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Strongly Nonlinear Perturbations of Nonnegative Boundary Value Problems with Kernel

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1. Introduction

The purpose of this paper is to discuss nonlinear boundary values problems of the form

$$Au + g(x, u) = f(x) \text{ in } \Omega, \tag{1.1}$$

where the linear operator A is a nonnegative self-adjoint operator on the real Hilbert space $L_2(\Omega)$, and the kernel of A is one dimensional. In the applications A will be defined by a differential operator subject to boundary conditions. The results extend easily to certain monotone nonlinear operators A. (See Remark 7 of Sect. 3.)

As examples, the reader should bear in mind the following. Let Ω be a nice bounded open subset of $\mathbb{R}^n$ and $g: \mathbb{R} \to \mathbb{R}$ continuous.

Example 1. Neumann problem for the Laplacian:

$$\begin{aligned} -\Delta u + g(u) &= f(x) \quad \text{in } \Omega, \\ \partial u/\partial n &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Example 2. Neumann problem for the biharmonic operator:

$$\begin{aligned} \Delta^2 u + g(u) &= f(x) \quad \text{in } \Omega, \\ \Delta u = \partial\, \Delta u/\partial n &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

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EXAMPLE 3. Dirichlet problem at lowest eigenvalue:

$$\begin{aligned} -\Delta u - \lambda_1 u + g(u) &= f(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

EXAMPLE 4. Dirichlet problem for biharmonic at lowest eigenvalue:

$$\begin{aligned} \Delta^2 u - \lambda_1 u + g(u) &= f(x) \quad \text{in } \Omega, \\ u = \partial u/\partial n &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where λ_1 is the lowest eigenvalue of Δ^2 with Dirichlet boundary conditions. Here we assume that λ_1 has multiplicity one.

These examples have the following features. For Neumann problems, the kernel consists of the constant functions. In particular, the elements of the kernel are strictly of one sign. In example 3, the lowest eigenvalue is of one sign, but not strictly so. The last example has kernel spanned by an element which may change sign. In all cases, the kernel consists of real analytic functions on Ω.

In the case where g is continuous and $g(-\infty) < g(s) < g(+\infty)$ necessary and sufficient conditions for solvability of these problems were obtained by Landesman and Lazer [9]. They showed that if θ is a basis for the kernel, then there is a solution if and only if

$$g(+\infty)\int_{\theta>0}\theta.+g(-\infty)\int_{\theta<0}\theta.>\int f\theta.>g(-\infty)\int_{\theta>0}\theta.+g(+\infty)\int_{\theta<0}\theta. \quad (1.2)$$

Actually, Landesman and Lazer treated only the second-order case but the results have been extended by several authors. (See [4] for references.)

All results of this type assumed that the function g is bounded or at least of slow growth at infinity. It is the purpose of this paper to show that in the case where A is *nonnegative*, such restrictions on the growth of g can be dropped. If we let

$$g(+\infty) = \varliminf_{s\to+\infty} g(s), \quad (1.3)$$

$$g(-\infty) = \varlimsup_{s\to-\infty} g(s), \quad (1.4)$$

then, provided that

$$g(+\infty) > g(-\infty), \quad (1.5)$$

each of the boundary value problems discussed above is solvable if (1.2) holds. In addition, if

$$g(-\infty) < g(s) < g(+\infty) \qquad \text{for all} \quad s \in \mathbb{R}, \quad (1.6)$$

then (1.2) is necessary. Notice that the only remaining restriction on g is (1.5) and that no growth conditions are imposed at infinity. In particular, $g(\pm\infty) = \pm\infty$ are allowed.

