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Pressure dependent yield criteria for polymers

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Summary

Different criteria have been proposed to include the influence of pressure (or mean normal stress) on the yield behavior of polymers. It is difficult to distinguish among them using the type of experiments that produce data used in two-dimensional plots of yield loci. This is due to the fact that the maximum range of values of mean normal stress is relatively small in such experiments. Marked differences between these criteria do occur however as the hydrostatic pressure or mean stress is altered substantially. Experiments that show the effect of applied pressure on tensile and/or compressive yield strength provide one means for describing such differences. This paper considers two forms of a pressure modified von Mises criterion and shows a comparison with available experimental information.

NOMENCLATURE

$\sigma_1, \sigma_2, \sigma_3$	principal stresses
σ_m	mean normal stress = $\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$
C	absolute value of compressive yield stress at atmospheric pressure
T	absolute value of tensile yield stress at atmospheric pressure
P	applied hydrostatic pressure
S	yield stress (general) under pressure P
S_c, S_t	compressive and tensile yield stresses under pressure P
R	normalized yield stress (general) under pressure P, ($R = S/T$)
R_c, R_t	normalized compressive and tensile yield stresses under pressure P, ($R_c = S_c/T$, $R_t = S_t/T$)
Y	ratio of C/T
M	normalized mean normal stress = σ_m/T
H	normalized hydrostatic pressure = P/T.

INTRODUCTION

There is no question that the yielding of polymeric solids depends upon the mean normal stress as well as the deviatoric components of the applied state of stress. In contrast and with apparently few exceptions the magnitude of the mean stress has little if any effect on the yield behavior of metals.

Raghava *et al.*[1] have proposed that a pressure modified von Mises criterion provides the best overall agreement with observed yield behavior of polymers. This criterion has been used by Stassi D'Alia[2] and, through the work and comments of Mehldahl[3], it apparently originated with Schleicher[4].

Earlier suggestions concerned with the yielding of polymers were put forth by Whitney[5], who used a pressure modified Tresca criterion, by Sternstein[6] and Bauwens[7], who used a pressure modified von Mises criterion (that differs from the one proposed by Raghava), and Bowden and Jukes[8], who added the Mohr–Coulomb (or Coulomb –Navier) criterion as a possibility. As some confusion might arise because of the various names appended to the many criteria, the interested reader can check the work of Paul[9] for a historical review and explanation.

It would appear, as discussed by Raghava[10], that neither the modified Tresca nor the Mohr–Coulomb criterion possesses the generality desired; thus, no further reference to those criteria is made in this paper. Rather, the major comparison is made between the criteria suggested by Sternstein[6] and Raghava[1].

Yield studies are most often conducted using stress states wherein at least one of three principal stresses is considered to be zero. What results of course is the usual yield locus plotted in two-dimensional stress space. Uniaxial tensile and compression tests, as well as internally pressurized thin wall tubes, which may be loaded axially or by torsion, provide essential data points the plot of which may be compared with predictions based upon a particular yield criterion. Such experiments are nearly always performed where the *external* pressure is atmospheric. This limits the range of mean normal stress experienced by the test material. Now if the yield behavior of the material is influenced by the mean normal stress, and even though the mean normal stress is not constant for the range of tests mentioned above, the maximum variation of the mean stress taken in terms of absolute values is of the order of the tensile yield plus compressive yield divided by two. In essence, it is quite small and because of this, a comparison of yield loci predicted from Sternstein's[6] criterion or Raghava's[1] shows little difference. There is only one way that these two criteria could demonstrate differences at ambient pressures; this has to do with the ratio of the absolute values of compressive to tensile yield strengths. If this ratio approaches 1.5 or greater, the criteria show decided theoretical differences especially in the third quadrant of yield locus plots[1,10]. The ellipse, which both describe, exhibits marked differences in the "third" quadrant and it should be noted that this behavior has nothing to do with anisotropy. To date, the greatest observed ratio of yield strengths is of the order of 1.3 which means that variations of theoretical yield loci based upon Raghava's or Sternstein's criterion are relatively small; one could conclude that little is gained from using one form over the other. However, typical yield locus studies are not conducted under varying states of externally imposed hydrostatic pressure because of the tremendous experimental difficulties involved; thus the range of σ_m is very restricted. Yet, it is these tests that would demonstrate the differences in these two forms of the pressure modified von Mises criterion. There does exist a fair amount of experimental information on the effect of imposed external pressure on the *tensile* yield strength of various polymers; much less information exists regarding the change in compressive yield strength under pressure. However, it is possible to compare the yield criterion proposed by Raghava with that of Sternstein by utilizing this pressure effect on tensile yield and comparing predictions with data available in the literature. This had been alluded to briefly[1] but further considerations resulted in the more complete thoughts put forth in this paper. Note that throughout the remainder of this paper, we consider an applied hydrostatic pressure as positive but that the stresses it induces are negative (*i.e.* compressive).

