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LETTERS TO THE EDITOR

A NOTE ON THE DAMPING OF LARGE AMPLITUDE BEAM VIBRATIONS

The efficacy of viscoelastic layers, in constrained and unconstrained configurations, on reducing vibration levels in beam structures has been studied extensively for linear motions. However, few inroads have been made for large amplitudes. Such work has been done for clamped–clamped compact beams consisting of a viscoelastic core constrained by metallic face sheets by Hyer, Anderson and Scott [1, 2]. Here moderate amplitude forced motions of a beam with a symmetric cross-section consisting of a metallic core with symmetrically attached top and bottom viscoelastic layers are considered. The following theory is proposed. It is assumed that the viscoelastic layers are so thin that they transmit membrane forces only. The core is taken to be a Timoshenko beam and non-linearity is introduced through the requirement that the beam ends be a fixed distance apart.

Axes xyz are used, with x along the neutral axis and y and z along the symmetry axes. The plane of loading is the xz-plane. Considering an element of the beam, summing forces in the x- and z-directions, taking y-axis moments, and neglecting rotatory inertia associated with shear, one obtains

$$\frac{\partial}{\partial x}[(N^t + N^b + N^c) - Q^c\theta] = \rho^c A^c \ddot{u}_0 + \rho^t A^t \ddot{u}_0 + \rho^b A^b \ddot{u}_0^b, \tag{1}$$

$$\frac{\partial}{\partial x}[(N^t + N^b)(\phi + \theta) + N^c\theta - Q^c] + a(x)q(t) = \rho^t A^t \ddot{w}_0^t + \rho^b A^b \ddot{w}_0^b + \rho^c A^c \ddot{w}_0, \tag{2}$$

$$-\partial M^c/\partial x + Q^c = \rho^c I^c \ddot{\theta} + \tfrac{1}{2}\rho^t A^t(h_1 + h_2)\ddot{u}_0^t - \tfrac{1}{2}\rho^b A^b(h_1 + h_2)\ddot{u}_0^b, \tag{3}$$

where

$$N^c(t, b) = \int_{c(t,b)} \sigma_{xx}^{c(t,b)}\,dz, \qquad M^c = \int_c z\sigma_{xx}^c\,dz, \qquad Q^c = \int_c \sigma_{xz}^c\,dz \tag{4}$$

and u_0 and w_0 denote mid-layer displacements, the superscripts c, t and b stand for core, top and bottom, respectively, θ and ϕ are slopes due to bending and shear, respectively, so that $\partial w_0/\partial x = \phi + \theta$, ρ denotes density, A is area and I^c is the moment of inertia about the y-axis of a unit length of the core. Further, a dot stands for a time derivative, $a(x)$ and $q(t)$ are the spatial and time dependencies of the external load, respectively, h_1 is the thickness of the core and h_2 is the thickness of the layers.

The relevant displacements are taken to be $w_0(x, t)$ and

$$u^c = u_0(x, t) - z\theta, \tag{5}$$

$$u^{t(b)} = u_0(x, t) \pm \tfrac{1}{2}h_1\theta \pm h_2(\phi + \theta) \approx u_0(x, t) \pm \tfrac{1}{2}h_1\theta. \tag{6}$$

The only non-zero strain components are assumed to be

$$\varepsilon_{xz} = \tfrac{1}{2}(\partial w/\partial x - \theta), \qquad \varepsilon_{xx} = \partial u/\partial x + \tfrac{1}{2}(\partial w/\partial x)^2. \tag{7}$$

Green's non-linear measure is used only in the core, since non-linear membrane effects in the face sheets are felt to be negligible. The stresses, and hence stress resultants, in the core can be calculated by using Hooke's law. A further ingredient is that the layer material is adequately described by the Kelvin law

$$\sigma_{xx}^{t(b)} = E^{t(b)}\varepsilon_{xx}^{t(b)} + \zeta^{t(b)}\dot{\varepsilon}_{xx}^{t(b)}, \tag{8}$$

where $E^{t(b)}$ and $\zeta^{t(b)}$ are material parameters. Use of this restrictive law is justified when as here, the ultimate application is to harmonic time-dependencies.

Substituting equations (4) through (8) into equations (1), (2) and (3), and injecting Timoshenko's shear coefficient k, gives

$$\frac{\partial}{\partial x}\left[B^c\frac{\partial u_0}{\partial x}+\frac{E^c A^c}{2}\left(\frac{\partial w_0}{\partial x}\right)^2+2A^t\zeta^t\frac{\partial \dot{u}_0}{\partial x}\right]=m\ddot{u}_0, \tag{9}$$

$$\frac{\partial}{\partial x}\left\{\left[2E^tA^t\frac{\partial u_0}{\partial x}+2A^t\zeta^t\frac{\partial \dot{u}_0}{\partial x}\right]\frac{\partial w_0}{\partial x}+E^cA^c\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right]\theta\right.$$

$$\left.-K\left(\frac{\partial w_0}{\partial x}-\theta\right)\right\}+a(x)q(t)=m\ddot{w}_0, \tag{10}$$

