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A VAN KAMPEN THEOREM FOR π_2

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If the path connected topological space X has a countable open cover $\mathcal{U}$ with path connected elements, then $\pi_2(X,*)$ is computed as a colimit determined by the second homotopy groups of the intersection of elements of $\mathcal{U}$ and the indices of the fundamental group injections of these intersections into the fundamental group of X. Aside from assuming that the inclusions induce such monomorphisms, certain other inclusions are also required to induce monomorphisms of fundamental groups and restrictions are placed on the arrangement of the elements of $\mathcal{U}$.

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Van Kampen Theorem, fundamental group, nerve of a cover, second homotopy group, Mayer–Vietoris Theorem

Under certain circumstances the second homotopy group of a topological space X is determined by the first and second homotopy groups of elements in an open cover $\mathcal{U}$ of X. This paper describes conditions on $\mathcal{U}$ which enable $\pi_2(X,*)$ to be computed as a colimit. It is in three parts. First, the case $\mathcal{U}=\{U_1, U_2\}$ is presented in complete detail as a basis for the generalizations which follow. Second, these generalizations are stated in such a way that the reader can quickly determine whether or not they are applicable to any particular situation at hand. Third, proofs are outlined.

Most of these results appeared in the author's thesis [1], which was written under the direction of Eldon Dyer whose assistance is gratefully acknowledged. In particular it was he who suggested this approach and Example 1.

1. The case $\mathcal{U}=\{U_1, U_2\}$

Theorem 1. *Suppose X is a topological space and $\mathcal{U}=\{U_1, U_2\}$ is an open cover of X with U_1, U_2, and $U_3=U_1\cap U_2$ path connected. Let $*\in U_3$ and let $k_i: U_3\hookrightarrow U_i$, $i=1,2$ denote the inclusions. Finally, suppose that $\pi_1(k_i)$ is monic for $i=1,2$. Then $\pi_2(X,*)$ is determined as the push-out*

$$\begin{array}{ccc} \bigoplus_{I_3} \pi_2(U_3,*) & \longrightarrow & \bigoplus_{I_1} \pi_2(U_1,*) \\ \downarrow & & \downarrow \\ \bigoplus_{I_2} \pi_2(U_2,*) & \longrightarrow & \pi_2(X,*) \end{array}$$

in the category of abelian groups where $I_j = \pi_1(X,*)/\pi_1(U_j,*)$ *for* $j = 1,2,3$.

Proof. The classical van Kampen Theorem yields $\pi_1(X,*)$ as the push-out

$$\begin{array}{ccc} \pi_1(U_3,*) & \longrightarrow & \pi_1(U_1,*) \\ \downarrow & & \downarrow \\ \pi_1(U_2,*) & \longrightarrow & \pi_1(X,*) \end{array}$$

in the category of groups. It is a theorem on amalgamated products [4, p. 9] that the hypotheses imply that the inclusion induced maps

$$\pi_1(U_i,*) \to \pi_1(X,*)$$

are monic for $i = 1,2,3$. Thus, for each i, $\pi_1(U_i,*)$ may be regarded as a subgroup of $\pi_1(X,*)$. Consider the inclusion induced pull-backs (restrictions):

$$\begin{array}{ccc} \hat{U}_i & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ U_i & \hookrightarrow & X \end{array} \qquad (i = 1,2,3)$$

where $\tilde{X} \to X$ is the universal covering projection. It is clear that $\hat{U}_i \to U_i$ is a covering projection. Furthermore, since π_1 commutes with pull-backs of covering projections, the diagrams

$$\begin{array}{ccc} \pi_1(\hat{U}_i,*) & \longrightarrow & \pi_1(\tilde{X},*) = 0 \\ \downarrow & & \downarrow \\ \pi_1(U_i,*) & \longrightarrow & \pi_1(X,*) \end{array} \qquad (i = 1,2,3)$$

imply that $\pi_1(\hat{U}_i,*) = 0$ (in each component). Thus, in each component, $\hat{U}_i$ is the universal cover $\tilde{U}_i$. Since $\pi_1(U_i,*)$ permutes the sheets in each component of $\hat{U}_i$, we have $\hat{U} = \bigsqcup_{I_j} \tilde{U}_j$ where $\sqcup$ denotes disjoint union and $I_j = \pi_1(X,*)/\pi_1(U_j,*)$ is a set which indexes the orbits of the action. The Hurewicz isomorphism theorem yields

$$H_1(\hat{U}_j;\mathbf{Z}) = \bigoplus_{I_3} H_1(\tilde{U}_j;\mathbf{Z}) \cong \bigoplus_{I_3} \pi_1(\tilde{U}_j,*) = 0.$$