ANALYTICAL DEVELOPMENTS

As proposed elsewhere[1,2,10], the yield criterion suggested by Raghava (which accounts for pressure effects *and* differences in compressive and tensile yield strengths) can be written as:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + 6\,(C - T)\sigma_m = 2CT \qquad (1)$$

where σ_1, σ_2, σ_3 are principal stresses, σ_m is the mean stress ($\sigma_m = (\sigma_1 + \sigma_2 + \sigma_3)/3$) and C and T are the ***absolute*** values of the compressive and tensile yield stresses as determined under *atmospheric* conditions of pressure.

We might ask what occurs if a hydrostatic pressure is applied to a polymer which is subsequently subjected to axial loading (either tensile or compressive). The stress state is described by:

$$\sigma_2 = \sigma_3 = -P,\ \sigma_1 = S - P$$

where S is the stress needed to induce yielding under the applied hydrostatic pressure P. If the above values are inserted into eqn. (1), either of the following may be derived:

$$S^2 + (C - T)\,(S - 3P) = CT \qquad (2)$$

or

$$S^2 + (C - T)\,(3\sigma_m) = CT \qquad (3)$$

These equations can be normalized to produce

$$R_c = -\frac{(Y-1)}{2} - \frac{1}{2}[(Y+1)^2$$

$$+ 12H(Y-1)]^{1/2} \quad (4a)$$

$$R_t = -\frac{(Y-1)}{2} + \frac{1}{2}[(Y+1)^2$$

$$+ 12H(Y-1)]^{1/2} \quad (4b)$$

$$R = \pm [Y - 3M(Y-1)]^{1/2} \quad (5)$$

where

$$R = \frac{S}{T},\; Y = \frac{C}{T}, M = \frac{\sigma_m}{T},\; \text{and}\; H = \frac{P}{T}.$$

Note that eqns. (4a) and (4b) come from eqn. (2) wherein two unequal roots are found. Equation (4a) gives the normalized root for compressive yield stress (R_c) while eqn. (4b) is for the tensile yield (R_t) as functions of normalized hydrostatic pressure H.

In constrast, eqn. (3) gives in eqn. (5) equal and opposite roots ($R = R_c = R_t$) in terms of *magnitude* and as functions of the normalized mean stress M. The algebraic manipulations leading to these various equations may be found in the Appendix.

Expressions analogous to those in eqns. (2)–(5) may also be produced using the yield criterion proposed by Sternstein. The basic relationship can be expressed as

$$[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2}$$

$$+ \frac{\sqrt{2}(C-T)\, 3\sigma_m}{(C+T)} = \frac{2\sqrt{2}\, CT}{(C+T)}. \quad (6)$$

