Section 2 we present preliminary results and show how to construct groups resembling extraspecial groups. This is carried out using properties of root systems. In Section 3 we prove the theorem for certain families of groups where we make use of large abelian subgroups of G. Then in Section 4 we handle all the other Chevalley groups $G = G(q)$ satisfying $m_p(G) = 1$. In this section we make use of the extraspecial groups as well as other methods. Finally Section 5 treats the finite number of Chevalley groups having exceptional Schur multipliers.

We assume the reader is familiar with the basic properties of Chevalley groups and root systems. At certain times we need detailed information on the structure of certain parabolic subgroups. This information either follows easily from the commutator relations or can be found in [4] or [7].

If $G = G(q)$ is a Chevalley group defined over $\mathbb{F}_q$, then associated with G is a root system Δ. Let B be a Borel subgroup of G, and $U = O_p(B)$. Then $B = UH$ with H an abelian p'-group. The Weyl group $W = N/H$ is a group generated by reflections $s_1,..., s_n$ and W acts on the root system Δ. Where there is no problem with coset representatives we will consider $s_1,..., s_n$ as elements in G. Let w_0 be the element of W having greatest length as a word in $s_1,..., s_n$. Next choose a fundamental system of positive roots $\alpha_1,..., \alpha_n$ of Δ, and define $U_{\alpha_i} = U \cap U^{w_0 s_i}$. If $r \in \Delta$ and $(\alpha_i)w = r$ for some $w \in W$, we write $U_r = (U_{\alpha_i})^w$. Then U_r is well-defined and is the root subgroup of G associated with the root r. For convenience we will write $U_i = U_{\alpha_i}$.

2. Preliminaries

Lemma 2.1. *Let G be a perfect group, F a field and suppose that $l(F)$ is the smallest integer $t > 1$ such that G has a projective irreducible F-representation of degree t. If $F < K$, then $l(F) \geqslant l(K)$.*

Proof. Suppose V is a representation space of degree $l(F)$ of an irreducible projective representation of G. Then there is central extension $\bar{G}$ of G such that $\bar{G}$ acts irreducibly on V. $K \otimes_F V$ is a representation module of degree $l(F)$ for $\bar{G}$ over K. If W is an irreducible submodule of $K \otimes_F V$ then $\dim(W) > 1$. For suppose $\dim(W) = 1$. Then $\bar{G}'$ acts trivially on W and it follows that $\bar{G}'$ is trivial on a subspace of V. As V is irreducible, $\bar{G}'$ is trivial on V and $\dim(V) = 1$, a contradiction. We now have $l(F) \geqslant \dim(W) \geqslant l(K)$, proving the lemma.

Lemma 2.1 shows that in considering minimal degrees of projective irreducible representations we may assume that the field is algebraically closed.

Lemma 2.2. *Let V be an n-dimensional vector space over a field $\mathbb{F}_q$, $q = p^a$.*

Let F be an algebraically closed field of characteristic other than p. If φ is a nontrivial linear character of V over F, then $\ker(\varphi)$ *contains a unique hyperplane of V.*

Proof. If V_0 is a hyperplane in V then there are precisely $q - 1$ nontrivial linear characters φ of V having $V_0 \leqslant \ker(\varphi)$. There are $(q^n - 1)/(q - 1)$ hyperplanes in V and no nontrivial linear character of V can have two distinct hyperplanes in its kernel. So there are $q^n - 1$ nontrivial linear characters φ of V having a unique hyperplane in $\ker(\varphi)$. As $|V| = q^n$, this proves the lemma.

DEFINITION. A p-group Q is of extraspecial-type if $1 < Z(Q) = Q' = \Phi(Q)$ and $Z(Q/Q_0) = Z(Q)/Q_0$ whenever $1 < Q_0 < Z(Q)$.

Remarks.

(1) Q is of extraspecial type if and only if $1 < Z(Q) = Q' = \Phi(Q)$ and $[g, Q] = Z(Q)$ for all $g \in Q - Z(Q)$.

(2) If Q is of extraspecial type, then $Z(Q)$ is elementary.

LEMMA 2.3. *Suppose Q is of extraspecial-type, $|Q| = p^{r+s}$ and $|Z(Q)| = p^s$. If F is algebraically closed and* $\operatorname{char} F = 0$ *or* $(\operatorname{char} F, q) = 1$, *then Q has exactly p^r linear characters over F and $p^s - 1$ nonlinear irreducible characters over F. Moreover r is even, each nonlinear irreducible character χ has degree $p^{r/2}$, and χ vanishes off $Z(Q)$.*

Proof. Suppose χ is a nonlinear irreducible character of Q over F. As $Z(Q)$ is elementary $Q_0 = Z(Q) \cap \ker \chi$ has index p in $Z(Q)$. We consider χ as an irreducible character of the extraspecial group $\bar{Q} = Q/Q_0$. Let $\bar{g} \in \bar{Q} - Z(\bar{Q})$. There exists an $\bar{h} \in \bar{Q}$ such that $[\bar{g}, \bar{h}] \neq 1$. Since $[\bar{g}, \bar{h}] \in Z(\bar{Q})$, $\chi(\bar{g}) = \chi(\bar{g}^{\bar{h}}) = \chi(\bar{g}[\bar{g}, \bar{h}]) = \alpha \cdot \chi(\bar{g})$ where $1 \neq \alpha \in F$. Thus $\chi(\bar{g}) = 0$ and χ vanishes on $\bar{Q} - Z(\bar{Q})$. We then have

$$p^{r+1} = |\bar{Q}| = \sum_{\bar{g} \in \bar{Q}} |\chi(\bar{g})|^2 = \sum_{\bar{g} \in Z(\bar{Q})} |\chi(\bar{g})|^2 = p\chi(1)^2,$$

and $\chi(1) = p^{r/2}$. As χ is determined by its action on $Z(Q)$, the lemma follows.

Next we indicate a general procedure for finding a p-group Q of extraspecial type in Chevalley groups defined over fields of characteristic p. These subgroups have the form $O_p(P)$ for P a suitable parabolic subgroup of G.

Let $G = G(q)$ be a Chevalley group defined over $\mathbb{F}_q$ generated by its root subgroups and such that $Z(G) = 1$. Let W be the Weyl group of G and Δ the associated root system. (We exclude $G = {}^2F_4(q)$ or $PSU(n, q)$, n odd.)

Let r be the root of highest height in Δ, and let $w_r \colon x \to x - 2[(x, r)/(r, r)]r$. We define

$$R(r) = \{s \in \Delta^+ \colon w_r(s) \neq s\}.$$

LEMMA 2.4 (Lemma 1, Section 2 of [10]).

(1) *$r \in R(r)$.*

(2) *If $s, t \in R(r)$ and $s + t \in \Delta^+$, then $s + t \in R(r)$.*

(3) *For each $r \neq s \in R(r)$, there exists a unique $t \in R(r)$ such that $s + t \in R(r)$. For this t, $s + t = r$.*

Proof. (1) follows from $w_r(r) = -r$. Let $s \in \Delta^+$. Then

$$w_r(s) = s - 2[(s, r)/(r, r)]r$$

and $s \in R(r)$ if and only if $(s, r) \neq 0$. Moreover if $r \neq s \in R(r)$, then $0 \neq 2[(s, r)/(r, r)] = p - q$ where p, q satisfy $s - pr, \ldots, s, \ldots, s + qr$ are roots and $s - (p + 1)r$, $s + (q + 1)r$ are not roots. Since r is of highest height, $q = 0$. If $p \geqslant 2$, then $s - 2r \in \Delta^-$ and $2r - s \in \Delta^+$. This contradicts the fact that r has highest height. Therefore $p = 1$. So for $s \in R(r)$, $w_r(s) = s - r$ and $r - s \in \Delta^+$. Also if $\{\alpha_1, \ldots, \alpha_n\}$ is a fundamental system of roots of Δ, then $\alpha_k \in R(r)$ if and only if $2[(\alpha_k, r)/(r, r)] = 1$.

Let $s, t \in R(r)$ and $s + t \in \Delta^+$. Then $w_r(s + t) = (s - r) + (t - r) = (s + t) - 2r \neq s + t$. Thus $s + t \in R(r)$ and (2) holds.

Suppose $r \neq s \in R(r)$, $s = \sum a_i \alpha_i$. Then

$$1 = 2[(s, r)/(r, r)] = \sum a_i[2(\alpha_i, r)/(r, r)].$$

As $2(\alpha_i, r)/(r, r)$ is a non-negative integer there is a unique $a_k \neq 0$ such that $\alpha_k \in R(r)$. For this k, $a_k = 1$.

We now prove (3). Let $r \neq s \in R(r)$. Then $r - s \in \Delta^+$ and setting $t = r - s$ we have $t \in R(r)$ and $s + t = r$. Suppose $s + \lambda \in R(r)$, $t \neq \lambda \in R(r)$. Then $r \neq s + \lambda$. Applying the results of the last paragraph we get a contradiction.

LEMMA 2.5. *Let $G = G(q)$ be a Chevalley group of normal type. If the Dynkin diagram of Δ has a double bond assume $(2, q) = 1$, and if Δ is of type G_2 assume $(3, q) = 1$. If $|R(r)| = l$, then $Q = \langle U_s : s \in R(r) \rangle$ is of extra-special-type, $|Q| = q^l$, and $|Z(Q)| = q$.*

Proof. This is an easy consequence of the Chevalley commutator identities and Lemma 2.4. Indeed Q is the central product of the subgroups $U_s U_{r-s} U_r$ for $r \neq s \in R(r)$ and each of these is of extraspecial-type and of order q^3.

3. THE ABELIAN CASE

Clearly Lemma 2.5 together with Lemma 2.3 can be used to obtain lower bounds for $l(G, p)$ for many Chevalley groups G. Indeed in Section 4 p-groups of extraspecial type will be used as the basis of an inductive procedure for obtaining lower bounds for $l(G, p)$ for certain Chevalley groups G.

However for some of the classical groups another method gives a better bound. These groups are handled in this section.

LEMMA 3.1. *Let* $G = PSL(n, q)$ *and assume* $m_p(G) = 1$. *If* $n = 2$, *then* $l(G, p) \geqslant (1/d)(q-1)$, *where* $d = (2, q-1)$. *If* $n > 2$, *then* $l(G, p) \geqslant q^{n-1} - 1$.

Proof. G permutes the 1-dimensional subspaces of an n-dimensional vector space V over $\mathbb{F}_q$. Let P be the stabilizer of a fixed 1-space of V. Then P is a parabolic subgroup of G. There is a normal elementary subgroup Q of P with $|Q| = q^{n-1}$. Suppose $n > 2$. Then P contains a subgroup RH_0 where $R \simeq SL(n-1, q)$, H_0 is cyclic of order $q-1$, $[R, H_0] = 1$, and RH_0 acts faithfully on Q, with H_0 inducing scalar multiplication. Also RH_0 is transitive on $Q^{\#}$. If $n = 2$, then P is Frobenius of order $(1/d)\, q(q-1)$, $d = (2, q-1)$.

Now suppose M is a representation module of a projective irreducible representation of G over a field F of characteristic 0 or relatively prime to q. Then there is a perfect central extension $\bar{G}$ of G such that $\bar{G}$ acts irreducibly on M. By hypothesis $Z(\bar{G})$ is a p'-subgroup. So if $\bar{Q}/Z(\bar{G}) = Q$, then $\bar{Q} = Q_0 \times Z(\bar{G})$, where Q_0 is $\overline{RH}_0$-isomorphic to Q.

By Lemma 2.1 we may assume F is algebraically closed. Clearly Q_0 is not contained in the kernel of $\bar{G}$ on M. So there is some Q_0-submodule M_0 of M such that M_0 affords a nontrivial 1-dimensional representation of Q_0. Suppose $n > 2$. As RH_0 is transitive on $Q^{\#}$, the preimage of RH_0 in $\bar{G}$ is transitive on $Q_0^{\#}$ and hence transitive on the nontrivial irreducible representations of $Q_0^{\#}$ (see [2], Lemma 1). Thus $\dim(M) \geqslant q^{n-1} - 1$. If $n = 2$, the preimage of P has 1 or 2 nontrivial orbits on $Q_0^{\#}$, of length $(1/d)(q-1)$, $d = (2, q-1)$. Hence $\dim(M) \geqslant (1/d)(q-1)$.