Since $\hat{U}_1$, $\hat{U}_2$, $\hat{U}_3$ amount to restrictions of $\tilde{X} \to X$, $\{\hat{U}_1, \hat{U}_2\}$ forms an open cover of $\tilde{X}$ with $\hat{U}_1 \cap \hat{U}_2 = \hat{U}_3$. The Mayer–Vietoris sequence yields the exact sequence

$$\to H_2(\hat{U}_3;\mathbf{Z}) \to H_2(\hat{U}_1;\mathbf{Z}) \oplus H_2(\hat{U};\mathbf{Z}) \to H_2(\tilde{X};\mathbf{Z}) \to 0$$

which is equivalent to the push-out

$$\begin{array}{ccc} H_2(\hat{U}_3;\mathbf{Z}) & \longrightarrow & H_2(\hat{U}_1;\mathbf{Z}) \\ \downarrow & & \downarrow \\ H_2(\hat{U}_2;\mathbf{Z}) & \longrightarrow & H_2(\tilde{X};\mathbf{Z}) \end{array}$$

in the category of abelian groups. Since all the fundamental groups are zero, there are Hurewicz (and universal covering) isomorphisms:

$$H_2(\hat{U}_i;\mathbf{Z})=\bigoplus_{I_i} H_2(\tilde{U}_i;\mathbf{Z})\cong\bigoplus_{I_i} \pi_2(\tilde{U}_i,*)\cong\bigoplus \pi_2(U_i,*) \qquad (i=1,2,3)$$

and

$$H_2(\tilde{X},\mathbf{Z})\cong\pi_2(\tilde{X},*)\cong\pi_2(X,*).$$

These isomorphisms enable the homology groups in the push-out to be replaced by second homotopy groups as required. ■

Theorem 2. *Suppose X is a topological space and $\mathcal{U}=\{U_1, U_2\}$ is an open cover of X with U_1, U_2 and $U_3=U_1\cap U_2$ path connected. Let $*\in U_3$ and let $k_i: U_3\hookrightarrow U_i$, $i=1,2$ denote the inclusions. Finally, suppose that $\ker\pi_1(k_1)=\ker\pi_1(k_2)=D$. Then $\pi_2(X,*)$ occurs in the exact sequence*

$$\bigoplus_{I_3} H_2(V;Z)\to\bigoplus_{I_1} \pi_2(U_1,*)\oplus\bigoplus_{I_2} \pi_2(U_2,*)\to\pi_2(X,*)\to\bigoplus_{I_3} D_{ab}\to 0$$

where D_{ab} is D abelianized, $I_i=\pi_1(X,*)/\pi_1(U_i,*)$ for $i=1,2,3$, and V is the covering of U_3 determined by $\pi_1(V,*)=D\subseteq\pi_1(U_3,*)$.

Proof. There is a push-out

$$\begin{array}{ccc} D & \xrightarrow{\ a\ } & \pi_1(U_1,*) \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle \pi_1(k_1)} \\ \pi_1(U_2,*) & \xrightarrow{\pi_1(k_2)} & \pi_1(X,*) \end{array}$$

where the maps a and b are monic. As before the pull-backs

$$\begin{array}{ccc} \pi_1(\hat{U}_i,*) & \longrightarrow & \pi_1(\tilde{X},*)=0 \\ \downarrow & & \downarrow \\ \pi_1(U_i,*) & \xrightarrow{\ c_i\ } & \pi_1(X,*) \end{array}$$

yield $\pi_1(\hat{U}_1,*)=\pi_1(\hat{U}_2,*)=0$ in each component. However, $\pi_1(\hat{U}_3,*)=\ker c_3=D$. A Mayer–Vietoris sequence is obtained as before and the theorem follows after applying the Hurewicz isomorphism theorem as much as possible. ■

The following example exhibits spaces X and Y with respective covers $\{U_1, U_2, U_3=U_1\cap U_2\}$ and $\{V_1, V_2, V_3=V_1\cap V_2\}$ such that U_i and V_i have the same homotopy type for $i=1,2,3$ and such that the action of $\pi_1(U_3,*)$ on $\pi_*(U_i,*)$ is identical with the action of $\pi_1(V_3,*)$ on $\pi_*(V_i,*)$ for $i=1,2$. However, $\pi_2(X,*)\neq\pi_2(Y,*)$. Thus, some restriction on $\pi_1(k_i)$ is necessary since in general $\pi_2(X,*)$ is not determined even by all the homotopy groups of elements in an open cover and the fundamental group action.

Example 1. Let $A = K(\mathbf{Z}_2, 1) = P_\infty(\mathbf{R})$ and let $X = K(\mathbf{Z}_2, 2)$. The cohomology ring of $P_\infty(\mathbf{R})$ is a polynomial ring in a single generator, η, in dimension 1. Let $f: A \to X$ be such that $f^*(\iota) = \eta^2$ where ι denotes the fundamental class. X_1 is the space obtained from the mapping cylinder of f by attaching a cone on A. That is, X_1 is the push-out

$$\begin{array}{ccc} A & \hookrightarrow & CA \\ \bar{f}\downarrow & & \downarrow \\ X & \longrightarrow & X \cup CA = X_1 \end{array}$$

where $\bar{f}$ is the cofibration replacing f in the mapping cylinder construction. X_2 is the push-out

$$\begin{array}{ccc} A & \longrightarrow & CA \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \vee SA =: X_2 \end{array}$$

where Y is obtained by attaching the vertex of a cone on A to the basepoint of X. SA denotes the suspension of A.