Along parallel lines as used above, the following can be derived:

$$\pm S + \frac{(C-T)(S-3P)}{(C+T)} = \frac{2CT}{(C+T)} \quad (7)$$

$$\pm S + \frac{(3\sigma_m)(C-T)}{(C+T)} = \frac{2\,CT}{(C+T)}. \quad (8)$$

Thus eqns. (7) and (8) derive from eqn. (6), as eqns. (2) and (3) were derived from eqn. (1). In normalized form there results

$$R_c = -\frac{3}{2}H(Y-1) - Y \quad (9a)$$

$$R_t = \frac{3}{2}H\left(\frac{Y-1}{Y}\right) + 1 \quad (9b)$$

$$\pm R = \frac{2Y}{(Y+1)} - \frac{3M(Y-1)}{(Y+1)} \quad (10)$$

As with the previous relations, eqn. (7) yields two unequal and opposite roots (*i.e.* eqns. (9a) and (9b) whereas eqn. (8) yields roots of equal magnitude and opposite sign. What might be helpful here is to realize that in the final analysis one is concerned with using these various equations to determine the compressive yield stress (S_c) or the tensile yield stress (S_t) as functions of applied hydrostatic pressure (P) or, more fundamentally, the mean normal stress (σ_m). Note too that when Y equals unity, both criteria degenerate to equivalent statements.

PREDICTIONS OF COMPRESSIVE YIELD STRESS (S_c) AND TENSILE YIELD STRESS (S_t) AS FUNCTIONS OF HYDROSTATIC PRESSURE OR MEAN STRESS

For convenience, the normalized expressions will be utilized for comparative purposes. Here we shall compare the results predicted by eqns. (4a) and (4b) with eqns. (9a) and (9b) and, similarly, those predicted by eqn. (5) with eqn. (10). To proceed, one need only determine the ratio Y by experiment; for our purposes at this juncture, let us assume that $Y = 1.3$. Figure 1 portrays the comparisons of eqns. (4a) and (4b) with eqns. (9a) and (9b). Several points are worth noting:

1. The behavior expressed by eqns. (4a) and (4b) shows a parabolic trend where the yield stress "bends back" towards the abscissa as pressure increases. In comparison, eqns. (9a) and (9b) show a linear increase of yield stress with pressure.

2. These plots all originate at a value of H equal to zero. Negative values would imply pressures less than atmospheric and it is felt that such conditions are not meaningful in this discussion.

3. The change in the absolute values from C

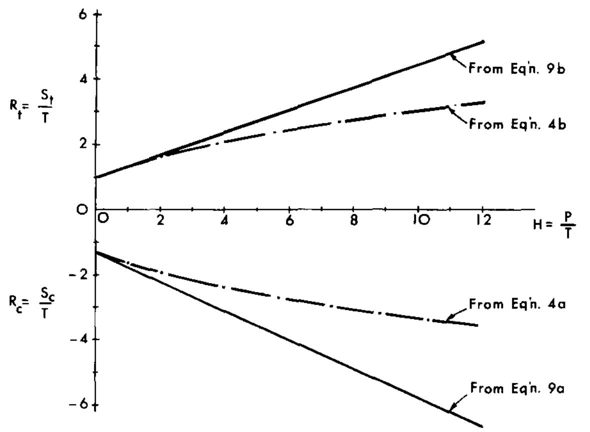

Fig. 1. Theoretical variation of tensile and compressive yield stress with increasing hydrostatic pressure. Note that "normalized" quantities are plotted and the curves, from eqns. (4a), (4b), (9a) and (9b), are based upon a C/T ratio of 1.3.

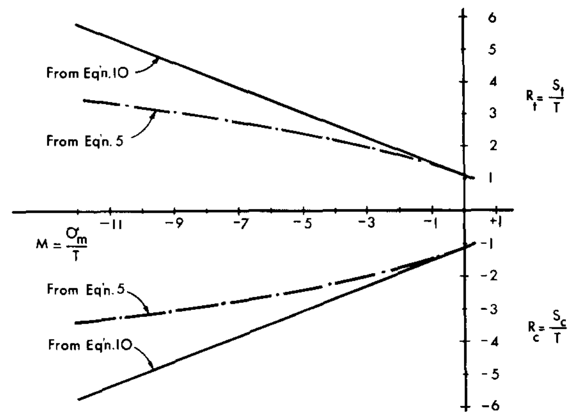

Fig. 2. Theoretical variations of tensile and compressive yield stress with mean normal stress. Note that "normalized" quantities are plotted and the curves, from eqns. (5) and (10), are based upon a C/T ratio of 1.3.

and T to S_c and S_t is the same in eqns. (4a) and (4b) whereas with eqns. (9a) and (9b), S_c increases by a greater amount than does S_t for a given pressure.