It is worth noting that when one is at a higher eigenvalue, the Landesman–Lazer condition (1.2) is no longer sufficient for solvability in the presence of a strong nonlinearity. For example, consider the Neumann problem

$$-\Delta u - \lambda_j u + g(u) = f(x) \quad \text{in } \Omega,$$
$$\partial u/\partial n = 0 \quad \text{on } \partial\Omega,$$

where λ_j is any eigenvalue other than the lowest, and

$$\begin{aligned} g(s) &= s^2 \quad &\text{if} \quad s > 0, \\ &= 0 \quad &\text{if} \quad s \leqslant 0. \end{aligned}$$

Write the differential equation as

$$-\Delta u + N(u) = f(x) \tag{1.7}$$

and observe that $i = \inf_s N(s) > -\infty$.

Since θ must assume both positive and negative values on sets of nonzero measure, the solvability condition (1.2) holds for all f. Choose $f \in C_0^\infty(\Omega)$ with $\int_{|\Omega|} f > i \mid \Omega \mid$. Integrating (1.7) over Ω yields $i \mid \Omega \mid < \int_\Omega N(u) = \int f > i \mid \Omega \mid$ which is impossible. Thus there can be no solution even though (1.2) is satisfied. Strongly nonlinear problems at higher eigenvalues remain poorly understood.

Recently, several other authors have studied strongly nonlinear nonnegative problems as above. We mention in particular, the work of Kazdan and Warner [8], Brezis *et al.* [1–3], and Hess [5–7]. The first-named authors use maximum principle techniques to obtain sharp results in the second-order case. The second authors have developed an abstract approach based on a theory of sums of monotone operators. It was the work of these authors which motivated our interest in the present problems. In particular, we wish to thank Professor Nirenberg for stimulating conversations and correspondence.

The work of Hess [7] is most closely related to ours as it avoids both monotonicity and growth conditions. He has an abstract sufficient condition (Theorem 3.3) for solvability which is verified in the case of Example 1. This verification would easily extend to any problem for which $\theta \neq 0$ on $\bar{\Omega}$. However, when θ changes sign, the problems are more difficult. One may view the main step in our proof as a verification that if the Landesman–Lazer condition (1.2) is satisfied, then so is Hess' abstract condition. We will, however, give a self-contained proof.

A key ingredient in the proof of our result is an a priori bound on solutions of

(1.1) which is derived by estimating separately the projection of u on $\ker A$ and on $(\ker A)^\perp$. This technique is reminiscent of the method of splitting in functional analysis, and both authors would like to acknowledge their debt to Professor Cesari, who taught them this technique.

2. The Main Result

Let A be a nonnegative self-adjoint operator on the real Hilbert space $L_2(\Omega)$, where Ω is a bounded open set in $\mathbb{R}^m$. Suppose in addition that $\dim \ker A = 1$ and choose $\theta \neq 0$, $\theta \in \ker A$. Let $V = D(A^{1/2})$ with the graph norm and let $a: V \times V \to \mathbb{R}$ be the associated quadratic form

$$a(u, v) = (A^{1/2}u, A^{1/2}v), \qquad \forall u, v \in V.$$

For any $f \in L_2(\Omega)$ we associate the linear functional on V defined by

$$V \ni v \to (v, f)_{L_2(\Omega)} .$$

Then $\sup_{x \in V\setminus\{0\}}(|(v, f)|/\| v \|_V)$ defines a norm on $L_2(\Omega)$. Let V' be the completion of $L_2(\Omega)$ in this norm. The form $(\ , \ )_{L_2(\Omega)}$ extends from $V \times L^2$ to $V \times V'$ and with this pairing, V and V' are duals. We make the following basic assumptions about A:

$$V \cap L_\infty(\Omega) \text{ is dense in } V. \tag{2.1}$$

$$\text{The imbedding } V \to L_2(\Omega) \text{ is compact.} \tag{2.2}$$

In practice, a and V are often more accessible than A and $D(A)$. In the examples of the Introduction, they are given as follows:

Example 1. $a(u, v) = \int_\Omega \nabla u \cdot \nabla v$, $V = H_1(\Omega)$.

Example 2. $a(u, v) = \int \Delta u \, \Delta v$, $V = H_2(\Omega)$.