That the elements in these two covers have the same respective homotopy types is obvious. Since in X_2, $\pi_1(A, *)$ acts through the basepoint, this action is trivial. Clearly, $\pi_2(X_2, *) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$. It remains to be shown that $\pi_1(A, *)$ acts trivially in X_1 and that $\pi_2(X_1, *) = \mathbf{Z}_4$.

By hypothesis $\pi_1(A, *) = \mathbf{Z}_2$; $\pi_i(A, *) = 0$, $i \neq 1$; $\pi_i(CA, *) = 0$; $\pi_2(X, *) = \mathbf{Z}_2$; $\pi_i(X, *) = 0$, $i \neq 2$. The long exact homology sequence of the pair (X, A) yields the short exact sequence:

(1) $$0 \to \mathbf{Z}_2 \to \pi_2(X, A, *) \to \mathbf{Z}_2 \to 0$$

and the fact that $\pi_1(X, A, *) = 0$. The relative Hurewicz isomorphism theorem yields

(2) $$H_1(X, A; \mathbf{Z}) = 0$$

and

(3) $h: \pi_2(X, A, *) \to H_2(X, A; \mathbf{Z})$ is onto.

The universal coefficient theorem for homology together with (2) yields

(4) $$H_2(X, A; \mathbf{Z}) \otimes \mathbf{Z}_2 \cong H_2(X, A; \mathbf{Z}_2).$$

The long exact cohomology sequence of the pair (X, A) is

$$\leftarrow H^2(A; \mathbf{Z}_2) \xleftarrow{f^*} H^2(X; \mathbf{Z}_2) \leftarrow H^2(X, A; \mathbf{Z}_2) \leftarrow H^1(A; \mathbf{Z}_2) \leftarrow 0.$$

Since f^* is an isomorphism

$$H_2(X, A; \mathbf{Z}_2) \cong H^2(X, A; \mathbf{Z}_2) \cong H^1(A; \mathbf{Z}_2) \cong \mathbf{Z}_2.$$

By (4), $H_2(X, A; \mathbf{Z})$ is cyclic. The Hurewicz isomorphism and the long exact homology sequence yields the short exact sequence

$$0 \to \mathbf{Z}_2 \to H_2(X, A; \mathbf{Z}) \to \mathbf{Z}_2 \to 0.$$

Thus, $H_2(X, A; \mathbf{Z}) \cong \mathbf{Z}_4$ and (1) together with (3) yields $\pi_2(X, A) \cong \mathbf{Z}_4$. Recall that

$$\ker(h) = \{x - \alpha x \mid x \in \pi_2(X, A, *),\ \alpha \in \pi_1(A, *)\}.$$

Since $\pi_2(X, A, *) \cong H_2(X, A; \mathbf{Z})$, $\ker(h) = 0$ and the action of $\pi_1(A, *)$ on $\pi_2(X, A, *)$ is trivial. By (1) $\pi_2(X, *) \to \pi_2(X, A, *)$ is monic so that the action of $\pi_1(A, *)$ on $\pi_1(X, *)$ is also trivial. The relative Hurewicz isomorphism may be written as the composition

$$\pi_2(X, A, *) \xrightarrow{\text{onto}} \pi_2(X \cup CA, *) \to H_2(X \cup CA; \mathbf{Z}) \xrightarrow{\cong} H_2(X, A; \mathbf{Z}).$$

This may be rewritten

$$\mathbf{Z}_4 \xrightarrow{\text{onto}} \pi_2(X \cup CA) \longrightarrow \mathbf{Z}_4.$$

Since this is an isomorphism,

$$\pi_2(X_1, *) = \pi_2(X \cup CA) \cong \mathbf{Z}_4.$$

2. Statement of results

If $\mathcal{U}$ contains more than two elements (but is locally finite and countable), then matters are complicated for two reasons. First, if we agree to call each path component of the intersection of elements of $\mathcal{U}$ a *part* of $\mathcal{U}$, then it is not sufficient to assume that all inclusions of parts of $\mathcal{U}$ into X induce monomorphisms of fundamental groups (see Example 2). Second, if $\cap \mathcal{U} = \emptyset$, then it may not be possible to define the functor required to compute π_2. Sufficient conditions are given to guarantee that the appropriate functors exist. While particular circumstances may admit weaker hypotheses, the conditions chosen simplify the theorems and their proofs in general.

In summary, the approach is to construct a CW-complex analogous to the nerve of $\mathcal{U}$. To this complex we associate a graph which when embedded in X provides a minimal selection of basepoints and paths between them. The edges and vertices of this graph are labeled by the parts of $\mathcal{U}$. Our hypotheses are that either this graph is a tree or else its labels satisfy a condition specified below.

Definition. If $\mathcal{U}$ is a locally finite open cover of X with path connected elements, then $c\mathcal{U}$ is the CW-complex defined inductively as follows: to each open set in $\mathcal{U}$ associate a vertex. For $n \geq 1$ represent each path component of the intersection of $n-1$ elements in $\mathcal{U}$ by an n-simplex attached with the appropriate $(n-1)$-simplices as boundary. (In particular cases, this complex will not be a simplicial complex because the intersection of some pairs of simplices will not be connected.)