Figure 2 is a plot of eqns. (5) and (10) using $Y = 1.3$ and variations in M (or σ_m). Note that perfect symmetry about the abscissa prevails for either equation and that M or σ_m gets progressively more compressive as one moves to the left along the abscissa. Care must be exercized however or a decided misconception can arise from this Figure. If one applies a certain hydrostatic pressure then conducts a uniaxial tensile test until yielding occurs, the value of M associated with this case would be $(S_t - 3P)/3$ whereas a uniaxial compressive test would give a comparable M value of $(S_c - 3P)/3$. Now since S_c is by definition negative whereas S_t is positive the value of M associated with the compression test is *more* negative than its tensile counterpart. Thus one would expect the magnitude of R_c (or S_c) to have a larger absolute value than R_t (or S_t) for a particular hydrostatic pressure since the values of M (or σ_m) are *not* equivalent. Of course the same results could be found from Fig. 1. As an example, assume T = 5000 p.s.i. and a pressure 40,000 p.s.i. is applied to a specimen. From Fig. 1, $H = 8$ and from eqns. (4a) and (4b), one finds $R_t = 2.8$ and $R_c = 3.1$ thus S_t = 14000 p.s.i. and S_c = 15500 p.s.i.

From Fig. 2, using the same values of T and P and referring to the plot of eqn. (5), if R_t = 2.8 then M is about −7 so that

$$-7 = \frac{S_t - 3P}{3T} \text{ or } -21(5000) - 14000 = -3P$$

or $P = 119000/3 \approx 40{,}000$ as it must be from the assumptions. It should be obvious that M must be greater if one is to get an S_c of −15,500 p.s.i. for the same applied pressure. Similar calculations could be performed using the plots of eqns. (9a), (9b) and (10) but the point under consideration would be the same as was just demonstrated.

COMPARISON OF THEORY AND EXPERIMENTAL RESULTS

One of the unfortunate consequences, regarding much of the data reported in the literature wherein pressure effects upon yielding were studied, is that the value of compressive yield under atmospheric conditions was not measured. In addition, a number of methods for defining "yielding" have been used. Christiansen *et al.*[11] have compiled a fairly extensive list of data regarding pressure and yield stress for a number of polymers. Although that specific paper[11] was apparently never published, some of the contents were[12]; the original sources[13–16] from which those data[11] were compiled are noted. The main thrust of the unpublished work[11], and to some extent the published work[12], was to investigate the applicability of the equation numbered (6) in this paper and to determine if pressure effects could be related to certain

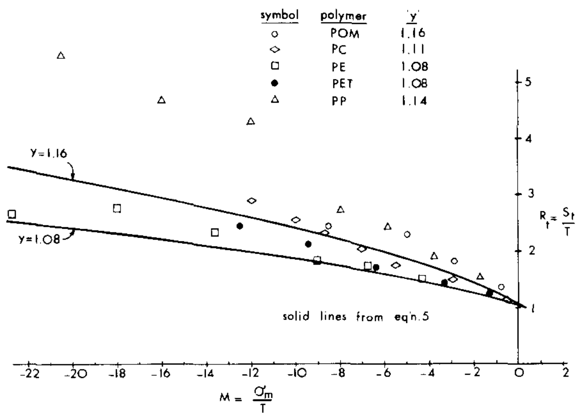

Fig. 3. Effect of mean normal stress on the tensile yield stress for various polymers. Data points are from ref. 11 while solid lines depict eqn. (5) for the range of C/T ratios involved.