Example 3. $a(u, v) = \int_\Omega (\nabla u \cdot \nabla v - \lambda_1 uv)$. $V = \mathring{H}_1(\Omega)$.

Example 4. $a(u, v) = \int_\Omega (\Delta u \, \Delta v - \lambda_1 uv)$, $V = \mathring{H}_2(\Omega)$.

In each case, (2.1) and (2.2) hold as well-known properties of Sobolev spaces provided Ω satisfies a cone condition.

Suppose $g \in C(\bar{\Omega} \times \mathbb{R})$ and define $g(x, \pm\infty)$ by

$$g(x, +\infty) = \varliminf_{s \to +\infty} g(x, s), \tag{2.3}$$

$$g(x, -\infty) = \varlimsup_{s \to -\infty} g(x, s). \tag{2.4}$$

We need to assume that g ultimately increases. That is

$$g(x, +\infty) > g(x, -\infty) \qquad \text{for all} \quad x \in \bar{\Omega}. \tag{2.5}$$

Actually it is necessary for (2.3) and (2.4) to hold uniformly in the sense

$$\begin{array}{l}\text{For any } \epsilon > 0, \text{ and } M \in C(\bar{\Omega}, \mathbb{R}) \text{ with } g(x, +\infty) > M(x)\\ \text{for all } x \in \bar{\Omega}, \text{ there is a } \rho \text{ such that for all } x \in \bar{\Omega}, \text{ and } s \geqslant \rho.\\ \text{Similarly } g(x, -\infty) < M(x), \text{ then } g(x, -s) - \epsilon < M(x)\\ \text{for all } x \in \bar{\Omega} \text{ and } s \in [\rho, \infty).\end{array} \tag{2.6}$$

THEOREM. *Suppose A, a, g, θ are as above (satisfying (2.1), (2.2), (2.5), (2.6)) and that $f \in V'$. Then Eq. (1.1) has a weak solution $u \in V$ provided*

$$\int_{\theta>0} g(x, +\infty)\,\theta + \int_{\theta<0} g(x, -\infty)\,\theta$$

$$> \langle f, \theta\rangle > \int_{\theta>0} g(x, -\infty)\,\theta + \int_{\theta<0} g(x, +\infty)\,\theta. \tag{2.7}$$

Note. By weak solution we mean that $u \in V$, $ug(x, u) \in L_1(\Omega)$, $g(x, u) \in L_1(\Omega) \cap V'$, and

$$a(u, v) + \langle g(x, u), v\rangle = \langle f, v\rangle \qquad \forall v \in V. \tag{2.8}$$

Proof. By (2.5) and (2.6), we may choose a function $\xi \in C(\bar{\Omega})$ such that

$$g(x, s) > \xi(x) > g(x, -\sigma) \qquad \forall x \in \bar{\Omega}, \quad \forall s, \sigma \in [\rho, \infty).$$

Replacing f by $f - \xi$ and $g(x, s)$ by $g(x, s) - \xi(x)$ conditions (2.6) and (2.7) are still satisfied and without loss of generality we may assume that

$$g(x, s) > 0 > g(x, -\sigma) \qquad \forall x \in \Omega,\ s, \sigma \in [\rho, \infty). \tag{2.9}$$

Let

$$b(u, u) = a(u, u) + \int_\Omega ug(x, u) - \langle f, u\rangle \tag{2.10}$$

which is defined for all $u \in V$ such that $ug(x, u) \in L_1(\Omega)$. The main step in the proof is to show that there is an $R > 0$ such that

$$\| u \|_V > R \Rightarrow b(u, u) > 0. \tag{2.11}$$

Here $\| \cdot \|_V$ is the norm in the Hilbert space V.

Notice that a weak solution u must satisfy $b(u, u) = 0$ so this yields an a priori estimate $\| u \|_V \leqslant R$ for all solutions.