LEMMA 3.2. *Let* $G = PSp(2n, q)$ *with* q *even, and suppose* $m_2(G) = 1$. *Then* $l(G, p) \geqslant \frac{1}{2}q^{n-1}(q-1)(q^{n-1}-1)$.

Proof. Let V be the natural $2n$-dimensional vector space over $\mathbb{F}_q$ for $Sp(2n, q)$. Then G permutes the 1-spaces of V and we let P be the stabilizer of a fixed 1-space. Then the structure of P is known (e.g., see [4], Section 3). There is a normal elementary subgroup Q of order q^{2n-1} and $P = Q(R \times H_0)$, where $R \simeq Sp(2(n-1), q)$, H_0 is cyclic of order $q-1$, and H_0 acts fixed-point-free on Q. Also if r is the root of highest height in Δ^+, then $P = N_G(U_r)$, $U_r \leqslant Q$, Q is indecomposable under the action of R, and R acts on Q/U_r in the usual way. We also note that $G = PSp(2n, q) \simeq PSO(2n+1, q)'$, and Q has a vector space structure on which R acts as $SO(2n-1, q)'$, U_r is the radical, and H_0 induces scalar action.

We now proceed as in (3.1). Let M be a faithful irreducible representation module over an algebraically closed field of odd characteristic for a perfect central extension $\bar{G}$ of G.

By hypothesis $\bar{Q} = Q_0 \times Z(\bar{G})$, $\bar{U}_r = U_0 \times Z(\bar{G})$ and Q_0 is P-isomorphic to Q. Let Z be an irreducible $\bar{P}$-composition factor of M with $U_0 \nleqslant \ker(Z)$. Then $Z \mid Q_0 = Z_1 \oplus \cdots \oplus Z_k$ with the Z_i's homogeneous and permuted transitively by $\bar{P}$. Actually $Z(\bar{G})$ is represented as scalar multiplication on Z, so we can consider P as permuting the Z_i's. Let $\bar{L}$ be the stabilizer of Z_1 and $L = \bar{L}/Z(\bar{G})$.

By Lemma 2.2 L stabilizes the unique hyperplane Q_1 of Q_0 contained in $\ker_{Q_0}(Z_1)$. Then $Q_0 = Q_1 \times U_0$ and Q_1, U_0 are L-invariant. Suppose $\bar{r}\bar{h} \in \bar{L}$, where $\bar{r} \in \bar{R}$ and $\bar{h} \in \bar{H}_0$. Then $\bar{r}$ stabilizes Q_1 and since $R \simeq SO(2n-1, q)'$, $\bar{r}$ is trivial on Q_0/Q_1. Now $\bar{r}\bar{h}$ fixes Z_1 and hence is trivial on $Q_0/\ker_{Q_0}(Z_1)$. However H_0 is fixed-point-free on Q_0. It follows that $\bar{h} \in Z(\bar{G})$ and $\bar{L} \leqslant \bar{R}$.

With respect to the quadratic form on Q_0, $Q_1^{\perp} \geqslant U_0$ and so $Q_1^{\perp} = U_0$ and Q_1 is nondegenerate. Then $\bar{L}$ is contained in a subgroup of $SO(2n-1, q)'$ isomorphic to $O^{\pm}(2n-2, q)$. Checking orders we have

$$| \bar{R} : \bar{L} | \geqslant \tfrac{1}{2}q^{n-1}(q^{n-1}-1) \qquad \text{and} \qquad | \bar{P} : \bar{L} | \geqslant \tfrac{1}{2}q^{n-1}(q^{n-1}-1)(q-1).$$

Thus $\dim(Z) \geqslant \frac{1}{2}q^{n-1}(q^{n-1}-1)(q-1)$ and the lemma is proved.

LEMMA 3.3. *Let $G = PSO^{\pm}(2n, q)'$ with $n \geqslant 4$ or $PSO(2n+1, q)'$ with $n \geqslant 3$ and q odd. Assume that $m_p(G) = 1$.*

(1) *If $G = PSO^{+}(2n, q)'$ and $q \neq 2, 3, 5$, then*
$$l(G, p) \geqslant (q^{n-1}-1)(q^{n-2}+1).$$

(2) *If $G = PSO^{+}(2n, q)'$ with $q = 2, 3$, or 5, then*
$$l(G, p) \geqslant q^{n-2}(q^{n-1}-1).$$

(3) *If $G = PSO^{-}(2n, q)'$, then $l(G, p) \geqslant (q^{n-1}+1)(q^{n-2}-1)$.*

(4) *If $G = PSO(2n+1, q)'$ and $q > 5$, then $l(G, p) \geqslant q^{2(n-1)}-1$.*

(5) *If $G = PSO(2n+1, q)'$ and $q = 3$ or 5, then $l(G, q) \geqslant q^{n-1}(q^{n-1}-1)$.*

Proof. Let V be the natural orthogonal space corresponding to G and let $P \leqslant G$ be the stabilizer of an isotropic 1-space V_1 of V. We first describe the structure of P. The group P contains a normal elementary subgroup Q of order q^l, where $l = \dim(V) - 2$. Write $V = V_0 \perp V_2$ where V_2 is a hyperbolic plane containing V_1, and decompose V_0 as $V_0 = V_3 \perp V_4$, where V_4 is a hyperbolic plane. Then $P = QRH_1$ where R is the subgroup of $SO(V)'$ that is trivial on V_2 and that induces the group $SO(V_0)'$ on V_0, and $H_1 = \langle h \rangle$ is cyclic of order $q-1$ and normalizes R. The element h can be described as follows. h is trivial on V_0 and on both V_2 and V_4 h induces the matrix $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)$ where $\langle \alpha \rangle = F_q^{\#}$ and where the matrix is given with respect to fixed hyperbolic pairs for V_2 and V_4. Clearly H_1 is fixed-point-free on V_2. Also $h^2 = h_4 h_2$ where $h_i \in SO(V_i)'$, so that $R\langle h^2 \rangle = R\langle h_2 \rangle$ and $[R, \langle h_2 \rangle] = 1$. The group $H_0 = \langle h_2 \rangle$ centralizes R and hence induces scalar action on Q. R acts on Q

preserving a nondegenerate quadratic form. Finally, suppose q is odd and t is the involution in H_1. It is easily seen that either $SO(V_3)'$ and $SO(V)'$ both have trivial centers or they both have centers of order 2. In the latter case, the product of the involution in $Z(SO(V_3)')$ and t is the involution in $Z(SO(V)')$, so that in $G = PSO(V)'$, these involutions are the same. So if $Z(SO(V)') > 1$, then $|R \cap H_1| = 2$.

Let M be a faithful irreducible representation module over an algebraically closed field of characteristic different from p for a perfect central extension $\bar{G}$ of G.

Let Z be an irreducible $\overline{RQ}$-composition factor of M, such that $Q_0 \not\leqslant \ker(Z)$, where $\overline{RQ}/Z(\bar{G}) = QR$, $\bar{Q}/Z(\bar{G}) = Q$, and $\bar{Q} = Q_0 \times Z(G)$. Then $Z \mid Q_0 = Z_1 \oplus \cdots \oplus Z_k$ where the Z_i's are the distinct homogeneous components of Q_0 on Z. Let $\bar{L}$ be the stabilizer in $\overline{RQ}$ of Z_1. Clearly $\bar{Q} \leqslant \bar{L}$, so set $L = \bar{L}/\bar{Q} \leqslant R$. By 2.2 $\ker_{Q_0}(Z_1)$ contains a unique hyperplane Q_1 of Q_0. Thus $\bar{L}$ stabilizes Q_1 (here we use the fact that $\bar{L}$ induces a subgroup of R on Q_0).

Suppose that $\operatorname{rad}(Q_1) \neq 0$. Then $\operatorname{rad}(Q_1)$ is an isotropic 1-space of Q_0 stabilized by $\bar{L}$ and we can determine the subgroup L of R. We have described the stabilizer, T, in R of $\operatorname{rad}(Q_1)$. In particular it follows from that discussion that T contains a normal subgroup T_0 having index $q-1$ in T such that T_0 is trivial on $Q_0/(\operatorname{rad}(Q_1))^\perp$ and T/T_0 is fixed-point-free on $Q_0/\operatorname{rad}(Q_1))^\perp$. Now L fixes Z_1, so L is trivial on $Q_0/\ker_{Q_0}(Z_1)$. This implies that $L \leqslant T_0$ and consequently $\dim(M) \geqslant k = |R:L| \geqslant (q-1)\,|R:T|$. If $G = PSO^+(2n, q)'$, $PSO^-(2n, q)'$, or $PSO(2n+1, q)'$, then $(q-1)\,|R:T|$ is, respectively $(q^{n-1}-1)(q^{n-2}+1)$, $(q^{n-1}+1)(q^{n-2}-1)$, or $(q^{2(n-1)}-1)$. So we are done if $\operatorname{rad}(Q_1) \neq 0$.

Now suppose $\operatorname{rad}(Q_1) = 0$. Since $\overline{QR} \trianglelefteq \bar{P}$, there is an irreducible $\bar{P}$-composition factor, Z', M such that Z is $\overline{RQ}$-isomorphic to a factor of Z'. Write $Z' \mid Q_0 = Z_1' \oplus \cdots \oplus Z_m'$, where the Z_i' are homogeneous. We may assume that Z_1 is isomorphic to a factor of Z_1'. Then Q_1 is the unique hyperplane of Q_0 contained in $\ker_{Q_0}(Z_1')$. Let $\bar{L}_1$ be the stabilizer in $\bar{P}$ of Z_1', so that $\bar{L}_1 \cap \bar{R} = \bar{L}$. Set $L_1 = \bar{L}_1/Z(\bar{G})$ and $L_0 = L_1 \cap RH_0$, where H_0 is as in the first paragraph.

We have $Q_0 = Q_1 \perp Q_1^\perp$, with $Q_1^\perp$ an anisotropic 1-space. Suppose $rh \in L_0$, with $r \in R$ and $h \in H_0$. As h is scalar on Q_0, r stabilizes Q_1, and hence r stabilizes $Q_1^\perp$. Consequently r induces ± 1 on $Q_1^\perp$. However rh is trivial on $Q_0/\ker_{Q_0}(Z_1')$, so rh is trivial on $Q_1^\perp$ and h induces ± 1 on Q_0. Thus $L_0 \leqslant R\langle t\rangle$, where $t = 1$ if $4 \nmid q-1$ and t is the involution in H_0 otherwise. Thus $\dim(M) \geqslant m = |\bar{P}:\bar{L}| \geqslant |RH_0 : L_0| = |RH_0 : R\langle t\rangle|\,|R\langle t\rangle : L_0|$.

If q is even then $t = 1$, $|RH_0 : R| = q-1$, and $L_0 \leqslant R$. In R the stabilizer of Q_1 is $SO(Q_1)'$. Since q is even, $G = PSO^\pm(2n, q)'$, $|R| = q^{(n-1)(n-2)}(q^{n-1} \mp 1)(q^{2(n-2)}-1)\cdots(q^2-1)$ and

$$|SO(Q_1)'| = q^{(n-2)^2}(q^{2(n-2)}-1)\cdots(q^2-1).$$

Thus $\dim(M) \geqslant (q-1)q^{n-2}(q^{n-1} \mp 1)$. So $\dim(M) \geqslant (q^{n-1} \mp 1)(q^{n-2} \pm 1)$ unless $q = 2$ and $G = PSO^+(2n, q)$. In the last case the bound in (2) holds. From now on we may assume that q is odd.