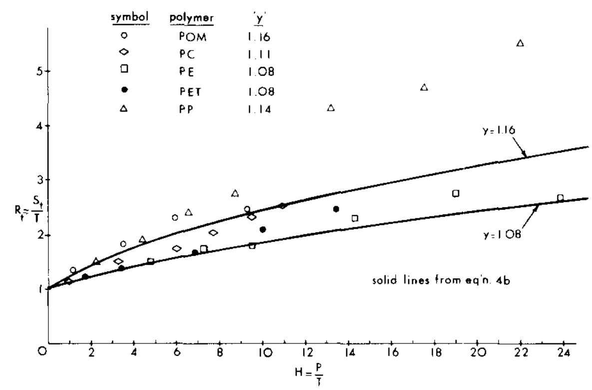

Fig. 5. Effect of hydrostatic pressure on the tensile yield stress for various polymers. Data points are from ref. 11 while solid lines depict eqn. (4b) for the range of C/T ratios involved.

properties of the various polymers. As only values of T and S_t (as a function of applied pressure) were available, values of Y (or in essence, C) were computed[11]. In view of other studies[5–8,10], some of those computed values appear low. This seems especially true in regard to polytetrafluoroethylene (PTFE) where Y was given as 1.03. With a value of Y equal to unity, both eqns. (1) and (6) degenerate to the usual von Mises criterion (pressure independent) and the entire point of this paper is negated. For that reason, the few data points for PTFE are not considered here. In addition, no tensile yield stress was given for polystyrene since it "failed in a brittle manner" under atmospheric pressure. Without such a measurement, the remaining data could not be used for our purposes. Useful data then remained for five polymers, namely, polyoxymethylene (POM), polyethylene (PE), a type of polycarbonate (PC), polyethylene terephthalate (PET), and polypropylene (PP). The range of values for Y (or C/T) for these five materials was from 1.08 (PE and PET) to 1.16 (POM). Figure 3 shows a plot of eqn. (5) using Y values of 1.08 and 1.16; this should bound all test points. Also included are the actual test data showing the behavior of tensile yield stress, S_t, normalized to give values of R_t, as a function of the normalized mean stress, M. Figure 4 displays the plots of eqn. (10) for the Y values of 1.08 and 1.16; again the test points are included. Two Figures were felt necessary here in order to avoid overlapping of the plotted lines; this could cause confusion and would show less clearly the comparisons that are to be drawn from these Figures.

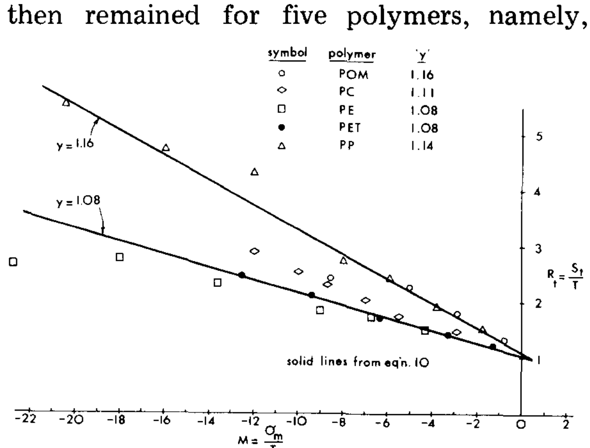

Fig. 4. Effect of mean normal stress on the tensile yield stress for various polymers. Data points are from ref. 11 while solid lines depict eqn. (10) for the range of C/T ratios involved.

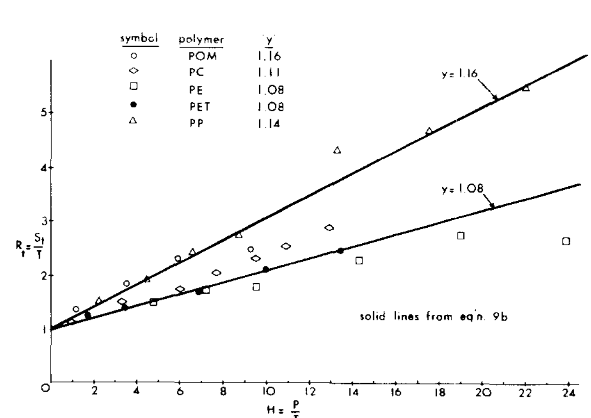

Fig. 6. Effect of hydrostatic pressure on the tensile yield stress for various polymers. Data points are from ref. 11 while solid lines depict eqn. (9b) for the range of C/T ratios involved.