A simplex in this complex is a *free face* if it is not a face of another simplex. The local finiteness of $\mathcal{U}$ guarantees that each simplex is contained in a free face. Each simplex in $c\mathcal{U}$ corresponds to a part of $\mathcal{U}$.

Definition. By a *graph* Y we mean a collection of vertices vY and oriented *edges* eY together with functions $o, t: eY \to vY$ giving *origin* and *terminal* points, respectively and an *inversion* $^{-}: eY \to vY$, such that for each $y \in eY$, $\bar{y} \neq y$, $\bar{\bar{y}} = y$ and $o(\bar{y}) = t(y)$. The graph $g\mathcal{U}$ of a locally finite open cover $\mathcal{U}$ has vertices corresponding to the free faces of $c\mathcal{U}$. Whenever two free faces v, w of $c\mathcal{U}$ intersect, their intersection $v \cap w$ is another CW-complex. The edges in $g\mathcal{U}$ between the vertices v, w correspond to the free faces in $v \cap w$. (Technically, each such free face is represented twice to allow inversion.) Each vertex and edge of this graph corresponds to a part of $\mathcal{U}$. If e is an edge of $g\mathcal{U}$, we denote the corresponding part by X_e.

Definition. A *path* in a graph is a finite sequence $\{e_i\}_{i=1}^n$ of edges such that $o(e_{i+1}) = t(e_i)$ for $i = 1, \ldots, n-1$. It is a *circuit* if $t(e_n) = o(e_1)$ as well. An edge e is *unnecessary because of the path* $\{e_i\}_{i=1}^n$ if the path has $o(e_1) = o(e)$, $t(e_n) = t(e)$, and $X_{e_i} \subseteq X_e$ for $i = 1, \ldots, n$. (Geometrically, unnecessary edges are ones which are unnecessary for shifting basepoints.) A subgraph $\mathcal{G} \subseteq g\mathcal{U}$ is a *generating graph* if $v\mathcal{G} = vg\mathcal{U}$ and each edge in $eg\mathcal{U} - e\mathcal{G}$ is unnecessary because of a path in $\mathcal{G}$. A generating graph $\mathcal{G}$ is *minimal* if no edge in $\mathcal{G}$ is unnecessary because of a path in $\mathcal{G}$.

The vertices of a minimal generating graph $\mathcal{G}$ correspond to free faces in $c\mathcal{U}$ which in turn correspond to the highest levels of non-empty intersection of elements of $\mathcal{U}$. If basepoints corresponding to each vertex in $\mathcal{G}$ are selected, at least one point in each part of $\mathcal{U}$ will be chosen. An edge, $e \in e\mathcal{G}$, will correspond to a path in X_e between two selected points. A minimal generating graph contains precisely those edges which are necessary to provide all the basepoint shifts.

It is possible to homeomorphically embed any minimal generating graph $\mathcal{G}$ into X by selecting basepoints corresponding to vertices and arcs (in the appropriate parts) corresponding to edges. Recall that path-connectedness implies arc-connectedness. Also, it may be necessary to reselect (once) basepoints after the arcs have been inserted among the initial selection.

Suppose $\mathcal{U}$ has a minimal generating graph, $\mathcal{G}$, which is a tree. Under this assumption it is possible to define the necessary functor immediately. Let any of the selected basepoints serve as the basepoint $*$ for X. Given a part A of $\mathcal{U}$ choose its basepoint $*_A$ as follows. In $\mathcal{G}$, let the distance between vertices be defined as the (minimal) number of edges in a path between them. Then $*_A$ is the selected point in A whose corresponding vertex is closest to the vertex corresponding to $*$.

We view $\mathcal{U}$ as a category whose objects are the parts of $\mathcal{U}$ and whose morphisms are all the inclusions. The functor F_1 from $\mathcal{U}$ into the category of groups is defined on parts, A by

$$F_1(A)=\pi_1(A,*_A)$$

and on morphisms $i: A \hookrightarrow B$ to be the composition

$$\pi_1(A,*_A)\xrightarrow{\pi_1(i)}\pi_1(B,*_A)\longrightarrow\pi_1(B,*_B)$$

where the second map is the conjugation induced by the arc corresponding to the edge in $\mathscr{G}$ between the vertices corresponding to $*_A$ and $*_B$.

Suppose further that for each part A of $\mathscr{U}$ the inclusion $A \hookrightarrow X$ induces a monomorphism of fundamental groups. Define the index set I_A by

$$I_A=\pi_1(X,*)/\pi_1(A,*_A).$$

The functors F_n, $n \geq 2$, are defined for parts A of $\mathscr{U}$ by

$$F_n(A)=\bigoplus_{I_A}\pi_n(A,*_A)$$

and on morphisms $i: A \hookrightarrow B$ to be the composition

$$\bigoplus \pi_n(A,*_A)\xrightarrow{\oplus\pi_n(i)}\bigoplus\pi_n(B,*_A)\longrightarrow\bigoplus\pi_n(B,*_B)$$

where the second map is the same as before.

The final fundamental group condition is expressed by the following definition.