Write $u = u_1 + e\theta$, where $e\theta$ is the orthogonal projection of u on ker A. Then

$$\| u \|_V \cong (\| u_1 \|_V^2 + | e |^2)^{1/2}$$

so it suffices to show that $b(u, u) \leqslant 0$ implies a bound on $\| u_1 \|_V$ and $| e |$. We bound b from below as follows.

$$a(u, u) = \| u_1 \|_V^2 ,$$

$$|\langle f, u \rangle| \leqslant \| u_1 \|_V \| f \|_{V'} + e\langle f, \theta \rangle \leqslant c(\| u_1 \|_V + | e |),$$

where c will be used to denote a constant independent of u. The first estimate for the $\int_\Omega ug(x, u)$ term comes from (2.9) which implies that $sg(x, s) \geqslant 0$ if $| s | \geqslant g$. Let $\mu = \inf_{x,s} sg(x, s) > -\infty$ so that

$$\int_\Omega ug(x, u) \geqslant \mu \, | \Omega |,$$

and consequently

$$b(u, u) \geqslant \| u_1 \|_V^2 - c(\| u_1 \|_V + | e | + 1).$$

Therefore

$$b(u, u) \leqslant 0 \Rightarrow \| u_1 \| \leqslant c(| e |^{1/2} + 1). \tag{2.12}$$

To complete the proof of (2.11) we will show that there is an $N > 0$ so that $\| u \|_V \leqslant c(| e |^{1/2} + 1)$ and $| e | > N$ imply $b(u, u) > 0$. Together with (2.12) this shows that $b(u, u) \leqslant 0$ implies $\| u \|_V \leqslant R$, thereby proving (2.11). Assume $\| u_1 \|_V \leqslant c(| e |^{1/2} + 1)$. Since $a \geqslant 0$ we have

$$b(u, u) \geqslant \int (u_1 + e\theta)\, g(x, u_1 + e\theta)\, dx - \langle f, u_1 + e\theta \rangle. \tag{2.13}$$

The idea of the estimate is that for $| e |$ very large, the right-hand side is approximately equal to

$$\int_\Omega e\theta g(x, e\theta) - e\langle f\theta \rangle,$$

which for e positive is roughly equal to

$$e \left[\int_{\theta > 0} \theta g(x, +\infty) + \int_{\theta < 0} \theta g(x, -\infty) - \langle f, \theta \rangle \right],$$

which is positive by virtue of condition (2.7). For e negative one finds

$$e\left[\int_{\theta>0}\theta g(x,-\infty)+\int_{\theta<0}\theta g(x,-\infty)-\langle f,\theta\rangle\right],$$

which is positive by the other half of (2.7). Our task is to make these arguments quantitative.

Because of the uniformity of the approach to $g(x, +\infty)$ as described in (2.6) and the strict inequality in (2.7), we may choose $r > g$ so that for any Lebesque measurable τ on Ω with $|\tau(x)| > r$ and sign $\tau(x) =$ sign $\theta(x)$ for almost all $x \in \Omega$ we have

$$\int_{|\theta|>0}\theta g(x,\tau)\,dx-\langle f,\theta\rangle>0-\int_{|\theta|>0}\theta g(x,-\tau)-\langle f,\theta\rangle>0.$$

Note that $|\{|\theta|>\delta\}| \to |\{|\theta|>0\}|$ as $\delta \to 0$ so with $\mu = \inf_{s,x} sg(x, s)$, we may choose $\delta \in (0, 1]$, $n > 1$ and $\eta > 0$ so that for all τ as above

$$\left(1-\frac{1}{n}\right)\int_{|\theta|>e\delta/n}\theta g(x,\tau)-\langle f,\theta\rangle>\eta. \tag{2.14.i}$$

$$-\left(1-\frac{1}{n}\right)\int_{|\theta|>e\delta/n}\theta g(x,-\tau)+\langle f,\theta\rangle>\eta. \tag{2.14.ii}$$