Suppose $G = PSO^+(2n, q)'$. Then $R \simeq SO^+(2(n-1), q)'$ and $|R| = \frac{1}{2}q^{(n-1)(n-2)}(q^{n-1}-1)(q^{2(n-2)}-1)\cdots(q^2-1)$. If $4 \mid q^n - 1$, then $Z(SO(V)') > 1$ and as mentioned earlier $|R \cap H_1| = 2$. It follows that $t \in R$. If $4 \nmid q^n - 1$, then $4 \nmid q - 1$ and $|H_0|$ is odd. So in either case $t \in R$ and $L_0 \leqslant R$. Now L_0 acts trivially on $Q_1^{\perp}$ and induces a subgroup of $SO^{\pm}(Q_1)'$, on Q_1. Thus $|L_0| \leqslant \frac{1}{2}q^{(n-2)^2}(q^{2(n-2)} - 1)\cdots(q^2-1)$, and $|R\langle t\rangle : L_0| = |R : L_0| \geqslant q^{n-2}(q^{n-1}-1)$. Thus $\dim(M) \geqslant |RH_0 : R|\, q^{n-2}(q^{n-1}-1)$, and $|RH_0 : R| = \frac{1}{2}(q-1)$ or $\frac{1}{4}(q-1)$, depending on whether $4 \nmid q-1$ or $4 \mid q-1$. Consequently the bound in (1) holds if $q > 5$ and the bound in (2) holds if $q \leqslant 5$. This proves the lemma for the case $G = PSO^+(2n, q)'$.

Next suppose that $G = PSO^-(2n, q)'$. Then $R \simeq SO^-(2(n-1), q)'$ and $|R| = \frac{1}{2}q^{(n-1)(n-2)}(q^{n-1}+1)(q^{2(n-2)}-1)\cdots(q^2-1)$. If $4 \mid q+1$, then $|H_0| = \frac{1}{2}(q-1)$ is odd and $1 = t \in R$. If $4 \nmid q+1$, then $1 \neq t$ and $t \notin R$. In the first case we proceed as above and get $\dim(M) \geqslant \frac{1}{2}(q-1)q^{n-2}(q^{n-1}+1)$, so that the bound in (c) holds. Suppose then that $4 \nmid q+1$, so that $R\langle t\rangle = R \times \langle t\rangle$. Then $L_0 \leqslant R_1\langle t\rangle$, where R_1 is the stabilizer in R of Q_1. Let $R_2 \leqslant R_1$ be the kernel of the action of R_1 on $Q_1^{\perp}$. Then $|R_1 : R_2| = 2$, R_2 is trivial on $Q_1^{\perp}$, and R_2 induces $SO(Q_1)'$ on Q_1. Thus

$$|R_1 \times \langle t\rangle| = 4\,|R_2| = 2q^{(n-2)^2}(q^{2(n-2)}-1)\cdots(q^2-1).$$

Also $t \notin L_0$, so $|L_0 \times \langle t\rangle : L_0| = 2$. Then $\dim(M) \geqslant |RH_0 : R\langle t\rangle|\,|R \times \langle t\rangle : R_1 \times \langle t\rangle|\,|R_1 \times \langle t\rangle : L_0 \times \langle t\rangle|\,|L_0 \times \langle t\rangle : L_0| \geqslant \frac{1}{4}(q-1)(\frac{1}{2})q^{n-2}(q^{n-1}+1)2 = \frac{1}{4}(q-1)q^{n-2}(q^{n-1}+1) \geqslant (q^{n-1}+1)(q^{n-2}-1)$. Again we have the bound in (c) holding, proving the lemma for $G = PSO^-(2n, q)'$.

The last case is $PSO(2n+1, q)'$. Here $R \simeq SO(2(n-1)+1, q)'$ and

$$|R| = (1/2)\, q^{(n-1)^2}(q^{2(n-1)}-1)\cdots(q^2-1).$$

If $t = 1$, then $L_0 \leqslant R$ and $|RH_0 : R| = \frac{1}{2}(q-1)$. Then L_0 is trivial on $Q_1^{\perp}$ and induces a subgroup of $SO^{\pm}(2(n-1), q)'$ on Q_1. Consequently $|L_0| \leqslant (\frac{1}{2})q^{(n-1)(n-2)}(q^{n-1} \mp 1)(q^{2(n-2)}-1)\cdots(q^2-1)$. Therefore $\dim(M) \geqslant \frac{1}{2}(q-1)q^{n-1}(q^{n-1} \pm 1)$. This gives the bound in (4) unless $q = 3$ in which case the bound in (5) holds. Next suppose $t \neq 1$. Then $t \notin R$, $4 \mid q-1$, $R\langle t\rangle = R \times \langle t\rangle$ and we proceed as in the previous paragraph. Namely $L_0 < L_0 \times \langle t\rangle \leqslant R_1 \times \langle t\rangle$, and $|R_1 : R_2| = 2$ where $R_2 \simeq SO^{\pm}(2(n-1), q)'$. We obtain $\dim(M) \geqslant \frac{1}{4}(q-1)(\frac{1}{2})q^{n-1} \pm 1)2$. Since $4 \mid q-1$, $\frac{1}{4}(q-1) \geqslant 1$ and we obtain the bound in (4) or (5). This completes the proof of (3.3).

4. The Extraspecial Case

In this section we will use groups of extraspecial type together with other arguments to obtain the required bound for the groups $G = G(q)$ that have not yet been considered and that satisfy $m_p(G) = 1$. We first handle the rank 1 groups.

Lemma 4.1.

(a) *If* $G = PSL(2, q)$, $q \neq 9$, *then* $l(G, p) \geqslant (1/d)(q-1)$, *where* $d = (2, q-1)$.

(b) *If* $G = PSU(3, q), q > 2$, *then* $l(G, p) \geqslant q(q-1)$.

(c) *If* $G = Sz(q), q > 8$, *then* $l(G, p) \geqslant (q-1)(q/2)^{1/2}$.

(d) *If* $G = {}^2G_2(q)$, *then* $l(G, p) \geqslant q(q-1)$.

Proof. Let Q be a Sylow p-subgroup of G, so that $N(Q) = QH$ where H is cyclic. If $G = PSL(2, 2)$ or $PSL(2, 3)$ the result holds. Let $\bar{G}$ be a perfect central extension of G and let $\bar{G}$ act faithfully and irreducibly on a vector space M over an algebraically closed field of characteristic other than p. The assumptions on q imply $m_p(G) = 1$, and hence $\bar{Q} = Q_0 \times Z(\bar{G})$ where Q_0 is $\overline{N(Q)}$-isomorphic to Q. Write $M \mid Q_0 = M_1 \oplus \cdots \oplus M_k$ where the M_i's are the homogeneous Wedderburn components of Q_0 on M.

If $G = PSL(2, q)$, then Q_0 is elementary of order q and H has d orbits of equal size on $Q_0{}^{\#}$, where $d = (2, q-1)$. Since Q_0 is not trivial on M, $k \geqslant (1/d)(q-1)$ and (a) holds.

If $G = PSU(3, q)$, $q > 2$, then Q is of extraspecial-type of order q^3 and H is transitive on $Z(Q)^{\#}$. By Lemma 2.3 $i \neq j$ implies that $M_i \mid Z(Q_0)$ and $M_j \mid Z(Q_0)$ are inequivalent. Thus $\bar{H}$ permutes the M_j's and $k \geqslant q-1$. Moreover Lemma 2.3 implies that $\dim(M_j) \geqslant q$, and $\dim(M) \geqslant q(q-1)$, proving (b).

If $G = Sz(q)$ then we use the results in [12], Section 4 to obtain the structure of $Q_0 \cong Q$. We have $Q_0 = Z(Q_0) = \Phi(Q_0) = \Omega_1(Q_0)$. The elements of Q_0 can be labeled $g = g(\alpha, \beta)$ where α, $\beta \in \mathbb{F}_q$ and $g(\alpha, \beta)\, g(\gamma, \delta) = g(\alpha + \gamma, \alpha\gamma^{\theta} + \beta + \delta)$ where θ is the field automorphism $x \to x^r$, $r = 2^n$, and $q = 2^{2n+1}$. Then $[g(\alpha, \beta), g(\gamma, \delta)] = g(\alpha, \gamma\alpha^{\theta} - \alpha\gamma^{\theta})$. We claim that if $T \leqslant Z(Q_0)$ and $|Z(Q_0){:}\, T| = 2$, then Q_0/T is the central product of Z_4 with an extraspecial group of order q. To see this it suffices to show that $Z(Q_0/T)$ has order 4. As q is an odd power of 2, $Z(Q_0/T) > Z(Q_0)/T$. Suppose $g \in Q_0 - Z(Q_0)$ and $gT \in Z(Q_0/T)$. We will show that $Z(Q_0/T) = \langle gT, Z(Q_0)/T\rangle$. As H is transitive on $(Q_0/Z(Q_0))^{\#}$ we may assume that $g = g(1, 0)$. Then the above commutator relation implies that $T = \{g(0, \gamma - \gamma^{\theta}){:}\, \gamma \in \mathbb{F}_q\}$. Suppose that $\alpha \in \mathbb{F}_q{}^{\#}$ and $g(\alpha, \beta)T \in Z(Q_0/T)$. The same commutator relation shows that for each $\gamma \in \mathbb{F}_q$, there is a $\delta \in \mathbb{F}_q$ such that $\gamma\alpha^{\theta} - \alpha\gamma^{\theta} = \delta - \delta^{\theta}$. Note that $x^{\theta^2} = x^{1/2}$. Letting $\gamma = \alpha^{-\theta}$, we have $1 - \alpha^{1/2} = \delta - \delta^{\theta}$ and $\alpha = 1 + x$ where $x \in [\mathbb{F}_q, \theta] = \{\zeta - \zeta^{\theta} : \zeta \in \mathbb{F}_q\}$. Then for $\gamma \in \mathbb{F}_q$, $\gamma\alpha^{\theta} - \alpha\gamma^{\theta} = \gamma - \gamma^{\theta} + \gamma x^{\theta} - x\gamma^{\theta} = \delta - \delta^{\theta}$ for some δ. Thus $\gamma x^{\theta} - x\gamma^{\theta} \in [\mathbb{F}_q, \theta]$ for each γ. If $x \neq 0$, then as above $x = 1 + y$ for some $y \in [\mathbb{F}_q, \theta]$, and hence $1 \in [\mathbb{F}_q, \theta]$ a contradiction. Thus $x = 0$, $\alpha = 1$, and $g(\alpha, \beta)T = g(1, 0)T$. This proves the claim.

We may assume that $Z(Q_0)$ is nontrivial on M_1. Set $T = \ker_{Z(Q_0)}(M_1)$ and obtain $|Z(Q_0): T| = 2$. Then it follows from the claim and Lemma 2.3 that $\dim(M_1) \geqslant (q/2)^{1/2}$. Thus $\dim(M_1) \geqslant (q-1)(q/2)^{1/2}$ and (c) holds.

Finally we consider $G = {}^2G_2(q)$. The structure of Q is well-known. If $q > 3$, $|Q| = q^3$, $Q' = \Phi(Q)$, $|Q'| = q^2$, and $|Z(Q)| = q$. Also Q' is elementary and $Q' = Q'' \times Z(Q)$ where $Q'' = C_Q(t)$ and t is the involution in H. Finally $|H| = q - 1$, H is fixed-point-free on Q/Q' and on $Z(Q)$, and $H/\langle t \rangle$ is fixed-point-free on Q''. If $q = 3$, then $G \simeq P\Gamma L(2, 8)$, Q is metacyclic of order 3^3 and the result follows as in (b). So we may suppose $q > 3$. Let $\bar{Q}' = Q_1 \times Z(Q_0) \times Z(\bar{G})$, where $\bar{Q}_1 = Q''$, and consider

$$M \mid (Q_1 \times Z(Q_0)) = M_1 \oplus \cdots \oplus M_l$$

where the M_i's are the distinct homogeneous components of $Q_1 \times Z(Q_0)$. Since $\bar{G}$ acts faithfully on M, there are some M_i's not having $Z(Q_0)$ in its kernel. These M_i's are permuted by $\bar{Q}\bar{H}$. If Q_1 is in the kernel of each of these M_i's then $Q_1 \trianglelefteq Q_0$, which is not the case. So there is some M_i with neither Q_1 nor $Z(Q_0)$ in its kernel. Let φ be the character of $Q_1 \times Z(Q_0)$ afforded by an irreducible $(Q_1 \times Z(Q_0))$-submodule of M_i. Then $\varphi = \varphi_1\varphi_2$ where φ_1 is a nontrivial linear character of Q_1 and φ_2 is a nontrivial linear character of $Z(Q_0)$. Suppose $\varphi^g = \varphi$ where $g = xh$ for $x \in \bar{Q}$, $h \in \bar{H}$. Then h fixes the character φ_2 of $Z(Q_0)$ and so $h \in Z(\bar{G})$. Thus $g \in \bar{Q}$. The stabilizer K of M_i in $\bar{Q}\bar{H}$ stabilizes φ and $\bar{Q}' \leqslant K \leqslant \bar{Q}$. If $g \in \bar{Q} - \bar{Q}'$, then $[Q_1, g] = Z(Q_0)$ (see [13], Section 3). Thus there is an element $q_1 \in Q_1$ such that $[q_1, g] \notin \ker \varphi_2$. We then have $\varphi(q_1) = \varphi_1(q_1)$ and $\varphi(q_1{}^g) = \varphi(q_1[q_1, g]) = \varphi_1(q_1)\varphi_2([q_1, g])$ and $g \notin K$. Thus $K = \bar{Q}'$ and if follows that $l \geqslant q(q-1)$, proving (d).