$$-\frac{\partial}{\partial x}\left(E^cI^c\frac{\partial\theta}{\partial x}\right)+K\left(\frac{\partial w_0}{\partial x}-\theta\right)=I\ddot{\theta}, \tag{11}$$

where

$$K=k^2bh_1G^c,\qquad m=\rho^cA^c+2\rho^tA^t,$$

$$I=\rho^cI^c+\tfrac{1}{2}\rho^tA^th_1(h_1+h_2),\qquad B^c=2A^tE^t+E^cA^c,$$

b is the width of the beam and E^c and G^c are Young's modulus and the shear modulus of the core material, respectively. In arriving at equation (9), a non-linear term $(\partial w_0/\partial x-\theta)\theta$ has been taken to be negligible, following Yu [3].

Galerkin's method is now used. It is assumed that

$$w_0=\Gamma(t)(1-\cos 2\pi x/L),\qquad u_0=\eta(t)\sin 4\pi x/L,\qquad \phi=T(t)\sin 2\pi x/L. \tag{12}$$

These trial functions, which have been used by other authors, satisfy the boundary conditions which call for vanishing of u_0, w_0 and θ at the ends of the beam. These modes were chosen with a certain amount of trepidation because they have the bad feature that the shear deformation angle ϕ is zero at the boundary. They do appear to be the best obvious choice, however, since a switch to $w_0=\Gamma(t)\sin\pi x/L$ and $u_0=\eta(t)\sin 2\pi x/L$ causes loss of bending effects in the core, loss of linear terms in one of the resulting differential equations, and poor behavior in the limit as shearing deformation vanishes (the Euler–Bernoulli beam theory is not recovered in the limit; see reference [4]). The inaccuracy of the shear deformation at the boundary is accepted on the rationale that shear effects in the core are small compared to bending effects. The method yields

$$A_2\Gamma^2+A_3\eta+A_4\dot{\eta}+A_5\ddot{\eta}=0, \tag{13}$$

$$K\pi T+F_2\eta T+F_3\eta\Gamma+F_4\Gamma^3+F_5\Gamma\dot{\eta}+F_6q(t)+F_7\ddot{\Gamma}+F_8T\Gamma^2=0, \tag{14}$$

$$B_1T+B_2\ddot{\Gamma}+B_3\Gamma+B_4\ddot{T}=0, \tag{15}$$

where

$$L^2A_2=2E^cA^c\pi^3,\qquad LA_3=-8\pi^2B^c,\qquad LA_4=-16\pi^2\zeta^tA^t,\qquad 2A_5=-mL,$$

$$B_1=\tfrac{1}{2}L(K-4\pi^2E^cI^c/L^2),\qquad B_2=-\pi I,\qquad L^2B_3=4\pi^3E^cI^c,\qquad 2B_4=IL,$$

$$LF_2=-2\pi^2E^cA^c,\qquad L^2F_3=-\pi^3(2E^tA^t+E^cA^c),\qquad 2L^3F_4=-3\pi^4E^cA^c,$$

$$L^2F_5=-2\pi^3\zeta^tA^t,\qquad F_6=L,\qquad 2F_7=-3mL,\qquad 2L^2F_8=3\pi^3E^cA^c.$$

In arriving at F_6, $a(x)$ has been set equal to 1.

The external force is taken to be

$$q(t) = H\cos\omega t + G\sin\omega t, \tag{16}$$

where H and G are constants. Assuming that no sub- or superharmonics arise (see references [1, 2] for a discussion of the frequency ranges for which this is reasonable), one takes

$$\Gamma(t) = W(\omega)\cos\omega t. \tag{17}$$

Substituting equation (17) into equations (13) and (15) gives linear ordinary differential equations for η and T, the steady-state solutions of which are

$$\eta = \bar{A}_2\cos(2\omega t - \alpha_2) - (A_2/2A_3)\,W^2, \qquad T = \gamma_1 W\cos\omega t, \tag{18}$$

where

$$\bar{A}_2 = [\omega^2 A_2^2 A_4^2 + A_2^2(A_3 - 4\omega^2 A_5)^2]/\Delta^2,$$
$$\Delta = (A_3 - 4\omega^2 A_5)^2 + 4\omega^2 A_4^2, \qquad \alpha_2 = \tan^{-1}[\omega A_4/(A_3 - 4\omega^2 A_5)],$$
$$\gamma_1 = (B_2\omega^2 - B_3)/(B_1 - B_4\omega^2).$$