Figures 5 and 6 present a similar comparison wherein R_t is plotted against the normalized hydrostatic pressure, H. There, eqns. (4b) and (9b) are used as they define values of the *tensile* yield stress as a function of pressure.

DISCUSSION

It is obvious that the forms of yield criteria, described by eqns. (1) and (6), give quite different predictions regarding the influence of increasing pressure (or mean normal stress) on the "yield stress" of concern. If one views *trends* as well as the fit of individual test points, the criterion proposed by Raghava appears to predict more reasonable behavior than does the criterion put forth by Sternstein. This can be seen in Figs. 3—6 where, except for the three points regarding PP at the highest pressures, the overall behavior of these five different polymers does show a somewhat better correlation on Figs. 3 and 5 compared with Figs. 4 and 6.

These Figures should also indicate why the usual predictions of yielding from yield loci plots would indicate a relative insensitivity between the two criteria under discussion. Since M would vary between ± 1, it is truly impossible to give great preference to one criterion over the other. However, since these are both presented as pressure dependent yield criteria, a better comparison of their real validity should be made under conditions of high hydrostatic pressure. This is exactly the point of the present paper.

In fairness, one could still argue that there is but little to choose from in terms of prediction and measurement as illustrated by Figs. 3—6. We can, however, point out the following:

1. It was mentioned earlier that the values of Y tabulated elsewhere[11] and used in this study were actually *calculated* and not derived from measurement. Compared with other published values[1], the values of Y used in this paper seem a bit on the low side. Now as Y increases, the predictions based upon eqns. (6), (9) or (10) are more drastically displaced from the abscissa than are those based upon eqns. (1), (4) or (5). Thus, since we question these *computed* Y values it is our conviction that an even better correlation would exist in Figs. 3 and 5 if more accurate values of Y had been available. This certainly indicates a need for further experimentation.

2. The tendency for a "bending back" to occur when yield stress is plotted as a function of applied pressure has been earlier pointed out by Nadai[17]. This trend can be inferred from the work of Sardar *et al.*[13] and, to some extent, from Ainbinder *et al.*[18]. Since this behavior is predicted by the yield criterion expressed by eqn. (1) and not by eqn. (6), it causes us to express greater confidence in the former over the latter.

In passing we might also add that in view of the number of methods used to define (and measure) yield stress, and in view of the sparsity of information available in regard to the effect of pressure on compressive yield strength, it is perhaps amazing that the fit of data to prediction is as good as is observed in Figs. 3—6. Future investigators should include in adequate detail the method used to *define* yield stress.

CONCLUSIONS

The macroscopic yield behavior of various polymers can be reasonably described by either of two forms of a pressure modified von Mises criterion under the usual pressure conditions at which yield studies are generally conducted. However, as the mean of the stress state becomes more compressive (or as the pressure condition becomes more pronounced) the criterion suggested by Raghava seems to provide a better correlation with experimental results than does the form used by Sternstein and others[6,7].

REFERENCES

1 R. Raghava, R.M. Caddell and G.S.Y. Yeh, *J. Mater. Sci.*, 8 (1973) 225.
2 F. Stassi D'Alia, *Meccanica*, 3 (1967) 178.
3 A. Mehldahl, *Brown Boveri Rev.*, 31 (1944) 260.
4 F. Schleicher, *Z. Angew. Math. Mech.*, 6 (1926) 199.
5 W. Whitney and R.D. Andrews, *J. Polymer Sci.*, C5, 16 (1967) 2981.
6 S.S. Sternstein and L. Ongchin, *ACS Polymer Prep.*, 10 (1969) 1117.
7 J.C. Bauwens, *J. Polymer Sci.*, A-2, 8 (1970) 893.
8 P.B. Bowden and J.A. Jukes, *J. Mater. Sci.*, 7 (1972) 52.
9 B. Paul, in H. Liebowitz (ed.), *Fracture*, Vol. II, Academic Press, New York, 1968, p. 446.