The integral of $ug(x, u)$ is analyzed in pieces. With n as in (2.14)

$$\int_\Omega ug(x,u)=\int_{\substack{|u_1|<|e\delta/n|\\|\theta|>\delta}}+\int_{\substack{|u_1|>e\delta/n\\|\theta|>\delta}}+\int_{|\theta|<\delta}.$$

The last two integrals are estimated as follows. Since $\int_\omega ug(x, u) \geqslant \mu\,|\,\omega\,| \geqslant \mu\,|\,\Omega\,|$, we have

$$\int_\Omega ug(x,u)\,dx\geqslant\int_{\substack{|u_1|<(e\delta/n)\\|\theta|>\delta}}ug(x,u)\,dx+\mu\,|\,\Omega\,|. \tag{2.15}$$

The integral on the right is estimated using the fact that for e large $|\,u_1\,| \times |\,e\theta\,|$ on the domain of integration.

Let $e_0 = \delta^{-1}(1-(1/n))^{-1}\Omega$. If $|\,e\,| > e_0$, then in the domain of integration on the right-hand side of (2.15) we have

$$\begin{gathered}
|\,u_1+e\theta\,|>r,\\
ug(x,u)\geqslant 0 \qquad (\text{since } |\,u\,|>r \text{ and } r>\rho),\\
\operatorname{sign} g(x,u_1+e\theta)=\operatorname{sign}(u_1+e\theta)=\operatorname{sign}(e\theta),\\
ug(x,u)>(1-(1/n))\,e\theta g(x,u_1+e\theta)
\end{gathered}$$

so that (2.15) yields

$$b(u, u) \geqslant \left(1 - \frac{1}{n}\right) e \int_{\substack{|u_1| < e\delta/n \\ |\theta| > \delta}} \theta g(x, u_1 + e\theta) dx - e\langle f, \theta\rangle - \langle f, u_1\rangle + \mu \,|\, \Omega \,|$$

$$\geqslant |\, e \,|\, \eta - \| f \| \, \| u_1 \| + \mu \,|\, \Omega \,| \quad \text{(by 2.14)}$$

$$\geqslant |\, e \,|\, \eta - c(|\, e \,|^{1/2} + 1) + \mu \,|\, \Omega \,|$$

for sufficiently large e.

Thus we have shown that

$$\| u_1 \|_V \leqslant c(|\, e \,|^{1/2} + 1)$$

and (2.16)

$$|\, e \,| \geqslant e_0 \text{ imply } b(u, u) > |\, e \,|\, \eta - c_1 \,|\, e \,|^{1/2} - c_2 \,.$$

Choose $N \geqslant e_0$ so larger that if $|\, e \,| \geqslant N$ the quantity on the right is positive. Then (2.12) and (2.16) together imply that if $b(u, u) \leqslant 0$ then

$$\| u_1 \|_V \leqslant c(|\, e \,|^{1/2} + 1) \qquad \text{and} \qquad |\, e \,| < N,$$

thereby proving inequality (2.11).

Given this basic estimate there are several ways to proceed with an existence proof. We use Galerkin's method. First observe that since $L^2(\Omega)$ is dense in V', it follows that V' is separable and therefore V is separable. Since (2.1) holds we may choose $V_n \subset V \cap L_\infty$ such that

$$\dim V_n < \infty, \qquad V_{n+1} \supset V_n\,, \qquad \bigcup_n V_n \text{ is dense in } V.$$

We will find $u_n \in V_n$ so that

$$b(u_n\,, v) = 0 \qquad \text{for all} \quad v \in V_n\,. \tag{2.17}$$

The solution constructed as the limit of a subsequence of the u_n. Define T_n: $V_n \to V_n'$ by $T_n u = Au + g(x, u) - f$. Equation (2.17) is equivalent to the identity $T_n u_n = 0$. Since g is continuous and V_n is a finite-dimensional subspace of L_∞ the map T_n is continuous. In addition $\langle T_n u, u\rangle > 0$ if $\| u \|_V \geqslant R$. It follows from the Brouwer fixed point theorem that there is a $u_n \in V_n$ with $\| u_n \|_V < R$ and $T_n(u) = 0$. (See [10, Chap. 1, Lemma 4.3].)

