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Qualitative Theory of the FitzHugh–Nagumo Equations

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1. INTRODUCTION

The pioneering work of Hodgkin and Huxley [12], and subsequent investigations, have established that good mathematical models for the conduction of nerve impulses along an axon can be given. These models take the form of a system of ordinary differential equations, coupled to a diffusion equation. Simpler models, which seem to describe the qualitative behavior, have been proposed by FitzHugh and Nagumo (see [3, 11] for background). This paper is devoted to the study of the FitzHugh–Nagumo (FN) system:

$$\begin{aligned} v_t &= v_{xx} + f(v) - u \\ u_t &= \sigma v - \gamma u, \end{aligned} \qquad t \geqslant 0, \quad x \in \mathbb{R}, \tag{1.1}$$

where σ, γ are positive constants, and $f(v)$ has the qualitative behavior indicated in Fig. 1 below.

There are two basic problems in the subject; namely, the threshold problem and the traveling wave problem. The first problem is to show that small solutions of (1.1) decay to zero as $t \to +\infty$. This corresponds to the biological fact that a minimum stimulus is needed to "trigger" a nerve; smaller stimuli lead to no signal transmitted down the axion. The second problem falls into two parts. The signals carried by the axon have a characteristic shape and speed. This leads one to investigate whether there are solutions $U = (v, u)$ of the form $U(t, x) =$

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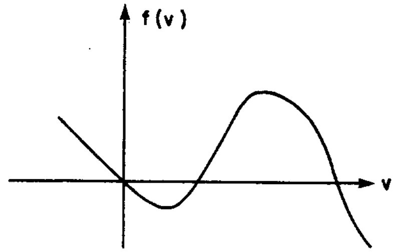

FIGURE 1

$\Phi(x - ct)$, where c is the speed and $\Phi(s)$ is the wave form. For the FN equations, the existence of such solutions has been proved by Conley [4], Carpenter [2], and Hastings [10], provided f, σ, and γ satisfy appropriate conditions. At the heart of these conditions are the assumptions that γ is small and that f has the qualitative form indicated in Fig. 1, where the positive "hump" is larger than the negative "hump." The second part of the traveling wave problem is to show that a large class of initial data do indeed "trigger" a traveling wave. Ideally, one would like to show that "most" large solutions converge to a traveling wave as $t \to +\infty$.

In this paper we present some new methods for the study of system (1.1), and, in particular, we give some qualitative information concerning both the threshold problem and the asymptotic behavior of large solutions. We now describe the contents of the paper in somewhat greater detail.

The first thing we do, in Section 2, is to present a clean framework for studying the local existence and regularity questions for solutions of systems of coupled ordinary differential equations and diffusion equations. Earlier attempts at this seemed awkward because they separated the two forms. Roughly, our idea is to consider the operator ∂_t as $\partial_t - a\partial_{xx}$, when $a = 0$; we thus treat these systems as coupled diffusion equations with some coefficients being zero. From this point of view it is natural to study the dependence of the solutions on the diffusion coefficients as they tend to zero; we do this in Section 2.3.

Our treatment allows us to consider initial data $U(0, x)$ in translation-invariant Banach space B of distributions in $BC^0(\mathbb{R})$ (the bounded uniformly continuous functions on $\mathbb{R}$), in which translation is norm-continuous. This wide freedom in choosing the spaces B permits us to prove the differentiability of solutions by choosing B to consist of functions differentiable in some sense. This freedom also comes up in our later developments where the natural framework for our theorems is in several different spaces B. We also discuss the differentiable dependence on initial conditions; this is needed later in our approach to the threshold problem.

In order to pass from locally defined (small time) solutions to globally defined solutions, we need a priori estimates on $\| U(t)\|_\infty$. In [5], Chueh, Conley, and Smoller introduce a technique, which we call contracting rectangles, to yield such an estimate for the FN equations. The basic property of a contracting rectangle R is that if $U(0, x)$ lies in R for all $x \in \mathbb{R}$, then $U(t, x) \in R$ for all $t \geqslant 0$ and $x \in \mathbb{R}$; i.e., R is an "invariant" set for the system (1.1). In Section 3.1 we study in detail the questions of existence and location of contracting rectangles, and in Section 3.3 we state the relevant global existence theorems for (1.1).

In Section 3.2 we give an important quantitative improvement on the basic property of contracting rectangles. This improvement asserts that, in fact, $U(t, x)$ lies in a smaller contracting rectangle for $t \geqslant \delta > 0$. This fact, together with the results of Section 3.1 allow us to prove, via Lyapunov's second method, that certain solutions tend to zero (Section 4.2), and that all large solutions eventually become smaller than some fixed number, the number being independent of the initial data (Section 5.1).