Lemma 4.2. *Let $G = G(q)$ and let $U_r \leqslant U$ be a root subgroup such that $|Z(U_r)| = q$. If $G(q) \not\simeq PSp(2n, q)$ with q odd (we allow $n = 1$), then H is transitive on $Z(U_r)^{\#}$. Otherwise H has two orbits of length $\frac{1}{2}(q-1)$ on $Z(U_r)^{\#}$.*

Proof. If G has rank 1 this is easily checked. Otherwise let Δ be the root system of G and let $\alpha_1, \ldots, \alpha_n$ be a fundamental system for Δ. Then there is some $w \in W$ and $1 \leqslant i \leqslant n$ such that $(r)w = \alpha_i$. Then U_r is conjugate to U_{α_i} and we may assume $r = \alpha_i$. Let j be chosen such that $c\alpha_i + d\alpha_j$ is a root for some $c > 0$, $d > 0$, and set $L = \langle U_{\pm\alpha_i}, U_{\pm\alpha_j} \rangle$. Then L is a rank 2

Chevalley group and $L/Z(L) \cong PSL(3, q)$, $PSp(4, q)$, $PSU(4, q)$, $PSU(5, q)$, $G_2(q)$, ${}^3D_4(q)$, or ${}^2F_4(q)$. By direct check it can be seen that $H \cap L$ is transitive on $Z(U_{\alpha_i}^{\#})$ except in the case $L/Z(L) \cong PSp(4, q)$, q odd. Indeed for the classical groups this can be seen by considering the geometry. For the cases $G_2(q)$ and ${}^3D_4(q)$ this can be worked out from the commutator relations or using the calculations in ([7], Section 9). The case of ${}^2F_4(q)$ is easy as $\{\langle U_{\pm\alpha_i}\rangle, \langle U_{\pm\alpha_j}\rangle\} = \{SL(2, q), Sz(q)\}$ and q is even.

We must now observe that if $L/Z(L) \cong PSp(4, q)$, q odd, then $G = PSp(2n, q)$, $PSO(2n + 1, q)'$, or $F_4(q)$ (this can be seen from the Dynkin diagram). If $G = F_4(q)$, then there is a fundamental root α_k such that $L_1 = \langle U_{\pm\alpha_i}, U_{\pm\alpha_k}\rangle$ satisfies $L_1/Z(L_1) \cong PSL(3, q)$. So in this case H is transitive on $Z(U_{\alpha_i})^{\#}$. Suppose $G = PSO(2n + 1, q)'$ with $n \geqslant 3$. If $\alpha_i \neq \alpha_n$, then there is a k such that $L_1 = \langle U_{\pm\alpha_i}, U_{\pm\alpha_k}\rangle$ satisfies $L_1/Z(L_1) = PSL(3, q)$. Suppose $\alpha_i = \alpha_n$. Let V be the natural module for $\hat{G} = SO(2n + 1, q)$ and let $\bar{L}_1 = \langle \overline{U_{\pm\alpha_n}, U_{\pm\alpha_{n-1}}} \rangle \cong SO(5, q)'$. Then there is a nondegenerate 5-space V_1 of V such that $\bar{L}_1$ is trivial on $V_1{}^{\perp}$. Considering orders we see that $\operatorname{stab}(V_1) > SO(V_1)' \times SO(V_1{}^{\perp})'$. In fact $\operatorname{stab}(V_1)/SO(V_1{}^{\perp})' \cong O(V_1)$. It follows that $\bar{H}$ is transitive on $\bar{U}_{\alpha_n}^{\#}$. Thus H is transitive on $U_{\alpha_i}^{\#}$. Finally, if $G = PSp(2n, q)$ and $i = n$, then H has 2 orbits on $U_i{}^{\#} = U_n{}^{\#}$ each of length $\frac{1}{2}(q - 1)$.

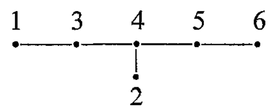
At this point we list the Dynkin diagrams for the groups $G(q) \neq {}^2F_4(q)$ not considered so far and produce certain p-groups of extraspacial-type.

C_n 

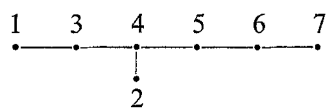
($G(q) = PSU(2n, q)$ or $PSP(2n, q)$, q odd)

BC_n 

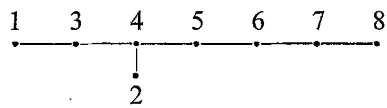
($G(q) = PSU(2n + 1, q)$)

E_6 

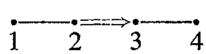
($G(q) = E_6(q)$)

E_7 

($G(q) = E_7(q)$)

E_8 

($G(q) = E_8(q)$)

F_4 

($G(q) = F_4(q)$ or ${}^2E_6(q)$)

G_2 

($G(q) = G_2(q)$ or ${}^3D_4(q)$).

In each root system Δ (omit BC_n) let r be the root of highest height (see the tables of roots in [1]). Let $\alpha_1, \ldots, \alpha_n$ be a fundamental system of roots in Δ^+. Then if $1 \leqslant k \leqslant n$, $P_k = \langle B, s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_n \rangle$ is a maximal parabolic subgroup and $Q_k = O_p(P_k) = \prod_s U_s$ where the product is taken over those $s = \sum c_i \alpha_i$ in Δ^+ with $c_k > 0$. Using the root structure given in Bourbaki [1] it is easily checked that in each of the above cases $R(r)$ (see Section 2) contains precisely one α_k. It then follows (see the proof of (2.4)) that $\langle U_s : s \in \Delta^+, s \in R(r) \rangle = Q_k$. The values for k are as follows:

$$\begin{array}{ll} \Delta = C_n & k = 1 \\ \Delta = E_6 & k = 2 \\ \Delta = E_7 & k = 1 \\ \Delta = E_8 & k = 8 \\ \Delta = F_4 & k = 1 \\ \Delta = G_2 & k = 2 \end{array}$$

If $G = G(q)$ is one of the remaining groups (other than $PSU(2n+1, q)$) let $Q = Q_k$, $P = P_k$, and $R = \langle U_{\pm\alpha_i} : i \neq k \rangle$. Then by Lemma 2.5 Q is of extraspecial-type provided that G is of normal type and $G \neq F_4(q^n)$, q even, $G \neq G_2(q)$, $q = 3^a$. If $G = PSU(2n+1, q)$, let $Q = Q_1 = O_p(P_1)$, $P = P_1$, and $R = \langle U_{\pm\alpha_i} : i > 1 \rangle$. If $G = PSU(2n+1, q)$, $PSU(2n, q)$, ${}^2E_6(q)$, or $F_4(2^a)$, then the structure of Q and P is described in [4], Sections 3–4. If $G = G_2(q)$ or ${}^3D_4(q)$ then the structure of Q and P is obtained in [7], Section 9 (for $G_2(q)$ this is easily obtained from the Chevalley commutator relations). If $G = {}^2F_4(q)$ let $Q = Q_1$ and $P = P_1$, where the ordering is such that $R = \langle U_2, U_{-2} \rangle \simeq Sz(q)$. We will use the structural properties obtained in Section 10 of [7]. We have the following:

LEMMA 4.3. *If $G \neq F_4(q)$, q even or $G_2(q)$, $q = 3^a$, then Q is of extra-special-type, $|Z(Q)| = q$, and $|Q|$ is as follows:*

$PSU(2n, q)$	$q^{4(n-1)+1}$
$PSp(2n, q)$, q odd	$q^{2(n-1)+1}$
$PSU(2n+1, q)$	$q^{2(2n-1)+1}$
$E_6(q)$	q^{21}
$E_7(q)$	q^{33}
$E_8(q)$	q^{57}
$F_4(q)$	q^{15}
${}^2E_6(q)$	q^{21}
$G_2(q)$	q^5
${}^3D_4(q)$	q^9

LEMMA 4.4. *Let $G \neq {}^2F_4(q)$ have rank at least* 2, $m_p(G) = 1$, *and* Q *as in* (4.3).

(a) *If* $G = PSp(2n, q)$ *for* q *odd, or if* $G = PSU(n, q)$, *then* $l(G, p) \geqslant \min\{2/s(q-1) \mid Q\colon Z(Q)|^{1/2}, (1/s)(q-1) \mid Q\colon Z(Q)|^{1/2} + l(R/Z(R), q)\}$, *where* $s = 2$ *if* $G = PSp(2n, q)$ *and* $s = 1$ *if* $G = PSU(n, q)$.

(b) *If* G *is an exceptional group and* $G \neq F_4(q)$ *then*

$$l(G, p) \geqslant q^{-1}(q^2-1) \mid Q\colon Z(Q)|^{1/2}.$$

Proof. Let $\bar{G}$ be a perfect central extension of G acting nontrivially on a finite dimensional vector space M over a field of characteristic other than p. Then $\bar{Q} = Q_0 \times Z(\bar{G})$ where Q_0 is P-isomorphic to Q. Also $Z(\bar{Q}) = Z(Q_0) \times Z(\bar{G})$. There is a root $s \in \Delta^+$ such that $s \neq r$ and s is conjugate to r by an element of W. Checking the root systems we see that $U_s \leqslant R = \langle U_{\pm\alpha_i}\colon i > 1\rangle$ in case (a) and that in cases (b) s may be chosen such that $U_s \leqslant Q$.

Write $M = M_1 \oplus M_2$ where $M_1 = C_M(Z(Q_0))$ and $M_2 = [Z(Q_0), M]$. Then M_1, M_2 are $\bar{P}$-invariant and $M_2 \neq 0$. Next we write $M_2 \mid Q_0 = V_1 \oplus \cdots \oplus V_k$ where the V_i are the distinct homogeneous Wedderburn components of Q_0 on M_2. On each V_i $Z(Q_0)$ is nontrivial and induces scalar multiplication, and by Lemma 2.3 $i \neq j$ implies that $V_i \mid Z(Q_0)$ and $V_j \mid Z(Q_0)$ are inequivalent. Now Lemma 4.2 shows that $k = q-1$ if $G \neq PSp(2n, q)$ and $k = q-1$ or $\frac{1}{2}(q-1)$ if $G = PSp(2n, q)$. Also Lemma 2.3 implies that $|Q : Z(Q)|^{1/2} = |Q_0 : Z(Q_0)|^{1/2}$ divides $\dim(V_i)$.