Since the inclusion $V \to L_2(\Omega)$ is compact we may choose a subsequence of the u_n (which we still denote by u_n) such that

$$u_n \to u \text{ weakly in } V,$$

$$u_n \to u \text{ strongly in } L_2(\Omega),$$

$$u_n \to u \text{ a.e. in } \Omega.$$

We next investigate the convergence of $g(x, u_n)$. Take $v = u_n$ in (2.17) to show that

$$\int_\Omega u_n g(x, u_n) \leqslant \| u_n \|_V^2 + \| u_n \|_V \| f \|_{V'}$$
$$\leqslant \text{constant independent of } n.$$

Since $sg(x, s) \geqslant 0$ if $| s | > \rho$ it follows that

$$\int_\Omega | u_n g(x, u_n)| \leqslant \text{constant independent of } n.$$

Since $u_n g(x, u_n) \to ug(x, u)$ a.e. Fatou's lemma implies that $ug(x, u) \in L_1(\Omega)$, and therefore that $g(x, u) \in L_1(\Omega)$.

Following Strauss [14], we show that $\{g(x, u_n)\}$ is weakly compact in $L_1(\Omega)$. For each $\epsilon > 0$ we must show that there is a $\delta > 0$ such that $| \omega | < \delta \Rightarrow \int_\omega | g(x, u_n)| < \epsilon$ for all n. For any positive k

$$g(x, s) \leqslant (1/k) \, | sg(x, s)| + \sup_{|s| \leqslant k} g(x, s). \tag{2.18}$$

Choose k so that $\int_\Omega | u_n g(x, u_n)| < k\epsilon/2$ for all n. Then choose $\delta > 0$ so that $2\delta \sup_{x \in \Omega, |s| \leqslant k} g(x, s) < \epsilon$. Then if $| \hat{\omega} | < \delta$,

$$\int_\omega | g(x, u_n)| < | \omega | \sup_{\substack{x \in \Omega \\ |s| \leqslant k}} | g(x, s)| + \frac{1}{k} \int_\omega | u_n g(x, u_n)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus passing to a subsequence (still denoted u_n) we may assume that

$$g(x, u_n) \to \gamma \text{ weakly in } L_1(\Omega).$$

To show that $\gamma = g(x, u)$ observe that $g(x, u_n) \to g(x, u)$ a.e. in Ω so that for any $\epsilon > 0$ we may choose a set $\nu \subset \Omega$ with $| \nu | < \epsilon$ and $g(x, u_n) \to g(x, u)$ uniformly on $\Omega \backslash \nu$. Then

$$g(x, u_n) \to g(x, u) \text{ weakly in } L_1(\Omega \backslash \nu),$$
$$g(x, u_n) \to \gamma \restriction \Omega \backslash \nu \text{ weakly in } L_1(\Omega \backslash \nu),$$

so $\gamma = g(x, u)$ a.e. on $\Omega \backslash \nu$. Since $| \nu |$ may be chosen arbitrarily small we have $\gamma \doteq g(x, u)$ a.e. in Ω.

For any n, m with $n > m$ we have $b(u_n, v) = 0 \; \forall v \in V_m$ since $v \in V_m \subset V_n$. Passing to the limit $n \to \infty$ using the weak convergence of u_n and $g(x, u_n)$ in V

and $L_1(\Omega)$, respectively, together with the fact that $v \in L_\infty(\Omega)$ we have $b(u, v) = 0$. It follows that this holds for all $v \in \bigcup_m V_m$. In particular,

$$\int_\Omega g(x, u)\, v\, dx = \langle f, v\rangle - a(u, v) \qquad \forall v \in \bigcup_m V_m .$$

Since the right-hand side is a continuous linear functional of $v \in V$ it follows that $g(x, u) \in V'$. This, in turn, shows that the map $v \mapsto b(u, v)$ is continuous from V to $\mathbb{R}$. Since the map vanishes on the dense set $\bigcup_m V_m$ it vanishes for all $v \in V$ which shows that u is a weak solution to our problem, and the proof of Theorem 1 is complete.