In Section 4 we gather our results on the threshold problem. The main idea is to show that weak initial stimuli yield no traveling waves. Stated mathematically, we show that if $U(0, x)$ is small, then $U(t) \to 0$ as $t \to +\infty$. The first approach in Section 4.1 proceeds by the standard method of linearization, but, the analysis of the linearized equations leads to some amusing computations. The result we obtain is that for $s > \frac{1}{2}$, if $\| U(0)\|_{H_s}$ is sufficiently small, then $\| U(t)\|_{H_s}$ tends to zero exponentially as $t \to +\infty$. In Section 4.2 we use the method of contracting rectangles to show that $\| U(t)\|_{C_0} \to 0$ as $t \to \infty$, where C_0 is the space of continuous functions which tend to zero as $x \to \pm \infty$. The nice feature of this result is that we obtain a simple explicit description of how small $U(0)$ must be. The defect is that an "extra hypothesis," namely, $-f'(0) > \sigma/\gamma$, must be put on the equations.

Using L_2 inequalities, in Section 4.3 we show that if $U(0) \in L_2 \cap C_0$, and we know that $v(t, x)$ is smaller than the first positive root of $f(v)$ for $t \geqslant t_1 > 0$, and all $x \in \mathbb{R}$, then $U(t) \to 0$ in $L_2 \cap C_0$. This result shows the importance, for the existence of traveling waves, of the fact that $f(v)$ is not negative for all $v > 0$.[1] We then apply this a posteriori decay theorem in Section 4.4 to obtain a sharp subthreshold theorem for small $\sigma > 0$.

In reality, a nerve axon is of finite extent, and the appropriate problem is a mixed initial boundary-value problem on a finite interval, with boundary conditions at the endpoints. In fact, nerves are initially at rest ($u = v = 0$ at $t = 0$), and are stimulated at an endpoint. In Section 4.4 we discuss a model of a semi-infinite at an endpoint. In Section 4.4 we discuss a model of a semi-infinite nerve on $x \geqslant 0$ stimulated at $x = 0$. We show that our previous results can be used to infer that "small" stimuli lead to decaying solutions. Here "small" can be taken in one of two senses: either it is a stimulus of small amplitude which can act over a relatively long time interval, or it is a large amplitude stimulus which lasts only a very short time.

[1] For scalar equations the importance of this sign change is well understood, see, for example, [1].

In the last section we prove two results on "large" solutions. The first uses contracting rectangles to show that there is a limiting size M for $U(t)$, in the sense that no matter how large $U(0)$ is, $\| U(t)\|_\infty \leqslant M$ for t large. We feel that this is a first, albeit small, step toward showing that all large solutions tend to traveling waves as $t \to +\infty$. Finally, in Section 5.2 we turn to the problem of nerves of finite length. We show that solutions of the FN equations on a finite x interval, with reasonable boundary conditions, must decay as $t \to +\infty$, if the interval is sufficiently short. Here the critical interval length increases with the height of the positive maximum of f. This again shows the crucial role played by the positive values of f for positive v.

2. General Theory

2.1. *Local Solvability*

Consider the nonlinear system of equations in two independent variables,

$$U_t = AU_{xx} + F(U), \tag{2.1}$$

where $U = (u_1, ..., u_n)$ is a real n-vector, F is a smooth $\mathbb{R}^n$-valued function, with $F(0) = 0$, and A is a diagonal matrix, $A = \text{diag}\{a_1, a_2, ..., a_n\}$ with each $a_i \geqslant 0$. If all the a_i are positive, then (2.1) is a parabolic system, while if some $a_i = 0$, we have a coupled system of parabolic equations and ordinary differential equations. This coupling of equations of different form has made the literature in the field appear somewhat awkward. The purpose of this section is to show how systems of the form (2.1) can be treated in a unified manner; that is, in a way which disregards the differences between the parabolic and ordinary differential equations. The main point is that for a given F, one has solvability in an interval $0 \leqslant t \leqslant t_0$, where t_0 depends only on the *sup norm* of the initial data $U(0, x)$.

We shall obtain solutions of (2.1) which are continuous functions of time with values in various Banach spaces B, where B depends, of course, on the initial data. To be precise we assume that B is a Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ subject to the following restrictions:

(2.2) B is a subset of the bounded continuous functions on $\mathbb{R}$ and for $W \in B$, $\| W\|_B \geqslant \| W\|_\infty$.

(2.3) B is translation-invariant; i.e., if $W \in B$, then $W \circ \tau \in B$ for any translation τ: $\mathbb{R} \to \mathbb{R}$, and $\| W \cdot \tau \|_B = \| W\|_B$.

(2.4) If f: $\mathbb{R}^n \to \mathbb{R}^n$ is a smooth function with $f(0) = 0$ then $f \circ W \in B$ for any $W \in B$ and for any $M > 0$ there are constants K_1 and K_2 such that

$$\| W\|_B \leqslant M \quad \text{and} \quad \| \tilde{W}\|_B \leqslant M \Rightarrow \|f \circ W - f \circ \tilde{W}\|_B \leqslant K_1 \| W - \tilde{W}\|_B,$$

and

$$\| W\|_\infty \leqslant M \Rightarrow \|f \circ W\|_B \leqslant K_2 \| W\|_B .$$

(2.5) *If* $\tau_h : \mathbb{R} \to \mathbb{R}$ is translation by h, $\tau_h(x) = x + h$, and $\tau_h W = W \circ \tau_h$, then for any $W \in B$, $\lim_{h\to 0} \| \tau_h W - W \|_B = 0$.