10 R. Raghava, *Ph. D. Dissertation*, University of Michigan, Ann Arbor, Michigan, 1972.
11 A.W. Christiansen, S.V. Radcliffe and E. Baer, personal communication, 1971.
12 A.W. Christiansen, E. Baer and S.V. Radcliffe, *Phil. Mag.*, 24 (1971) 451.
13 D. Sardar, S.V. Radcliffe and E. Baer, *Polym. Eng. Sci.*, 8 (1968) 290.
14 K.D. Pae and D.R. Mears, *Polymer Letters*, 6 (1968) 269.
15 G. Biglione, E. Baer and S.V. Radcliffe, *Proc. 2nd Intern. Conf. on Fracture, Bristol, England*, 1969.
16 D.R. Mears, K.D. Pae and J.A. Sauer, *J. Appl. Phys.*, 40 (1969) 4229.
17 A. Nadai, *Theory of Flow and Fracture of Solids*, McGraw-Hill, New York, 1950, p. 220.
18 S.B. Ainbinder, M.G. Laka and I Yu Maiors, *Mekh. Polim.*, 1 (1965) 65.

APPENDIX

Starting with eqn. (1) we have

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2 + 2(C-T)\sigma_m = 2CT\,. \qquad (A.1)$$

Now with $\sigma_2 = \sigma_3 = -P$, $\sigma_1 = S - P$ in eqn. (A.1) one gets

$$(S-P+P)^2+(-P+P)^2+(-P-S+P)^2 + 2(C-T)(S-P-P-P) = 2CT$$

or

$$S^2+S^2+2(C-T)(S-3P) = 2CT\,,$$

thus,

$$S^2+(C-T)(S-3P) = CT. \qquad (A.2)$$

or eqn. (2)

Equation (A.2) can be written as

$$S^2+S(C-T)-[3P(C-T)+CT] = 0\,. \qquad (A.3)$$

Using $R = S/T$, $Y = C/T$, $H = P/T$, eqn. (A.3) becomes

$$R^2+R(Y-1)-[3H(Y-1)+Y] = 0\,.$$

Thus,

$$R = -\tfrac{1}{2}(Y-1) \pm \tfrac{1}{2}[(Y-1)^2 + 12H(Y-1)+4Y]^{1/2}$$

or

$$R = -\tfrac{1}{2}(Y-1) \pm \tfrac{1}{2}[(Y+1)^2 + 12H(Y-1)]^{1/2}\,. \qquad (A.4)$$

The two roots of eqn. (A.4) give the expressions for R_c and R_t as expressed by eqns. (4a) and (4b).
Now since $(S-3P)$ equals $(3\sigma_m)$, eqn. (A.2) may be written as

$$S^2+(C-T)(3\sigma_m) = CT\,. \qquad (A.5)$$

or eqn. (3)

Now eqn. (A.5) may be written as

$$\left(\frac{S}{T}\right)^2 + \frac{3\sigma_m}{T}\left(\frac{C}{T}-1\right) = \frac{C}{T}$$

which in normalized form becomes

$$R^2+3M(Y-1) = Y \qquad (A.6)$$

where

$$M = \frac{\sigma_m}{T}\,.$$

Now eqn. (A.6) gives

$$R = \pm[Y-3M(Y-1)]^{1/2} \qquad (A.7)$$

which is eqn. (5).
The form used by Sternstein or Bauwens may be expressed as

$$\tau_0 + A\sigma_m = V \qquad (A.8)$$

where τ_0 is the octahedral shear stress and A and V are constants to be determined.