Inserting expressions (16), (17) and (18) into equation (14) yields the frequency-amplitude relation, on using the harmonic balance technique,

$$C_7 W^6 + C_5 W^4 + C_3 W^2 + C_1 = 0, \tag{19}$$

where

$$C_1 = -F_6^2 F^2, \qquad F = \sqrt{H^2 + G^2}, \qquad C_3 = (\pi K\gamma_1 - F_7\omega^2)^2,$$
$$C_5 = \sqrt{C_3}[(F_2\gamma_1 + F_3)(\beta_2\cos\alpha_2 - A_2/A_3) + 3(F_4 + \gamma_1 F_8)/2 + 2\omega F_5\beta_2\sin\alpha_2],$$
$$\beta_2 = [(\omega A_2 A_4)^2 + A_2^2(A_3 - 4\omega^2 A_5)^2]^{1/2}/\Delta,$$
$$\begin{aligned} C_7 = {} & 9(F_4 + F_8\gamma_1)^2/16 + (F_4 F_8\gamma_1)\,[\tfrac{3}{4}(F_3 + F_2\gamma_1)(\beta_2\cos\alpha_2 - A_2/A_3) \\ & + \tfrac{3}{2}F_5\omega\beta_2\sin\alpha_2] + (F_2\gamma_1 + F_3)^2 + [\tfrac{1}{4}\beta_2^2 - \tfrac{1}{2}(A_2/A_3)\beta_2\cos\alpha_2 + A_2^2/(4A_3^2)] \\ & - (A_2/A_3)(F_2\gamma_1 + F_3)\omega\beta_2 F_5\sin\alpha_2 + \omega^2 F_5^2\beta_2^2. \end{aligned}$$

Some specific numerical explorations will now be given; the numbers correspond to an experiment that was performed. The core material is steel, with $E = 2{\cdot}07 \times 10^{11}$ Pa, $G = 6{\cdot}89 \times 10^{11}$ Pa and $\rho = 7{\cdot}82 \times 10^3$ kg/m^3. Other items are the beam length $L = 30{\cdot}48 \times 10^{-2}$ m and width $2{\cdot}54 \times 10^{-2}$ m and the force level $F = 0{\cdot}45$ N. The viscoelastic material was taken to be ISPD 111. Shown in Figure 1 is the frequency-amplitude response curve for a

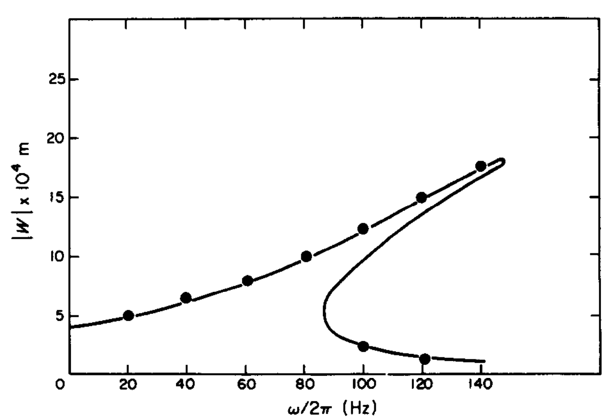

Figure 1. Damped response. $h_1 = 8{\cdot}89 \times 10^{-4}$ m; $b = 2{\cdot}54 \times 10^{-2}$ m; $L = 30{\cdot}48 \times 10^{-2}$ m; $h_2 = 2{\cdot}03 \times 10^{-4}$ m. $F = 0{\cdot}45$ N.

beam of thickness $h_1 = 8{\cdot}89 \times 10^{-4}$ m to which is attached single viscoelastic layers. Rather strong non-linear effects are present. Numerical results for the undamped case are so close to those for the damped beam as to be almost indistinguishable on the scale of Figure 1. (An experiment confirmed this observation. The points on Figure 1 refer to that experiment.) More detailed information is given in Figure 2, which shows amplitudes at 70 Hz (the damping coefficient of ISPD 111 is a maximum at this frequency) as a function of tape thickness, for beams with thicknesses $h_1 = 4{\cdot}57 \times 10^{-4}$ m, $5{\cdot}58 \times 10^{-4}$ m and $8{\cdot}89 \times 10^{-4}$ m.

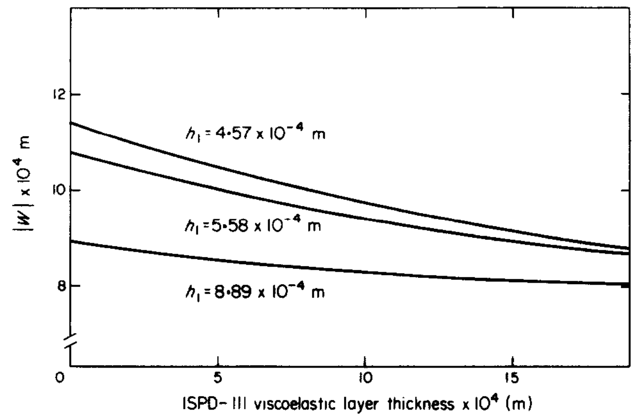

Figure 2. Amplitudes as functions of ISPD-111 viscoelastic layer thickness. $F = 0{\cdot}45$ N at 70 Hz.

To achieve a reduction of 5%, layers of at least the beam thickness must be applied! Thus the evidence points towards the inefficiency of unconstrained damping layers for control of moderate amplitude vibrations. This is due in part to the unconstrained nature of the damping layer but is also due to the dominance of the elastic forces on limiting amplitudes in this type of hardening, non-linear system.

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