First suppose we are in case (b). Then $U_s \leqslant Q$, $U_r = Z(U_r)$, and $\bar{U}_s = L \times Z(\bar{G})$ where $L \leqslant Q_0$ and $L \cap Z(Q_0) = 1$. If θ is the character of Q_0 afforded by M_2, then Lemma 2.3 shows that θ vanishes on $L^\#$ and hence $\theta \mid L = c\rho_L$, where ρ_L is the regular character of L. Then $c = (1/q)\dim(M_2) \geqslant (1/q)(q-1) \mid Q : Z(Q)|^{1/2}$ and L fixes each vector in a subspace of M_2 of dimension $(1/q)(q-1) \mid Q : Z(Q)|^{1/2}$. As L and $Z(Q_0)$ are conjugate, and as $Z(Q_0)$ fixes no nonzero vector in M_2, it follows that $\dim(M_1) \geqslant 1/q(q-1) \mid Q : Z(Q)|^{1/2}$. Then $\dim(M) = \dim M_1 + \dim M_2 \geqslant q^{-1}(q^2-1) \mid Q\colon Z(Q)|^{1/2}$ and the result follows.

Next suppose we are in case (a). Here $U_s \leqslant R$. Write $\bar{R} = R_1 \times Z$ where R_1 is a central extension of R and $Z \leqslant Z(\bar{G})$. Unless $R = PSL(2, 3)$, we may assume that R_1 is a perfect central extension. If $R = PSL(2, 3)$ we may assume $R_1 = SL(2, 3)$ or $PSL(2, 3)$. In either case $L \leqslant R_1$ and $L^{R_1} = R_1$, where $L \times Z(\bar{G}) = \bar{U}_s$.

So if L acts nontrivially on M_1, then $\dim(M_1) \geqslant l(R/Z(R), q)$ (note that $l(R/Z(R), q) = 1$) if $R = PSL(2, 3)$) and

$$\dim(M) \geqslant (1/s)(q-1) \mid Q : Z(Q)|^{1/2} + l(R/Z(R), q).$$

Suppose then that L acts trivially on M_1. Since L and $Z(Q_0)$ are conjugate, L can fix no nonzero vector in M_2.

From Lemma 4.2 it follows that H acts irreducibly on $Z(U_r)$. Since $(U \cap R)H$ normalizes $Z(U_r) = Z(Q)$ and since $U \cap R$ centralizes a non-identity element of $Z(Q)$, it follows that $U \cap R \leqslant C_R(Z(Q)) \trianglelefteq R$, and consequently R centralizes $Z(Q)$. Therefore $\bar{R}$ centralizes $Z(Q_0)$ and stabilizes $V_1,..., V_k$. In particular $\langle \overline{U_s}, \overline{U_{-s}} \rangle$ fixes each V_i. Set $H_1 = H \cap \langle U_s, U_{-s} \rangle$. Then considering the possibilities for the rank 1 group $\langle U_s, U_{-s} \rangle$ we see that H_1 is either transitive on $Z(U_s)^{\#}$ or H_1 has two orbits of length $\frac{1}{2}(q-1)$ on $Z(U_s)^{\#}$. Consider $\overline{Z(U_s)}\bar{H}_1 = L\bar{H}_1$ acting on V_i.

$$V_i \mid L = W_{i1} \oplus \cdots \oplus W_{it_i},$$

where the W_{ij} are the distinct homogeneous Wedderburn components of L on V_i. Now $\bar{H}_1$ permutes the W_{ij} and L is trivial on no W_{ij}. Thus each orbit of $\bar{H}_1$ on the W_{ij}'s has length $q-1$ or $\frac{1}{2}(q-1)$. So $\dim(V_i)$ is divisible by $q-1$ or $\frac{1}{2}(q-1)$ and hence $(q-1) \mid Q : Z(Q)|^{1/2}$ or $\frac{1}{2}(q-1) \mid Q : Z(Q)|^{1/2}$ divides $\dim(V_i)$. If $q > 3$, then $\dim(V_i) \geqslant 2 \mid Q : Z(Q)|^{1/2}$, $\dim(M_2) \geqslant (1/s)(q-1)\dim(V_i)$ and the result follows. Suppose $q = 2$. Then $|Z(U_s)| = |L| = 2$ and L induces scalar action on M_2. But then $[L, Q_0] = 1$ whereas $[Z(U_s), Q] \neq 1$. Suppose $q = 3$. Here $|Z(Q_0)| = 3 = |L|$. As above L cannot be scalar on M_2. Therefore $Z(Q_0)$ is not scalar on M_2 (i.e., $k > 1$). So $M_2 = V_1 \oplus V_2$ and $\dim(M) \geqslant 2\dim(V_1) \geqslant 2 \mid Q : Z(Q)|^{1/2}$. If $G = PSp(2n, 3)$, then $2 \mid Q : Z(Q)|^{1/2} = 2/s(q-1) \mid Q : Z(Q)|^{1/2}$, and we are done. The only remaining case is $G = PSU(n, 3)$. If $H \cap R$ is not transitive on $Z(U_s)^{\#}$, then by Lemma 4.2 $R = PSL(2, 3)$ and $n = 4$. But here $m_p(G) \neq 1$ ([8]). So $H \cap R$ is transitive on $Z(U_s)^{\#}$ and as before $2 = q - 1$ divides $\dim(V_i)$ and $\dim(M) \geqslant \dim(M_2) \geqslant 2\dim(V_1) \geqslant 4 \mid Q : Z(Q)|^{1/2}$ and the result holds. This completes the proof of Lemma 4.3.

LEMMA 4.5. *If $G = E_6(q), E_7(q), E_8(q), G_2(q)$ with $q \neq 4, 3^a$, or $D_4^3(q)$, then the Theorem holds.*

Proof. This follows directly from Lemma 4.4(b) and the facts in (4.3).

LEMMA 4.6.

(a) *If $G = PSp(2n, q)$ for q odd, then $l(G, p) \geqslant \frac{1}{2}(q^n - 1)$.*

(b) *If $G = PSU(2n, q)$ with $n \geqslant 2$ and $q \geqslant 4$ if $n = 2$, $l(G, p) \geqslant (q^{2n} - 1)/(q + 1)$.*

(c) *If $G = PSU(2n + 1, q)$ with $n \geqslant 2$, then $l(G, p) \geqslant (q^{2n} - 1) q/(q + 1)$.*

Proof. We use Lemma 4.4 and induction. Suppose $G = PSp(2n, q)$ for q odd. If $n = 1$, the result follows from Lemma 4.1(a). If $n > 1$, then Lemma 4.3(a) implies

$$l(G, p) \geqslant \min\{(q-1)\, q^{n-1}, \tfrac{1}{2}(q-1)\, q^{n-1} + \tfrac{1}{2}(q^{n-1}-1)\} = \tfrac{1}{2}(q^n - 1).$$

This proves (a).

Next we consider (b). Let $G = PSU(4, q)$ with $q \geqslant 4$. Lemma 4.4(a) shows that $l(G, p) \geqslant \min\{2(q-1)\, q^2, (q-1)\, q^2 + l(PSL(2, q), q)\}$. In the proof of Lemma 4.4(a) we actually showed that

$$l(G, p) \geqslant \min\{2(q-1)\, q^2, (q-1)\, q^2 + l(RH/Z(RH), q)\}.$$

Now $RH/Z(RH) \cong PGL(2, q)$ and as in Lemma 4.1 we have $l(PGL(2, q), p) \geqslant q - 1$ provided $q \neq 9$ (i.e., $m_p(PGL(2, q)) = 1$). Since $m_3(PSU(4, 9)) = 1$ ([8]) no 3-fold covering group of $RH/Z(RH)$ will appear in a perfect central extension of $PSU(4, 9)$. It follows that $l(RH/Z(RH), p) \geqslant q - 1$ in all cases and $l(G, p) \geqslant (q-1)(q^2+1)$. Inductively let $G = PSU(2n, q)$. Then $l(G, p) \geqslant \min\{2(q-1)\, q^{2n-2}, (q-1)\, q^{2n-2} + l(PSU(2n-2, q), q\} \geqslant \min\{2(q-1)\, q^{2n-2}, (q-1)\, q^{2n-2} + (q^{2n-2}-1)/(q+1) = (q-1)\, q^{2n-2} + (q^{2n-2}-1)/(q+1)\} = (q^{2n}-1)/(q+1)$. This proves (b).

The proof of (c) is similar (use Lemma 4.1(b)).

LEMMA 4.7. *If $G = G_2(q)$ with $q = 3^a > 3$, then $l(G, p) \geqslant q(q^2 - 1)$.*

Proof. Consider Q as in (4.3). Let α_1 be a short root and α_2 a long root. Then $Q = U \cap U^{s_1} = U_2 U_1^{s_2} U_2^{s_1 s_2} U_1^{s_2 s_1} U_2^{s_1}$. However Q is not of extraspecial-type because the commutator relations imply that $[U_1^{s_2}, U_1^{s_2 s_1}] = 1$. In fact $Q = U_1^{s_2} U_1^{s_2 s_1} \times U_2 U_2^{s_1} U_2^{s_1 s_2}$ and $U_2 U_2^{s_1} U_2^{s_1 s_2}$ is of extraspecial-type of order q^3. Also $\langle U_1, U_1^{s_1} \rangle = R \cong SL(2, q)$ acts in a natural way on $U_1^{s_2} U_1^{s_2 s_1}$ and on $U_2 U_2^{s_1} U_2^{s_1 s_2} / U_2^{s_1 s_2}$.

Let $\bar{G}$ be a perfect central extension of G and assume that $\bar{G}$ acts faithfully and irreducibly on a vector space M over an algebraically closed field of characteristic other than 3. Write $\bar{Q} = Q_1 \times Q_2 \times Z(\bar{G})$ where

$$Q_1 Z(\bar{G})/Z(\bar{G}) = U_1^{s_2} U_1^{s_2 s_1} \quad \text{and} \quad Q_2 Z(\bar{G})/Z(\bar{G}) = U_2 U_2^{s_1} U_2^{s_1 s_2}.$$

Let $M_1 = [M, Z(Q_2)]$. First suppose that $[M_1, Q_1] \neq 1$. Then Q_2 acts on $[M_1, Q_1]$ and (2.3) implies that q divides the dimension of $[M_1, Q_1]$. Also $\bar{R} Q_1$ acts on $[M_1, Q_1]$ and Q_1 acts without fixed points. As $\bar{R}$ is transitive on the nontrivial linear representations of Q_1, it follows that $q^2 - 1$ divides $\dim[M_1, Q_1]$. Hence $\dim(M) \geqslant q(q^2 - 1)$. So we may assume that Q_1 is trivial on M_1. Next write $M = M_1 \oplus C_M(Z(Q_2))$ and $C_M(Z(Q_2)) = M_2 \oplus M_3$, where $M_2 = [C_M(Z(Q_2)), Q_1] \neq 1$ and $M_3 = C_M(Z(Q_2)) \cap C_M(Q_1)$. The group $\bar{R}Q_1$ acts on M_2 and Q_1 acts without fixed points. It follows that M_2 contains each nontrivial linear representation of Q_1 an equal number of times and $\dim(M_2) = x(q^2 - 1)$, Let $\bar{U}_1^{s_2 s_1} = U_0 \times Z(\bar{G})$ and $\bar{U}_1 = U_{00} \times Z(\bar{G})$. It is easy to see that U_0 is contained in the kernel of precisely $q - 1$ nontrivial linear representations of Q_1 and that $U_{00}Q_1$ is of extraspecial-type with $Z(U_{00}Q_1) = U_0$. Thus $M_2 = C_{M_2}(U_0) \times [M_2, U_0]$, $\dim(C_{M_2}(U_0)) = x(q-1)$ and on $[M_2, U_0]$, U_{00} induces $x(q - 1)$ copies of the regular representation of U_{00}. We now have $\dim(C_{M_2}(U_0)) = x(q - 1)$ and $\dim(C_{M_2}(U_{00})) \leqslant 2x(q - 1)$.