Definition. The cover $\mathscr{U}$ *satisfies condition H* provided for each $U \in \mathscr{U}$, the inclusion

$$U \cap \bigcup(\mathscr{U}-\{U\}) \to U$$

induces a monomorphism of fundamental groups. (Since the proofs are by induction the following slightly weaker condition may be substituted: there exists a sequence $\{U_1, U_2, \ldots\}=\mathscr{U}$ such that for each $i=2,3,\ldots$

$$\pi_1\left(U_i \cap \bigcup_{j=1}^{i-1} U_j\right)\to\pi_1(U_i) \text{ is monic and } \bigcup_{j=1}^{i-1} U_j \text{ is path connected.)}$$

Note that if $c\mathscr{U}$ is 1-dimensional, each element of $\mathscr{U}$ will intersect at most one other element of $\mathscr{U}$. Thus, the intersection $U \cap \bigcup(\mathscr{U}-\{U\})$ will be a part of $\mathscr{U}$. Since we are going to assume that the inclusions of parts of $\mathscr{U}$ into X induce monomorphisms of fundamental groups, this extra condition is unnecessary in this case.

Theorem 3. *Suppose X is a path connected topological space and $\mathscr{U}$ is a locally finite countable open cover of X such that each element of $\mathscr{U}$ is path connected. Suppose further that $\mathscr{U}$ has a minimal generating graph $\mathscr{G}$ which is a tree. Then* $\operatorname{colim} F_1=\pi_1(X,*)$.

Theorem 4. *Suppose that X, $\mathcal{U}$ and $\mathcal{G}$ satisfy all the hypotheses of Theorem 3. Suppose further that for each part A of $\mathcal{U}$ the inclusion $A \hookrightarrow X$ induces a monomorphism of fundamental groups. If $\mathcal{U}$ also satisfies condition H, then*

$$\operatorname{colim} F_2 = \pi_2(X, *).$$

Theorem 5. *Suppose that X, $\mathcal{U}$ and $\mathcal{G}$ satisfy all the hypotheses of Theorem 4 and that $\pi_i(A, *_A) = 0$ for $i = 2, \ldots, n-1$ for each part A of $\mathcal{U}$. Then*

$$\operatorname{colim} F_n = \pi_n(X, *).$$

Corollary. *Suppose that X, $\mathcal{U}$, and $\mathcal{G}$ satisfy all the conditions of Theorem 4 and that each part A, of $\mathcal{U}$ is $(n-1)$-connected, $n \geq 1$. Then*

$$\pi_n(X, *) = \operatorname{colim} F'_n$$

where $F'_n(A) = \pi_n(A, *_A)$ and $F'_n(i)$ is the composition

$$\pi_n(A, *_A) \xrightarrow{\pi_n(i)} \pi_n(B, *_A) \longrightarrow \pi_n(B, *_B).$$

Proof. If $n = 1$, this is just Theorem 3. Otherwise, for each part A, $\pi_1(A, *_A) = 0$ and Theorem 3 implies that $\pi_1(X, *) = 0$. Thus, I_A has cardinality 1. ■

Now suppose that $\mathcal{U}$ does not have a minimal generating graph which is a tree.

Definition. A circuit $\{e_i\}_{i=1}^n$ in a generating graph is *inessential* if there exists a part A of $\mathcal{U}$ such that $X_{e_i} \subseteq A$ for $i = 1, \ldots, n$. A minimal generating graph is *essential* if it has no inessential circuits.

This condition may also be phrased topologically. First embed $\mathcal{G}$ in $c\mathcal{U}$ by associating to each vertex the barycenter of the corresponding free face and to each edge an arc through the barycenter of the corresponding simplex. $\mathcal{G}$ is essential if the induced map $\pi_1(\mathcal{G}, *) \rightarrow \pi_1(c\mathcal{U}, *)$ is monic. To see this, note that if $\mathcal{G}$ is inessential, then the loop in $c\mathcal{U}$ corresponding to any inessential circuit in $\mathcal{G}$ may be contracted to any vertex corresponding to an element of $\mathcal{U}$ containing A (the part in the definition).

Theorem 6. *If $\mathcal{U}$ and X satisfy the hypotheses of Theorem 3, $\mathcal{U}$ has an essential generating graph $\mathcal{G}$, and X is locally arc-connected, then $\pi_1(X, *)$ is the semi-direct product determined by the split extension*

$$0 \rightarrow \pi_1(\hat{X}, *) \rightarrow \pi_1(X, *) \leftrightarrows \pi_1(\mathcal{G}, *) \rightarrow 0$$

where $\hat{X}$ is a covering space of X which has an open cover $\hat{\mathcal{U}} = \bigsqcup_I \mathcal{U}$ with a generating graph which is a tree. (Thus, $\pi_1 \hat{X}$ is computable by Theorem 3.)