3. Remarks and Extensions

(1) *Kernels of one sign.* If $\theta \geqslant 0$ then condition (2.7) is simplified to

$$\int_\Omega g(x_1, +\infty)\, \theta > \langle f, \theta\rangle > \int_\Omega g(x_1, -\infty)\, \theta,$$

which has an appealing simplicity. For the case $\theta =$ constant which arises in several Neumann problems the condition further simplifies to

$$\text{Average } g(x, +\infty) > \text{Average } f > \text{Average } g(x, -\infty).$$

(2) *Regularity.* In examples 2 and 4 of the Introduction the authors do not know if there are smooth solutions provided $f \in C^\infty(\bar{\Omega})$. One way to obtain regular solutions is to prove an a priori sup norm estimate using maximum principles. This is the method of Kazdan and Warner [8] who succeed in solving examples 1 and 3. When g is monotone in u one can often prove that the weak solutions are somewhat more regular by standard energy methods. For instance, in example 4 one can find solutions in $H_3(\Omega)$. However, if the space dimension is $\geqslant 6$ this does not allow one to prove C^∞ regularity.

(3) *Approach to resonance.* One can solve the problems

$$(A + \lambda)u_\lambda + g(x, u_\lambda) = f$$

$\lambda \geqslant 0$ under the hypotheses of Theorem 1 with a uniform estimate $\| u_\lambda \|_V \leqslant$ constant independent of λ. The estimate is derived by writing $u_\lambda = (u_\lambda)_1 + e_\lambda \theta$ as in the proof of Theorem 1. For $\lambda > 0$ this is not natural, but for the limit $\lambda \to 0$ it is.

Since in Section 2, we have proved that $b(u, u) \geqslant \delta$ for $\| u \|_V = R$, it follows that if $N\colon V \to V'$ is continuous, compact, and maps bounded sets into bounded sets then there exists an ϵ_0 such that if $|\epsilon| < \epsilon_0$, $b(u, u) + \langle \epsilon Nu, u\rangle \geqslant \delta/2$. Thus the equation

$$Au + g(x, u) + \epsilon Nu = f(x)$$

has a weak solution.

(4) *More general kernels.* As when g is bounded some results may be obtained if dim ker $A > 1$. See [11, 13, 15] for the necessary ideas.

(5) *Nonlinearities depending on derivatives.* In Theorem 1, the function of g may depend on the derivatives of u up to order $m - k$, provided the inclusion $V \to H_{m-k}(\Omega)$ is compact. If $g \in C(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^l)$, where $l = \sum_{1\leqslant|\alpha|\leqslant m-k} 1$. Consider the equation

$$Au + g(x, u, \partial u/\partial x_1, ...) = f, \tag{3.1}$$

where the derivatives are of order $\leqslant(m - k)$ and g is bounded in its dependence on the *derivatives* of u in the sense that for any bounded set $I \subset \mathbb{R}$,

$$\sup_{\substack{x\in\Omega\\ t\in\mathbb{R}^l\\ s\in I}} |g(x, s, t)| < \infty.$$