Next we consider U_{00} acting on M_1. First we note that the elements of $(U_2^{s_1 s_2})^{\#}$ are all conjugate in P (actually in H) so $M_1 = M_{1,1} \oplus \cdots \oplus M_{1,q-1}$ where the $M_{1,j} \mid Q_2$ are homogeneous and conjugate under the action of $\bar{P}$. So there is an integer y such that $\dim(M_{1,j}) = yq$ and $\dim(M_1) = yq(q - 1)$. Suppose that $\dim(C_{M_1}(U_{00})) > \frac{3}{4}yq(q - 1)$. If $t \in \bar{G}$ and $tZ(\bar{G}) = s_1$, then $\dim(C_{M_1}(U_{00}^t)) > \frac{3}{4}yq(q - 1)$ and so $\dim(C_{M_1}(\langle U_{00}, U_{00}^t\rangle)) > \frac{1}{2}yq(q - 1)$. Let $K = \langle U_{00}, U_{00}^t\rangle$. Then $KZ(\bar{G})/Z(\bar{G}) = \bar{R}$ and if if $g \in Q_2 - Z(Q_2)$ then $\langle K, K^g\rangle Z(\bar{G})/Z(\bar{G}) = RU_2U_2^{s_1}U_2^{s_1 s_2}$. But $C_{M_1}(K) \cap C_{M_1}(K^g) \neq 0$, and this implies that $C_{M_1}(\langle K, K^g\rangle) \neq 0$, and so $C_{M_1}(Z(Q_2)) \neq 0$ which is not the case. Thus $\dim(C_{M_1}(U_{00})) \leqslant \frac{3}{4}yq(q - 1)$.

We now have $\dim(C_M(U_0)) = \dim M_3 + x(q - 1) + yq(q - 1)$ and $\dim(C_M(U_{00})) \leqslant \dim M_3 + 2x(q - 1) + \frac{3}{4}yq(q - 1)$. As U_0 and U_{00} are conjugate in $\bar{G}$ it follows that $2x(q - 1) + \frac{3}{4}yq(q - 1) \geqslant x(q - 1) + yq(q - 1)$ and $x \geqslant (\frac{1}{4})yq$. Consequently $\dim([M, U_0]) = \dim([M_2, U_0]) = x(q^2 - q) \geqslant (\frac{1}{4})yq^2(q - 1)$ and $\dim([M, Z(Q_2)]) = yq(q - 1)$, As $q > 3$, $\dim([M, U_0]) > \dim[M, Z(Q_2)]$.

Now the graph automorphism of G interchanges U_1 and U_2, $U_1^{s_2 s_1}$ and $U_2^{s_1 s_2}$, and the maximal parabolic subgroups P and P' of G, where $P' = \langle B, s_2\rangle$. Thus we could have started with the parabolic subgroup P' and considered $O_3(P')$. Arguing as above we would obtain $\dim[M, Z(Q_2)]) > \dim([M, U_0])$ and this is a contradiction. The proof of 4.7 is complete.

LEMMA 4.8.

(a) *If* $G = F_4(q)$ *and* $q = 2^a > 2$, *then* $l(G, 2) \geqslant \frac{1}{2}q^7(q^3 - 1)(q - 1)$.

(b) *If* $G = F_4(q)$ *and* q *odd, then* $l(G, p) \geqslant q^4(q^6 - 1)$.

(c) *If* $G = {}^2E_6(q)$ *and* $q > 2$, *then* $l(G, p) \geqslant q^8(q^4 + 1)(q^3 - 1)$.

Proof. To prove the lemma we use the results of [4], Section 4 giving the structure of the parabolic subgroup $P = P_4$. P contains a normal subgroup $T = O_p(P)$ and a subgroup R such that $R \cong SO(7, q)'$ if $G = F_4(q)$ and $R \cong SO^-(8, q)'$ if $G = {}^2E_6(q)$. Then $P = TRH$. There is an elementary subgroup $K < T$ such that $K \lhd P$, $|K| = q^7$ if $G = F_4(q)$ and $|K| = q^8$ if

$G = {}^2E_6(q)$. Moreover R acts in a natural way on K preserving a nondegenerate quadratic form. Also P acts irreducibly on T/K.

Let $\bar{G}$ be a perfect central extension of G acting faithfully and irreducibly on a vector space M over an algebraically closed field of characteristic other than p. Write $\bar{T} = T_0 \times Z(\bar{G})$, $\bar{K} = K_0 \times Z(\bar{G})$, and $\bar{U}_s = U_0 \times Z(\bar{G})$, where s is the root $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$. If $G = {}^2E_6(q)$ then $| U_0 | = q^2$ and U_0 is a 2-space in $K_0 \cong K$. If $G = F_4(q)$ for q odd, then U_0 is a 1-space in K_0, and is $G = F_4(q)$ for q even, then $U_0 = \mathrm{rad}(K_0)$.

If $G \neq F_4(q)$ with q even, that R acts irreducibly on T_0/K_0 and on K_0 and hence $K_0 \leqslant Z(T_0)$. Suppose $G = F_4(q)$ with q even. The graph automorphism of G interchanges the parabolic subgroups P_1 and P_4. The structure of $Q = O_2(P_1)$ is given in [4], Lemma 4.5, and it follows from this that $K_0 = Z(T_0)$. So in all cases $K_0 \leqslant Z(T_0)$.

Let M_1 be an irreducible $\bar{P}$-submodule of M such that $U_0 \nleqslant \ker_{\bar{P}}(M_1)$. Write $M_1 \mid K_0 = Z_1 \oplus \cdots \oplus Z_k$, where the Z_i's are the distinct homogeneous Wedderburn components of K_0 on M_1. Let K_1 be the unique hyperplane of K_0 such that $K_1 \leqslant \ker_{K_0}(Z_1)$ (Lemma 2.3). We may assume that $U_0 \nleqslant K_1$ and hence $K_0 = K_1 + U_0$.

Since $K_0 \leqslant Z(T_0)$, T_0 fixes each Z_i. In particular T_0 fixes Z_1 and T_0/K_1 acts on Z_1. We claim that T_0/K_1 is of extraspecial-type. To see this we first note that $T/K \cong \prod U_t$ where the product is direct and taken over all short roots t in Δ^+ such that t has a nonzero coefficient of α_4. For each such t, $s - t$ is another such root and so $U_t U_{s-t} U_s$ is of extraspacial type of order $| U_s |^3$. If $G = F_4(q)$, $| U_s | = q$ and if $G = {}^2E_6(q)$, then $| U_s | = q^2$. Thus T_0/K_1 is of extraspecial type of order q^9 or q^{18}, respectively. We now have that $\dim(Z_1) \geqslant q^4$ if $G = F_4(q)$ and $\dim(Z_1) \geqslant q^8$ if $G = {}^2E_6(q)$.

Let $\bar{L}$ be the stabilizer in $\bar{P}$ of Z_1 and $\bar{L}_1 = \bar{L} \cap \bar{R}$. Suppose $G = F_4(q)$. Then there is a subgroup H_0 of H such that $RH_0 = R \times H_0$, H_0 is cyclic or order $q - 1$. Indeed choose $h = h(\chi)$ such that $\chi(\alpha_1) = \chi(\alpha_2) = \chi(\alpha_3) = 1$ and $\chi(\alpha_4) = 5$ where $\langle \xi \rangle = \mathbb{F}_q^{\#}$; then $H_0 = \langle h \rangle$. If $t \in \Delta^+$ and $U_t \leqslant K$, then the coefficient of α_4 in t is 2 (see [4, Section 4]). So H_0 is scalar on K_0 of order $(q-1)/(2, q-1)$. If q is even we proceed as in the proof of Lemma 3.2 and obtain $| \bar{P} : \bar{L} | \geqslant (\frac{1}{2})q^3(q^3 - 1)(q - 1)$. Suppose q is odd. In this case we proceed as in the proof of Lemma 3.3 and obtain $| P : L | \geqslant q^6 - 1$ (note that here $| H_0 | = q - 1$ rather than $\frac{1}{2}(q - 1)$ as in (3.3)).

Suppose $G = {}^2E_6(q)$. If $\mathrm{rad}(K_1) \neq 0$, then proceeding as in the first part of the proof of 3.3 we have $| \bar{R} : \bar{L}_1 | \geqslant (q^4 + 1)(q^3 - 1)$. Now suppose $\mathrm{rad}(K_1) = 0$. If q is even, $\bar{L}_1 \leqslant \bar{R} \leqslant O^{\pm}(7, q) \cong Sp(6, q)$ and $| \bar{R} : \bar{L}_1 | \geqslant q^3(q^4 + 1) > (q^4 + 1)(q^3 - 1)$. If q is odd, then since $\bar{L}_1$ is trivial on $K_1{}^{\perp}$, $\bar{L}_1$ induces a subgroup of $SO^{\pm}(7, q)'$ on K_1. Checking orders we again have $| \bar{R} : \bar{L}_1 | \geqslant q^3(q^4 + 1) > (q^4 + 1)(q^3 - 1)$.

Since $| \bar{P} : \bar{L} | \geqslant | \bar{R} : \bar{L}_1 |$, we have $| \bar{P} : \bar{L} | \geqslant q^6 - 1, (\frac{1}{2})q^3(q^3 - 1)(q - 1)$, or $(q^4+1)(q^3-1)$ according as $G = F_4(q)$ q odd, $F_4(q)$ q even, or ${}^2E_6(q)$. But $|\bar{P} : \bar{L}|$ is the number of conjugates of Z_1 and hence $\dim(M) \geqslant \dim(M_1) \geqslant q^4(q^6-1)$, $(\frac{1}{2})q^7(q^3-1)(q-1)$, $q^8(q^4+1)(q^3-1)$ according as $G = F_4(q)$ q odd, $F_4(q)$ q even, or ${}^2E_6(q)$. This completes the proof of Lemma 4.8.

LEMMA 4.9. *If* $G = {}^2F_4(q)$, *then* $l(G, 2) \geqslant (q/2)^{1/2}q^4(q-1)$.

Proof. Let Let $\bar{G}$ be a central extension of G such that $Z(\bar{G}) \leqslant \bar{G}'$ and suppose that $\bar{G}$ acts faithfully and irreducibly on a module M over an algebraically closed field of odd characteristic or characteristic 0. Then $m_2(G) = 1$ (see Greiss [8]).

Write $\bar{Q} = Q_0 \times Z(\bar{G})$. Then $Q_0 \simeq Q$ and the structure of Q_0 is determined in Section 10 of [7]. Let $M = M_1 \oplus \cdots \oplus M_k$ be the decomposition of M into the distinct homogeneous Wedderburn components of Q_0'. We consider the action of P on Q to be the same as that of $\bar{P}$ on Q_0. $P = Q(R \times H_0)$ where $R \simeq Sz(q)$, H_0 is cyclic of order $q-1$, and H_0 acts fixed-point-free on Q. Write $\bar{U}_2^{s_1s_2s_1} = U_0 \times Z(\bar{G})$, $\bar{Q}' = Q_1 \times Z(\bar{G})$.

We may assume that $\Omega_1(U_0) \nleqslant \ker M_1$. Then $Q_1 = \ker_{Q_1}(M_1)\,\Omega_1(U_0)$. Since $|Q_1| = q^5$ and Q_1 is elementary abelian, there are $q^4(q-1)$ subgroups of Q_1 having index 2 and not containing $\Omega_1(U_0)$. We claim that these are conjugate in $\overline{QH}_0$. To see this we go to the group QH_0 (see $6H$ of [7]). Then

$$Q_1 = \Omega_1(U_2^{s_1}) \times U_1^{s_2s_1} \times \Omega_1(U_2^{s_1s_2}) \times U_1^{s_2s_1s_2} \times \Omega_1(U_2^{s_1s_2s_1}).$$

Let $T = \Omega_1(U_2^{s_1}) \times U_1^{s_2s_1} \times U_1^{s_2s_1s_2} \times \Omega_1(U_2^{s_1s_2}) \times T_0$ where T_0 is a subgroup of index 2 in $\Omega_1(U_2^{s_1s_2s_1})$. Let $L = N_{QH_0}(T)$. Since Q'/T has order 2, $L \leqslant Q$ (H_0 is fixed-point-free on Q). At this point we apply the results in [7], Section 10. We immediately see that $N(T) \geqslant \Omega_1(U_2^{s_1})\, U_1^{s_2s_1}U_2^{s_1s_2s_1}U_1^{s_2s_1s_2}\Omega_1(U_2^{s_1s_2})$. Suppose that $abcd \in N(T)$ with $a \in U_1$, $b \in U_2^{s_1}$, $c \in U_2^{s_1s_2}$, and $d \in U_1^{s_2}$. If $b \notin \Omega_1(U_2^{s_1})$, then using (10.11) (i) of [7] we have $[b, U_1^{s_2s_1s_2}] = \Omega_1(U_2^{s_1s_2s_1})$. As $a, c, d \in N(U_2^{s_1s_2s_1}) \leqslant N(\Omega_1(U_2^{s_1s_2s_1}))$, $a, c \in C(U_1^{s_2s_1s_2})$, and $[d, U_1^{s_2s_1s_2}] \leqslant U_2^{s_1s_2}$, we have $\Omega_1(U_2^{s_1s_2s_1}) \leqslant [abcd, U_1^{s_2s_1s_2}] \leqslant [abcd, T] \leqslant T$, a contradiction. Thus $b \in \Omega_1(U_2^{s_1}) \leqslant T$ and similarly $c \in \Omega_1(U_2^{s_1s_2}) \leqslant T$. So $ad \in N(T)$. If $a \neq 1$, then (10.13) (i) and (10.15) of [7] show that $[a, U_2^{s_1s_2}]T \geqslant \Omega_1(U_2^{s_1s_2s_1})$. Also $[d, U_2^{s_1s_2}] = 1$. As above this leads to a contradiction. Thus $a = 1$ and similarly $d = 1$. This proves that

$$N_{Q_0H}(T) = \Omega_1(U_2^{s_1})\, U_1^{s_2s_1}U_2^{s_1s_2s_1}U_1^{s_2s_1s_2}\Omega_1(U_2^{s_1s_2})$$

and so T has $q^4(q-1)$ conjugates in Q_0H, proving the claim.