Theorem 7. *Suppose X, $\mathcal{G}$, and $\mathcal{U}$ satisfy the conditions of Theorem 6. Suppose that* *for each part A of $\mathcal{U}$, the inclusion $A \hookrightarrow X$ induces a monomorphism of fundamental groups and that $\mathcal{U}$ satisfies condition H. Finally, suppose that $\pi_1(\mathcal{G}, *)$ is a countable group. Then $\pi_2(X, *)$ is determined by the groups $\pi_2(A, *_A)$ for each part A of $\mathcal{U}$, by the indices of the fundamental group injections and by $\mathcal{G}$.*

In the proof of Theorem 7 we will use the universal cover of $\mathcal{G}$ and the elements of $\mathcal{U}$ to construct an explicit covering space of X to which Theorem 4 is applicable. The details of this construction are given below. Exactly this same procedure enables a similar generalization of Theorem 5.

The next example illustrates the necessity of condition H.

Example 2. X is the surface obtained by rotating the circle $(x-1)^2+y^2=1$ about the y-axis. $\mathcal{U}$ is the three piece cover generated as this circle rotates from $-\varepsilon$ to $2\pi/3+\varepsilon$, $2\pi/3-\varepsilon$ to $4\pi/3+\varepsilon$, and $4\pi/3-\varepsilon$ to $2\pi+\varepsilon$ for some small positive number ε. All intersections are path connected and all three pieces intersect at the origin. Thus, $c\mathcal{U}$ is a triangle and $g\mathcal{U} = \mathcal{G}$ is a point. Each element of the cover and the intersection of any two elements is homeomorphic to the circle which generates $\pi_1(X, *) = \mathbf{Z}$. It follows that the inclusion of each part of $\mathcal{U}$ into X induces a monomorphism of fundamental groups. Since each part of $\mathcal{U}$ has no higher homotopy, F_n is identically zero for $n \geq 2$. However, condition H fails since each element of the cover intersects the other two in a figure 8 inducing the mapping $\mathbf{Z} * \mathbf{Z} \to \mathbf{Z}$. Nevertheless, we can use Theorem 4 to compute $\pi_2(X, *)$ by noting that upto homotopy, X is a torus with a disc sewn across the inside. The universal cover $\tilde{X}$ is an infinite spiraling circular cylinder with discs sewn in at regular intervals. Thus, $\pi_2(X, *) = \pi_2(\tilde{X}, *) = \bigoplus_{\mathbf{Z}} \mathbf{Z}$.

3. Proofs

Theorem 3 is the main theorem of [2, p. 43]. In this paper an isomorphism (and its inverse) from $\operatorname{colim} F_1 \to \pi_1(X, *)$ is given explicitly.

The following easy lemma will permit inductive arguments for Theorems 4 and 5.

Lemma. *If X is a topological space and $\mathcal{U} = \{U_1, U_2, \ldots\}$ is an open cover of X such that $U_i \subseteq U_{i+1}$ for all i, then*

$$H_n(X; \mathbf{Z}) = \operatorname*{colim}_i H_n(U_i; \mathbf{Z}).$$

Theorems 4 and 5 will follow immediately from the appropriate generalization of the Mayer–Vietoris Theorem. It is easier to obtain this theorem if we incorporate slightly more generality than we actually need. Suppose $p: Y \to X$ is an onto map. The functor $F_p: \mathcal{U} \to \mathrm{Top}$ is defined on parts A of $\mathcal{U}$ to be the pull-back

$$\begin{array}{ccc} F_p(A) & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ A & \hookrightarrow & X \end{array}$$

and on morphisms in the obvious way.

Theorem 8. *Suppose X, $\mathcal{U}$ and $\mathcal{G}$ satisfy the conditions of Theorem 4. Then if $p: Y \to X$ is a disjoint union of universal covering projections,*

$$H_2(Y;\mathbf{Z}) = H_2 \operatorname{colim} F_p \cong \operatorname{colim} H_2F_p.$$

Proof. The proof is by induction on the cardinality of $\mathcal{U}$. If $\mathcal{U}$ has a single element, then F_p is constant and the theorem is trivial. If $\mathcal{U}$ has two elements, then since $\mathcal{G}$ is a tree, $\bigcap \mathcal{U}$ is path connected and the theorem follows as in Section 1 of this paper.

Suppose $\mathcal{U}$ has n elements. There exists an element $U' \in \mathcal{U}$ such that $\bigcup \mathcal{U}'$ is path-connected for $\mathcal{U}' = \mathcal{U} - \{U\}$. Let $X' = \bigcup \mathcal{U}'$, $W = U' \cap X_1$. Clearly, $\{X_1, U'\}$ is an open cover for X. Consider the push-out

$$\begin{array}{ccc} W & \xrightarrow{b} & X_1 \\ c\downarrow & & \downarrow a \\ U' & \xrightarrow{d} & X \end{array}$$

Since U' is a part of $\mathcal{U}$, d induces a monomorphism of fundamental groups. That c induces a monomorphism of fundamental groups is precisely condition H. Thus b induces a monomorphism of fundamental groups.