Now in terms of principal stresses, eqn. (A.8) becomes

$$\tfrac{1}{3}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]^{1/2} + \frac{A}{3}(\sigma_1+\sigma_2+\sigma_3) = V\,. \qquad (A.9)$$

Considering the case of unaxial stress eqn. (A.9) becomes

$$[2\sigma_1^{\,2}]^{1/2} + A\sigma_1 = 3V,$$

or

$$\pm\sigma_1 + \frac{A\sigma_1}{\sqrt{2}} = \frac{3V}{\sqrt{2}}\,. \qquad (A.10)$$

Now eqn. (A.10) provides two relationships,

$$+\sigma_1 + \frac{A\sigma_1}{\sqrt{2}} = \frac{3V}{\sqrt{2}}$$

and

$$-\sigma_1 + \frac{A\sigma_1}{\sqrt{2}} = \frac{3V}{\sqrt{2}}\,.$$

Considering $T = +\sigma_1$ and $C = -\sigma_1$, one gets

$$T + \frac{AT}{\sqrt{2}} = \frac{3V}{\sqrt{2}},$$

$$C - \frac{AC}{\sqrt{2}} = \frac{3V}{\sqrt{2}}.$$

From these equations, the constants A and V are found to be

$$A = \frac{\sqrt{2}\,(C-T)}{(C+T)} \text{ and } V = \frac{2\sqrt{2}\,CT}{3(C+T)}.$$

Now if these values for A and V are substituted into eqn. (A.9), there results

$$[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2]^{1/2} + \frac{\sqrt{2}\,(C-T)3\sigma_m}{(C+T)} = \frac{2\sqrt{2}\,CT}{(C+T)}. \quad \text{(A.11)}$$

and eqn. (A.11) is eqn. (6).
Again with $\sigma_2 = \sigma_3 = -P$, $\sigma_1 = S - P$, eqn. (A.11) becomes

$$[2S^2]^{1/2} + \frac{\sqrt{2}\,(C-T)(S-3P)}{(C+T)} = \frac{2\sqrt{2}\,CT}{(C+T)},$$

or

$$[S^2]^{1/2} + \frac{(C-T)(S-3P)}{(C+T)} = \frac{2CT}{(C+T)} \quad \text{(A.12)}$$

or

$$\pm S + \frac{(C-T)(S-3P)}{(C+T)} = \frac{2CT}{(C+T)}. \quad \text{(A.13)}$$

or eqn. (7)

In normalized form, eqn. (A.13) becomes

$$\pm R + \frac{(Y-1)(R-3H)}{(Y+1)} = \frac{2Y}{(Y+1)}.$$

Now,

$$\pm R + \frac{R(Y-1)}{Y+1} = \frac{2Y}{Y+1} + \frac{3H(Y-1)}{Y+1}$$

or

$$\pm R + \frac{R(Y-1)}{Y+1} = \frac{3H(Y-1)+2Y}{(Y+1)}.$$

Consider the $-R$ root (which is R_c in our notation), then

$$-R(Y+1) + R(Y-1) = 3H(Y-1) + 2Y$$

or

$$R(-2) = 3H(Y-1) + 2Y;$$

thus

$$R_c = -\tfrac{3}{2}\,H(Y-1) - Y. \quad \text{(A.14)}$$

or eqn. (9a)

Considering the $+R$ root (*i.e.* R_t), then

$$R(Y+1) + R(Y-1) = 3H(Y-1) + 2Y,$$

or

$$2RY = 3H(Y-1) + 2Y;$$

thus

$$R_t = \tfrac{3}{2}H\left(\frac{Y-1}{Y}\right) + 1. \quad \text{(A.15)}$$

or eqn. (9b)

Again, since $(S-3P)$ equals $(3\sigma_m)$, eqn. (A.13) equals

$$\pm S + \frac{(C-T)\,3\sigma_m}{(C+T)} = \frac{2\,CT}{(C+T)}. \quad \text{(A.16)}$$

or eqn. (8)

In normalized form eqn. (A.16) is

$$\pm R = \frac{2Y}{(Y+1)} - \frac{3M\,(Y-1)}{(Y+1)}. \quad \text{(A.17)}$$

or eqn. (10)