It follows from the claim that $k \geqslant q^4(q-1)$. Also if $\bar{T} = T_0 \times Z(\bar{G})$ then $T_0 = \ker_{Z(Q_0)}(M_i)$ for some i. Then $\bar{U}_2^{s_1s_2s_1}$ stabilizes M_i and as in (4.1) (c) we have $\dim(M_i) \geqslant (q/2)^{1/2}$. Thus $\dim(M) \geqslant (q/2)^{1/2}q^4(q-1)$ and the lemma is proved.

5. The Case $m_p(G) \neq 1$

At this point we have handled all Chevalley groups $G = G(q)$ such that $m_p(G) = 1$. It remains to prove the theorem for the finite number of Chevalley groups $G = G(q)$ satisfying $m_p(G) \neq 1$. The basic reference for information concerning these groups will be Griess [8]. We keep the notation of Section 4.

Throughout $\bar{G}$ will denote a perfect central extension of $G = G(q)$ acting faithfully and irreducibly on a module M over an algebraically closed field of characteristic other than p.

LEMMA 5.1. *Suppose $G = G(q)$ and $m_p(G) \neq 1$. Then G is one of the following*:

(i) $PSL(2, 4)$, $PSL(2, 9)$, $PSL(3, 2)$, $PSL(3, 4)$, $PSL(4, 2)$.

(ii) $PSp(4, 2)$, $PSp(6, 2)$.

(iii) $SO(7, 3)'$.

(iv) $PSO^+(8, 2)'$.

(v) $G_2(3)$, $G_2(4)$.

(vi) $PSU(4, 2)$, $PSU(4, 3)$, $PSU(6, 2)$.

(vii) $Sz(8)$.

(viii) ${}^2E_6(2)$.

(ix) $F_4(2)$.

Proof. See Griess [8].

LEMMA 5.2.

(a) *If $G = PSL(2, 4)$, then $l(G, 2) = 2$.*

(b) *If $G = PSL(2, 9)$, then $l(G, 3) = 3$.*

(c) *If $G = PSL(3, 2)$, then $l(G, 2) = 2$.*

(d) *If $G = PSL(3, 4)$, then $l(G, 2) = 4$.*

(e) *If $G = PSL(4, 2)$, then $l(G, 2) = 7$.*

Proof. We first note that $PSL(2, 4) \simeq PSL(2, 5)$, $PSL(2, 9) \simeq A_6 \leqslant PSL(3,4)$, $PSL(3, 2) \simeq PSL(2, 7)$, $PSL(3, 4) \leqslant PSU(4, 3)$ and $PSL(4, 2) \simeq A_8$. Thus $l(G, p)$ is at most the numbers given in the lemma. Clearly (a) and (c) hold, as $PSL(2, 4)$ and $PSL(3, 2)$ are simple. If $p \nmid |Z(\bar{G})|$, then the proof of Lemma 3.1 shows that $\dim(M) \geqslant q^{n-1} - 1$, where $G = PSL(n, q)$. In each of the cases $l(G, p) \leqslant q^{n-1} - 1$. So to prove the lemma we may assume that $p \mid (Z(\bar{G})|$.

Suppose $G = PSL(2, 9)$. Since $3 \mid Z(\bar{G})$ the Sylow 3-subgroups of $\bar{G}$ are nonabelian and hence $\dim(M) \geqslant 3$. Suppose $G = PSL(3, 4)$. Then $2 \mid | Z(\bar{G})|$ and $\bar{G}$ perfect implies that $\dim(M)$ is even. If $\dim(M) = 2$, then $\bar{G} \leqslant SL(2, r)$ for some odd prime power r and this would contradict the structure of the Sylow 2-subgroups of G. Thus $\dim(M) \geqslant 4$ as needed.

Finally suppose $G = PSL(4, 2)$. Let $P \leqslant G$ be the stabilizer of a 1-space in the natural 4-dimensional module for $G = PSL(4, 2) = GL(4, 2)$. Then $P = QR$ where $Q = O_2(P)$ is elementary of order 8 and $R \simeq GL(3, 2)$ acts in the natural way on Q. Consider $\bar{Q}$ and let $\langle v \rangle$ be a Sylow 7-group of R. Then $\langle v \rangle$ is transitive on $Q^{\#}$ and since $| Q | = 2^3$ it easily follows that $\bar{Q}$ is abelian and $\bar{Q} = Q_0 \times Z(\bar{G})$ where $Q_0 \cong Q$ under the action of $\langle v \rangle$. At this point we follow the proof of (3.1) to obtain $\dim(M) \geqslant 7$.

LEMMA 5.3.

(a) *If* $G = PSp(4, 2)'$, *then* $l(G, 2) = 2$.

(b) *If* $G = PSp(6, 2)$, *then* $l(G, 2) = 7$.

Proof. If $G = PSp(4, 2)'$ then $G \simeq A_6 \simeq PSL(2, 9)$ and (a) holds. Suppose $G = PSp(6, 2)$. Then $G \geqslant SO^+(6, 2) \simeq GL(4, 2)$ and it follows from (5.2) (e) that $l(G, 2) \geqslant 7$. Since G is the derived group of the Weyl group of type E_7, $l(G, 2) = 7$.

LEMMA 5.4. *If* $G = SO(7, 3)'$, *then* $l(G, 3) \geqslant 27$.

Proof. $G = SO(7, 3)' \geqslant SO^+(6, 3)' \simeq PSL(4, 3)$. As $m_3(PSL(4, 3)) = 1$, we apply Lemma 3.1 and obtain $l(G, 3) \geqslant 3^3 - 1 = 26$. If $3 \mid | Z(\bar{G})|$, then $3 \mid \dim(M)$ and $\dim(M) \geqslant 27$. If $3 \nmid | Z(\bar{G})|$, then we proceed as in Lemma 3.3 and obtain $\dim(M) \geqslant 3^2(3^2 - 1) \geqslant 27$.

LEMMA 5.5. *If* $G = PSO^+(8, 2)'$, *then* $l(G, 2) = 8$.

Proof. If $2 \nmid | Z(\bar{G})|$, then as in (3.3) $\dim(M) \geqslant 2^2(2^3 - 1) > 8$. Suppose $2 \mid | Z(\bar{G})|$. Let $P \leqslant G$ be the stabilizer of an isotropic 1-space, so that $P = QR$, where Q is elementary of order 2^6 and $R \simeq SO^+(6, 2) \simeq GL(4, 2)$. Then (5.2) (e) shows that $\dim(M) \geqslant 7$. However as $| Z(\bar{G})|$ is even, $\dim(M)$ is even so $l(G, 2) \geqslant 8$. On the other hand, $PSO^+(8, 2)' \simeq D_4(2)$ is isomorphic to the group $(L/Z(L))'$ where L is the Weyl Group of type E_8. Thus $D_4(2)$ does have a projective representation of degree 8 and $l(G, 2) = 8$.

LEMMA 5.6.

(a) *If* $G = G_2(3)$, *then* $l(G, 3) \geqslant 14$.

(b) *If* $G = G_2(4)$, *then* $l(G, 4) \geqslant 60 = 4(4^2 - 1)$.

Proof. First consider $G = G_2(3)$. Suppose that $3 \nmid |Z(\bar{G})|$. In this case we follow the proof of Lemma 4.7 although a change must be made at the end of that argument. Namely, in the notation of 4.7 let $M = M_1 \oplus M_2 \oplus M_3$, where $M_1 = [M, Z(Q_2)]$ and $M_2 = [C_M(Z(Q_2)), Q_1]$. It was shown that $\dim(M_1) \geqslant q(q-1) = 6$ and $\dim(M_2) \geqslant q^2 - 1 = 8$. Thus $\dim(M) \geqslant 14$. Now suppose that $3 \mid |Z(\bar{G})|$. Then by [8] $\bar{G}$ is a covering group of G and generators and relations of $\bar{G}$ are known. Let $\bar{U}_2^{s_1s_2} = L \times Z(\bar{G})$, $L = \langle x \rangle$, $Z(\bar{G}) = \langle y \rangle$. Then $L \times Z(\bar{G}) = Z(\bar{Q})$ and $\bar{Q}/\langle xy \rangle \simeq \bar{Q}/\langle x^2y \rangle$ is extraspecial of order 3^5. If M_1 is a $\bar{Q}$-composition of M then one of the subgroups of order 3 in $Z(\bar{Q})$ is trivial on M_1. However $Z(\bar{G})$ induces scalar action on M and x cannot be trivial on all such M_1. So we may assume that $\langle xy \rangle$ or $\langle x^2y \rangle$ is trivial on M_1 and hence $\dim(M_1) \geqslant 3^2 = 9$. It is easy to see that xy and x^2y are conjugate by an element h of $\bar{H}$, so that $M_1 \oplus M_1{}^h \leqslant M$ has dimension at least 18. This proves (5.6) (a).

Now assume that $G = G_2(4)$. By Griess [8] $Z(\bar{G}) = 1$ or $|Z(\bar{G})| = 2$. If $Z(\bar{G}) = 1$, we use the argument in (4.5). Suppose $|Z(\bar{G})| = 2$. We will show how in this case we can again use the argument in (4.5). We note that Q is of extraspecial-type of order 4^5 and having center $U_2^{s_1s_2}$. Consider $\overline{U_2^{s_1s_2}}$. $(U_2^{s_1s_2})^{\#}$ is fused under the action of H, so $U_2^{s_1s_2}$ is quaternion or elementary abelian. We check that $\overline{U_2^{s_1s_2}}$ is elementary of order 8. It follows that R centralizes $\overline{U_2^{s_1s_2}}$ and this implies that $\overline{U_2^{s_1s_2}} \leqslant Z(Q)$. Consider $\overline{U_2^{s_1s_2}}$ as a module for H_0, where $\bar{P}/\bar{Q} \simeq P/Q = R \times H_0$. Write $U_0 = [\overline{U_2^{s_1s_2}}, H_0] = [\overline{U_2^{s_1s_2}}, H]$. Then $|U_0| = 4$ and $\overline{U_2^{s_1s_2}} = U_0 \times Z(G)$. We then have $U_0 \trianglelefteq \bar{P}$ and the elements of $U_0{}^{\#}$ are fused under the action of $\bar{H}_0 \leqslant \bar{H}$. Also $\overline{s_1s_2} = g$ is such that $\bar{U}_2{}^g = \overline{U_2^{s_1s_2}}$. Write $U_{00} = U_0^{g^{-1}}$ so that $\bar{U}_2 = U_{00} \times Z(\bar{G})$ and U_0 and U_{00} are conjugate. Write $M = M_1 \oplus \cdots \oplus M_k$ where the M_i's are distinct and homogeneous under the action of $\bar{Q}$. We may assume $U_0 \nleqslant \ker(M_1)$. Then $\ker_{\bar{Q}}(M_1) = \langle x, yz \rangle$, where $U_0 = \langle x, y \rangle$ and $\langle z \rangle = Z(\bar{G})$, and where $Q/\ker_{\bar{Q}}(M_1)$ is extraspecial. Also U_{00} on M_1 is a multiple of the regular representations. From these facts we can argue as in (4.5) to complete the proof.