It is straightforward to verify that if W is not path-connected, then no minimal generating graph for $\mathcal{U}$ is a tree. Define Y' and Y'' to be the pull-backs

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \hookrightarrow & X \end{array} \qquad \begin{array}{ccc} Y'' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ W & \hookrightarrow & X. \end{array}$$

The hypothesis that $\mathcal{G}$ is a minimal generating tree is sufficient to guarantee that the intersection of elements of $\mathcal{U}$ is path-connected [1, Theorem 2, p. 12]. If we cover W by the intersection of pairs of elements of $\mathcal{U}$ it turns out that this cover, $\mathcal{W}$, has fewer than n elements. Generating trees $\mathcal{G}'$ and $\mathcal{G}''$ for $\mathcal{U}'$ and $\mathcal{W}$ are obtained by restricting $\mathcal{G}$. With these covers and generating graphs, X', $\mathcal{U}'$, $\mathcal{G}'$, Y', and W, $\mathcal{W}$, $\mathcal{G}''$, Y'' satisfy the hypotheses of the theorem so that $H_2(Y';\mathbf{Z})$ and $H_2(Y'';\mathbf{Z})$ are colimits. If we now apply the Mayer–Vietoris Theorem as it appears in the proof of Theorem 1, Theorem 8 follows by diagram chasing. ∎

Definition. Let Ab denote the category of Abelian groups and define $F_H : \mathcal{U} \to \mathrm{Ab}$

as follows. On objects, $A, F_H(A) = \oplus_{I_A} H_2(\tilde{A}; \mathbf{Z})$ for $\tilde{A}$ the universal cover of A and I_A as before. On morphisms, define F_H in the obvious way.

Theorem 9. $H_2(\tilde{X}; \mathbf{Z}) = \operatorname{colim} F_H$.

Proof. This follows from Theorem 8 (by taking Y to be $\tilde{X}$) and from the fact that singular homology is additive (like in the proof of Theorem 1). ■

Theorem 4 is an immediate consequence of Theorem 9 since the Hurewicz isomorphism is a natural transformation and

$$\pi_2(X, *) \cong \pi_2(\tilde{X}; \mathbf{Z}) \cong H_2(\tilde{X}; \mathbf{Z}).$$

Theorem 5 can be proved by these same arguments since its hypotheses guarantee that the lower dimensional part of the Mayer–Vietoris sequence is zero.

In order to prove Theorems 6 and 7 we will show how to use $\mathcal{U}$ and $\mathcal{G}$ to construct a covering space, $\hat{X}$, for X. This covering space will have an open cover $\mathcal{W}$ induced by $\mathcal{U}$. It has as many open sets as elements in the set $\mathcal{U} \times \pi_1(\mathcal{G}, *)$. (This explains why $\pi_1(\mathcal{G}, *)$ must be countable.) A minimal generating graph for $\mathcal{W}$ will be $\tilde{\mathcal{G}}$, the universal cover of $\mathcal{G}$. It turns out that $\hat{X}$, $\mathcal{W}$, and $\tilde{G}$ satisfy the conditions of Theorem 4. If $c\mathcal{U}$ is 1-dimensional it is possible to use the map $c\mathcal{U} \to c\mathcal{U}$ to construct an analagous covering space (exactly as was done in Example 2). The appropriate analog of Theorem 6 may also be proved in this case. If $c\mathcal{U}$ is not 1-dimensional it is necessary to use $\mathcal{G}$ and its universal cover to construct $\mathcal{W}$. The idea is to embed $\mathcal{G}$ in X, notice which edges lie in which parts and then assemble $\mathcal{W}$ from parts of $\mathcal{U}$ placed on the appropriate edges of $\tilde{\mathcal{G}}$. If $\mathcal{G}$ is not essential, $\mathcal{W}$ will contain singularities and will not be a covering space. If $\mathcal{G}$ is essential, $\mathcal{W}$ is not too tangled.

The construction of $\mathcal{W}$ involves the introduction of a category C. Let $p: \tilde{\mathcal{G}} \to \mathcal{G}$ denote the universal covering projection. The objects of C are equivalence classes of pairs $[(A, v)]$ where $v \in v\tilde{\mathcal{G}}$ and A is any part of $\mathcal{U}$ such that $X_{pv} \subseteq A$. The equivalence relation is generated by $\sim$ where

$$(A, v) \sim (A, v')$$

if there is an edge e in $\tilde{\mathcal{G}}$ from v to v' such that $X_{pe} \subseteq A$. As for morphisms, $C([(A, v)], [(B, v')])$ is non-empty if and only if $A \subseteq B$ and $(B, v') \in [(B, v)]$. In this case $C([(A, v)], [(B, v')])$ contains only the inclusion $A \hookrightarrow B$. The functor $G: C \to \mathrm{Top}$ is defined on objects by $G[(A, v)] = A$. On morphisms it is the identity. The family of inclusions $G[(A, v)] \hookrightarrow X$ is a compatible family which induces an onto map $r: \hat{X} \to X$. It will be convenient to denote the image of $G[(A, v)]$ in $\hat{X}$ by $\psi G[(A, v)]$.

It is clear that the collection

$$\mathcal{W} = \{G[(U, v)] \mid U \in \mathcal{U},\ v \in v\tilde{\mathcal{G}}, X_{pv} \subseteq U\}$$

is a cover for $\hat{X}$. It is also true that if X, $\mathcal{U}$, and $\mathcal{G}$ satisfy the conditions of Theorem 7, then $\hat{X}$, $\mathcal{W}$, and $\tilde{\mathcal{G}}$ satisfy the conditions of Theorem 4.