Let

$$g(x, +\infty) = \liminf_{\substack{x\in\Omega\\ t\in\mathbb{R}^l\\ s\to\infty}} g(x, s, t),$$

$$g(x, -\infty) = \limsup_{\substack{x\in\Omega\\ t\in\mathbb{R}^l\\ s\to-\infty}} g(x, s, t)$$

and assume as in (2.7) that these limits are obtained uniformly for $x \in \Omega$, $t \in \mathbb{R}^k$. If $g(x, +\infty) > g(x, -\infty)$ and the Landesman–Lazer condition (2.8) holds then there exists a solution u of (3.1) in the sense that $u \in V$, $g(x, u, Du) \in V' \cap L_1$, $ug(x, u, Du) \in L_1$ and $a(u, v) + \int_\Omega g(x, u, Du)v = \langle f, v\rangle\ \forall v \in V$. For the proof, first observe that the proof of the basic estimate goes through without essential change, and second that in the Galerkin argument the subsequence u_n can be chosen so that for $|\alpha| \leqslant m - k$, $D^\alpha u_n \to D_\alpha u$ in $L^2(\Omega)$ and a.e. in Ω. Then $g(x, u_n, Du_n) \to g(x, u, Du)$ a.e. in Ω. In addition, the equi-integrability survives, so that we may choose u_n with $g(x, u_n, Du) \to \gamma$ weakly in $L_1(\Omega)$. It follows as before $g(x, u, D^\alpha u) = \gamma$, which is the last modification required.

(6) *Discontinuous* g. Provided one extends g to be multiple valued, Theorem 1 can be extended to g which are not continuous in their dependence on u. As an example we describe the modifications in Theorem 1 in case g is independent of x. Suppose $g \in L_\infty^{\text{loc}}(\mathbb{R})$ and satisfies (1.5). The limiting values $g(\pm\infty)$ are defined as in (2.3), (2.4) and we need to consider the upper and lower envelope

$$\bar{g}(s) = \lim_{\epsilon \to 0} \sup_{|t-s|<\epsilon} g(t)$$

$$\underline{g}(s) = \lim_{\epsilon \to 0} \inf_{|t-s|<\epsilon} g(t).$$

Then for any $f \in V'$ satisfying the Landesman–Lazer condition there is a $u \in V$ and $\gamma \in V' \cap L_1$ such that $u\gamma \in L_1$ and

$$Au + \gamma = f \text{ in } \Omega$$
$$\underline{g}(u) \leqslant \gamma \leqslant \bar{g}(u).$$

One should think of this as saying $\gamma \in \hat{g}(u)$, where $\hat{g}$ is the multiple-valued function

$$\hat{g}(s) = [\underline{g}(s), \bar{g}(s)].$$

The techniques for proving this result can be found in [12].

(7) *Quasi-linear equations.* The self-adjoint operator A in our theorem may be replaced by a monotone nonlinear map as follows. Suppose V is a Hilbert space compactly embedded in $L_2(\Omega)$ and V' the dual with the pariring $\langle \, , \rangle$ which extends $(\, ,)\theta._{(\Omega)}$. We suppose that V is the direct sum (not necessarily orthogonal) $V_1 \pm \mathbb{R}\theta$ and that hypotheses (2.1), (2.2), and (2.3) hold. Suppose $A = V \to V'$ is a bounded, monotone, and hemicontinuous map of $V \to V'$ which is semicoercive in the sense that

$$A\theta = 0, \langle Av, \theta \rangle = 0 \qquad \text{for all} \quad v \in V, \tag{3.2}$$

and

$$\langle Av_1 , v_1 \rangle \geqslant \text{constant} \, \| v_1 \|_V^2 , \tag{3.3}$$

for all $v_1 \in V_1$.

THEOREM 2. *If A, V, θ are as above, g satisfies* (2.7) *and $f \in V'$ satisfies* (2.8) *then there is a $u \in V$ such that $g(x, u) \in L_1(\Omega) \cap V'$, $ug(x, u) \in L_1$ and $Au + g(x, u) = f$.*

The proof is the same as that of Theorem 1, except that Minty's device is needed to handle the weak convergence of Au_n to Au.

(8) *More general L_2 spaces.* The underlying space $L_2(\Omega)$ may be replaced by the square integrable functions on a finite-measure space. In particular, one can treat nonlinear elliptic problems on compact manifolds.

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