LEMMA 5.7.

(a) *If* $G = PSU(4, 2)$, *then* $l(G, 2) = 4$.

(b) *If* $G = PSU(4, 3)$, *then* $l(G, 3) = 6$.

(c) *If* $G = PSU(6, 2)$, *then* $l(G, 2) \geqslant 21 = (2^6 - 1)/(2 + 1)$.

Proof. If $p \nmid |Z(\bar{G})|$, then the proof of (4.6) (b) yields the result. We therefore assume that $p \mid |Z(\bar{G})|$. Since $\bar{G}'$ acts irreducibly on M, $Z(\bar{G})$ is cyclic and $Z(\bar{G}) \leqslant \bar{G}'$ implies that $|Z(\bar{G})|$ divides $\dim(M)$.

If $G = PSU(4, 2)$, then $G \simeq PSp(4, 3)$ and we are done unless $\dim(M) = 2$.

However $\bar{G} \leqslant SL(2,F)$ for F a field would contradict the structure of the Sylow 2-subgroups of $\bar{G}$ (or of G). Thus (a) holds. Suppose $G = PSU(4, 3)$. Then $G \leqslant PSU(6, 2)$ ([6]) and $l(G, 3) \leqslant 6$. We need only show that $\dim(M) \geqslant 6$. As $3 \mid \dim(M)$, the only problem would be that $\dim(M) = 3$. If $\dim(M) = 3$, then $M \mid O_3(\bar{Q})$ is irreducible and hence $Z(O_3(\bar{Q}))$ is cyclic. However it is easy to prove that $O_3(Z(\bar{Q}))$ is not cyclic, proving (b).

Finally we suppose that $G = PSU(6, 2)$. Then $Z(\bar{Q})$ cyclic implies that $\mid O_2(Z(\bar{G}))\mid = 2$ [8]. Recall that P acts irreducibly on $Q/Z(Q)$ and $\mid Z(Q)\mid = 2$. It follows that $Z(O_2(\bar{Q}))$ is elementary of order 4, say $Z(O_2(\bar{Q})) = \langle x\rangle \times \langle t\rangle$ where $t \in Z(\bar{G})$. If $O_2(\bar{Q})' = Z(O_2(\bar{Q}))$, then $O_2(\bar{Q}')/T$ is extraspecial of order 2^9 whenever T is a subgroup of order 2 in $\langle x\rangle \times \langle t\rangle$. Since $\bar{Q}$ acts faithfully on M, M contains two distinct homogeneous components of $\bar{Q}$ each of dimension at least 2^4. Consequently $\dim(M) \geqslant 32 > 21$ and we are done. Now suppose that $\mid O_2(\bar{Q}')\mid = 2$. Since $\mid Q' \mid = 2$, it follows that $t \notin O_2(\bar{Q})'$ and we may assume $\langle x\rangle = O_2(\bar{Q}')$. Then $M = M_1 \oplus M_2$ where $M_1 = C_M(x)$ and $M_2 = [x, M]$. The group $O_2(\bar{Q})$ acts on M_2 and $\dim(M_2) \geqslant 16$. Also $O_2(\bar{Q})/\langle x\rangle$ acts nontrivially on M_1. Decompose M_1 into homogeneous components of $O_2(\bar{Q})/\langle x\rangle$, say $M_1 = M_{11} \oplus \cdots \oplus M_{1k}$.

If $\bar{R}$ stabilizes M_{1j} for some j, then set $Q_0 = \ker_{O_2(\bar{Q})}(M_{1j})$. Consequently $\bar{R}'Q_0$ is a group. If $t \notin \bar{R}'$, then a Sylow 2-subgroup of $\bar{G}$ has the form $L \times \langle t\rangle$, where L is a Sylow 2-subgroup in $\bar{R}'Q_0$. This contradicts Gaschütz's Theorem. Therefore $t \in \bar{R}'$ and $\bar{R}' \simeq Sp(4, 3)$ ([8]). In $G = \bar{G}/Z(\bar{G})$ the involution $\bar{x}$ is a transvection and conjugate to a central involution in R. Consequently there is an involution $v \in \bar{R} - \langle t\rangle$ such that x or xt is conjugate to v. Now $Sp(4, 3)$ has just 2 classes of involutions, so v and vt are fused in $\bar{R}$, and consequently x and xt are conjugate in $\bar{G}$. Therefore $M \geqslant [x, M] \oplus [xt, M]$ and $\dim(M) \geqslant 32 > 21$.

Finally, suppose $\bar{R}$ stabilizes no M_{1j}. Since $\bar{R} \nleqslant S_5$, $\mid \bar{R}\colon \mathrm{stab}_{\bar{R}}(M_{11})\mid > 5$, so that $k > 5$ and $\dim(M) \geqslant k + 16 > 21$. This completes the proof of (5.7).

LEMMA 5.8. *If* $G = Sz(8)$, *then* $l(G, 2) \geqslant 8$.

Proof. If $Z(\bar{G}) = 1$, we proceed as in (4.1) (c). Hence suppose $Z(\bar{G}) \neq 1$. Since the multiplier of G is $Z_2 \times Z_2$ and $Z(\bar{G})$ is cyclic, $\mid Z(\bar{G})\mid = 2$. As in (4.1) let Q be 2-Sylow in G. Then $\mid N_G(Q)\mid = QH$, where $\mid H \mid = 7$ and H acts transitively on $(Q/Z(Q))^{\#}$ and on $Z(Q)^{\#}$. Consideration of the action of $\bar{H}$ on $\overline{Z(Q)}$ shows that $Z(\bar{Q}) = Q_0 \times Z(\bar{G})$ where Q_0 is elementary of order 8 and $\bar{H}$-invariant. If $Q_0 \trianglelefteq \bar{Q}$, then we consider $\bar{Q}/Q_0$ and show that $\bar{Q}$ splits over $Z(\bar{G})$, contradicting Gaschütz's Theorem. Thus $Q_0 \ntrianglelefteq \bar{Q}$ and since $Q_0Z(\bar{G})\bar{H}$ is maximal in $\overline{QH}$, $N(Q_0) = Q_0Z(\bar{G})\bar{H}$. Thus under the action of $\bar{Q}$ there are eight distinct conjugates of Q_0. On the other hand there are precisely eight subgroups of $\overline{Z(Q)}$ complementing $Z(\bar{G})$, and we have $\bar{Q}$ transitive on these subgroups. If $M_1 \leqslant M$ is an irreducible $Z(\bar{G})$ submodule of M, then $\ker(M_1)$ complements $Z(\bar{G})$. Considering the conjugates of M_1 under $\bar{Q}$ we have the result.

LEMMA 5.9. *If* $G = {}^2E_6(2)$, *then* $l(G, 2) \geqslant 3 \cdot 2^9$.

Proof. If $2 \nmid |Z(\bar{G})|$ then we proceed as in (4.8). Suppose then that $2 \nmid |Z(\bar{G})|$. As the multiplier of ${}^2E_6(2)$ is $Z_2 \times Z_6$ [8] and $Z(\bar{G})$ is cyclic, 2 exactly divides $|Z(\bar{G})|$. Let Q be as in Lemma 4.3. Then Q is extraspecial of order 2^{21} and P acts irreducibly on $Q/Z(Q)$ ([4, Section 4]). We proceed as in (5.8). Let $\langle t \rangle = Z(\bar{G})$. If $|O_2(\bar{Q}')| = 4$, then $M \geqslant M_1 \oplus M_2$ where M_1, M_2 are acted on faithfully by an extraspecial group of order 2^{21}. In this case $\dim(M) \geqslant 2^{10} + 2^{10} > 3 \cdot 2^9$. Now suppose that $O_2(\bar{Q}') = \langle x \rangle$ and $M = [M, x] \oplus C_M(x)$. Then $\dim([M, x]) \geqslant 2^{10}$ and it suffices to show that $\dim(C_M(x)) \geqslant 2^9$. In Q there is an element $\bar{v}$ conjugate to $\bar{x}$. This can be seen by noting that $\langle \bar{x} \rangle = U_r$ (r as in Section 2) and that there is a root $s \neq r$ conjugate to r such that $U_s \leqslant Q$. Then x is conjugate to $d = v$ or vt. Then on $[M, x]$, $\langle d \rangle$ gives a multiple of the regular representation and hence $|C_M(d) \cap [M, x]| = 2^9$. As $d \sim x$ this means that we also have $|C_M(x)| \geqslant 2^9$ and $\dim(M) \geqslant 3 \cdot 2^9$.

LEMMA 5.10. *If* $G = F_4(2)$, *then* $l(G, 2) \geqslant 44$.

Proof. If $Z(\bar{G}) = 1$, then we proceed as in (4.8) and obtain $\dim(M) \geqslant \frac{1}{2} 2^7(2^3 - 1) > 44$. Suppose $Z(\bar{G}) > 1$. Then $|Z(\bar{G})| = 2$ and generators and relations are known for $\bar{G}$ [8]. If Δ is a root system of type F_4 and $G = \langle U_r : r \in \Delta \rangle$ such that the usual commutator relations hold, then $\bar{G}$ is generated by subgroups Y_r of order 2 where there are certain commutator relations holding as follows. Let $U_r = \langle u_r \rangle$ and $Y_r = \langle y_r \rangle$. If r, s do not form an angle of 135° then $[y_r, y_s]$ is the same as $[u_r, u_s]$ with the obvious change in notation. If r, s form an angle of 135° then one of r, s is long and the other is short. Say r is long. Then $[y_r, y_s] = y_{r+s} y_{r+2s} z$ where $\langle z \rangle = Z(\bar{G})$.

We use the notation of 4.8. The group $P_4 = TR$, where $T = O_2(P_4)$ and $R \cong SO(7, 2)'$. $T = L \times S$ where $L = \langle U_t : t \text{ long}, U_t \leqslant T \rangle$ is elementary of order 2^6 and $S = \langle U_t : t \text{ short}, U_t \leqslant T \rangle$ is extraspecial of order 2^9. Let s be the short root such that $U_s = Z(S)$. Then $LU_s = L \times U_s$ and R acts on $L \times U_s$ preserving a nondegenerate quadratic form with radical U_s.

Checking the root system Δ and the commutator relations satisfied in $\bar{G}$ we have the following: $\bar{L} = L_0 \times \langle z \rangle$, with $L_0 = \langle y_t : t \text{ long}, U_t \leqslant T \rangle$, $\bar{S} = S_0 \times \langle z \rangle$ with $S_0 = \langle y_t : t \text{ short and } U_t \leqslant T \rangle$, $Z(S_0) = \langle y_s \rangle$, $L_0 \times \langle y_s z \rangle$ is $\bar{R}$-invariant and $\bar{R}$ isomorphic to $L \times \langle u_s \rangle$ under the isomorphism sending $y_t \to u_t$ for t long and $u_s \to y_s z$.

Since $(m)z = -m$ for each $m \in M$, we have $M = M_1 \oplus M_2$, where $M_1 = C_M(y_s)$ and $M_2 = C_M(y_s z)$. Moreover M is faithful, so $M_1 \neq 0$, $M_2 \neq 0$. Now y_s acts as -1 on M_2 and S_0 is extraspecial with center $\langle y_s \rangle$. Thus S_0 acts on M_2 and $\dim(M_2) \geqslant 2^4$. Also $\bar{R}(L_0 \times \langle y_s z \rangle)$ acts on M_1 and arguing as in (3.2) we see $\dim(M_1) \geqslant \frac{1}{2} 2^3(2^3 - 1) = 28$. Thus $\dim(M) \geqslant 44$ and (5.10) is proved.

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