It turns out that $\tilde{G}$ is a minimal generating tree for $\mathcal{W}$. We will show that the usual action of $\pi_1(\mathcal{G}, *)$ on its universal cover extends to a properly discontinuous action on $\hat{X}$ which maps $\tilde{G}$ onto itself and whose orbit space is X. It follows that r is a covering projection and Theorem 7 is proved.

Theorem 6 is an immediate consequence of Lemma 1.1 of [3] since $\tilde{\mathcal{G}}$ is a tree. (Local arc-connectedness is an hypothesis of Lemma 1.1. It is also used in the proof that $\mathcal{W}$ is an open cover of $\hat{X}$.)

A detailed proof of Theorems 6 and 7 involves many straightforward verifications. We shall limit ourselves here to an indication of how the fact that $\mathcal{G}$ is essential is used to obtain the properly discontinuous action.

The category C can be constructed given any cover $\mathcal{U}$ and graph $\mathcal{G}$. In general, an open subset of X will not be evenly covered because the equivalence relation on C identifies too many pairs. The equivalence relation is defined in terms of paths in $\mathcal{G}$ and the hypothesis of essentiality limits these identifications. For example, the following lemma indicates that if $\mathcal{G}$ is essential then all identifications in the colimit may be accounted for by the shortest possible diagrams.

Lemma. *If $\mathcal{G}$ is essential and $x \in \psi G[(A_0, v_0)] \cap \psi G[(A_1, v_1)]$ then there exists a diagram*

$$\begin{array}{ccc} & [(A_2, v_2)] & \\ {}^{s}\swarrow & & \searrow^{t} \\ [(A_0, v_0)] & & [(A_1, v_1)] \end{array}$$

in C and elements $x_i \in G[(A_i, v_i)]$, $i = 0, 1, 2$ with $Gsx_2 = x_0$, $Gtx_2 = x_1$ and $\psi x_i = x$ for $i = 0, 1, 2$.

Proof. The proof is by induction. Suppose a diagram of length two relates elements in $\hat{X}$:

$$\begin{array}{ccccc} & [(B_2, v_2)] & & [(B_4, v_4)] & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \\ [(B_1, v_1)] & & [(B_3, v_3)] & & [(B_5, v_5)]. \end{array}$$

It is immediate that representatives can be chosen such that $pv_2 = pv_3 = pv_4$. If $v_2 \neq v_4$ then the paths relating (B_3, v_2) to (B_3, v_3) and (B_3, v_4) to (B_3, v_3) project to a circuit in $\mathcal{G}$ since p is a covering projection. This circuit is inessential (all its edges are in B_3). Thus $v_2 = v_4$. From the definition of a pair, $X_{pv_2} \subseteq B_2$, $X_{pv_4} \subseteq B_4$. Since $X_{pv_2} = X_{pv_4}$, $B_2 \cap B_4 \neq \emptyset$ and there is a pair (B_6, v_6) and a diagram

$$\begin{array}{ccccccc} & & & [(B_6, v_6)] & & & \\ & & \swarrow & & \searrow & & \\ & [(B_2, v_2)] & & & & [(B_4, v_4)] & \\ \swarrow & & & & & & \searrow \\ [(B_1, v_1)] & & & & & & [(B_5, v_5)] \end{array}$$

which relates the original elements. ∎

Lemma. *If A is part of $\mathscr{U}$ (and $\mathscr{G}$ is essential), then*

$$r^{-1}A = \bigsqcup_{v} \psi G[(A, v)].$$

Proof. It is easy to verify that $r^{-1}A = \bigcup_v \psi G[(A, v)]$. The difficulty lies in showing that this union is disjoint. If $x \in \psi G[(A, v)] \cap \psi G[(A, v')]$, then the previous lemma guarantees the existence of a diagram

$$\begin{array}{ccc} & [(B, w)] & \\ \swarrow & & \searrow \\ [(A, v)] & & [(A, v')]. \end{array}$$

These maps exist only if $(B, w) \in [(B, v)]$ and $(B, w) \in [(B, v')]$. The definition of $\sim$ provides a path in $\tilde{\mathscr{G}}$ from v to v' which projects into $B \subseteq A$. Thus, $[(A, v)] = [(A, v')]$. ∎

The action α of $\pi_1(\mathscr{G}, *)$ on $\hat{X}$ is defined as follows. If $v \in v\tilde{\mathscr{G}}$ and $g \in \pi_1(\mathscr{G}, *)$, denote by gv the usual action of $\pi_1(\mathscr{G}, *)$ on v. If $x \in \hat{X}$, then since $\mathscr{W}$ is an open cover of $\hat{X}$, $x \in \psi G[(U, v)]$ for some $U \in \mathscr{U}$. Since $\psi G[(U, v)] \cap \psi G[(U, pv)] = \emptyset$ (if $pv \neq v$), we may let $\alpha(g, x)$ be the image of x in $\psi G[(U, pv)]$ under the "identity map" $(G[(U, v)] = U = G[(U, pv)])$.

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