It is easy to see that the following spaces satisfy (2.2)–(2.5):

$BC^k = \{W: (d/dx)^j W$ is a bounded uniformly continuous function on $\mathbb{R}$ for $0 \leqslant j \leqslant k\}$, $k \geqslant 0$;

$BC^0 \cap L_p$, $p \geqslant 1$;

$W_p^k = \{W \in L_p : (d/dx)^j W \in L_p$ for $0 \leqslant j \leqslant k\}$, $k \geqslant 1$;

$C_0^k = \{W \in BC^k : \lim (d/dx)^j W = 0$ as $|x| \to \infty$ for $0 \leqslant j \leqslant k\}$.

Moreover, the intersection of a finite number of spaces having properties (2.2)–(2.5) again has these properties. On the other hand, the space $L_\infty \cap L_p$ does not have property (2.5), as one easily sees by taking W to be the characteristic function of an interval.

If j is in the Schwartz class, $\mathscr{S}(\mathbb{R})$ satisfies $\int_{-\infty}^{\infty} j(x)\, dx = 1$, and $j_\epsilon(x) \equiv \epsilon^{-1} j(\epsilon^{-1} x)$, then (2.5) holds if and only if

(2.6) For any $W \in B$, $j_\epsilon * W \to W$ in B as $\epsilon \to 0$.

Proof. That (2.5) implies (2.6) follows from the identity

$$j_\epsilon * W = \int j_\epsilon(h)(\tau_h W)\, dh.$$

The reverse implication is not needed below and the proof is left to the reader. □

Applying (2.6) in the special case $j = (4\pi)^{-1/2} e^{-x^2}$ it follows that the solution of $u_t = a u_{xx}$ with $u(0) = u^0 \in B$ is a continuous function of t with values in B for $0 \leqslant t < \infty$.

Of special interest to us will be the Banach space of continuous functions on $[0, T]$ with values in B. This space, denoted by $C([0, T] | B)$, is normed by $\sup_{0 \leqslant t \leqslant T} \| W(t) \|_B$.

In order to solve system (2.1) we use the Green's kernel $G(t)$ for the associated linear system; namely, $G(t) = \mathrm{diag}\{g_{a_1}(t), g_{a_2}(t), \ldots, g_{a_n}(t)\}$, where $g_a(t) = (4\pi a t)^{-1/2} \exp[-x^2(4at)^{-1}]$ if $a > 0$, while $g_0(t) = \delta(x)$. It is a simple matter to show that $U \epsilon([0, t] B)$ satisfies (2.1) (in the sense of distributions) and the initial condition $U(0) = U^0$ if and only if U is a solution of the integral equation

$$U(t) = G(t) * U^0 + \int_0^t G(t - s) * F(U(s))\, ds, \tag{2.7}$$

where $*$ denotes convolution on $\mathbb{R}$.

It is worth noting that for $U \in C([0, t] | B)$ the integrand in (2.7) is a continuous function of s with values in B, so that the integral is actually a Riemann integral.

Finally, if μ is a finite Borel measure on $\mathbb{R}$, and $W \in B$, then $\mu * W \in B$ and

$$\| \mu * W \|_B \leqslant (\text{total variation of } \mu) \| W \|_B .$$

This inequality is a consequence of the translation invariance of the norm in B. Applied to G, we see that for any $t \geqslant 0$, and $U \in B$, we have

$$\| G(t) * U \|_B \leqslant \| U \|_B .$$

With these preliminaries out of the way, we can solve the integral equation (2.7) for a short time interval by standard iteration techniques. The point to be made here is that the size of this time interval depends only on F and the sup norm of U^0. The usual methods have the time interval also depending on A and $\| U^0 \|_B$.

THEOREM 2.1. *For any $U^0 \in B$, there is a constant $t_1 > 0$, depending only on F and $\| U^0 \|_\infty$ such that the initial value problem for* (2.1) *with data $U(0) = U^0$ has a unique solution in $C([0, t_0] | B)$.*

Proof. We first show that there is a $t_0 > 0$ depending only on $\| U^0 \|_B$ and F such that (2.1) has a unique solution in $C([0, t]\, 1B)$ and $\| U \|_{C([0,t]1B)} \leqslant 2 \| U^0 \|_B$. For any $t_0 > 0$, let

$$\Omega = \{U \in C([0, t_0] | B) \colon \| U(t) - G(t) * U^0 \|_B \leqslant \| U^0 \|_B ,\ 0 \leqslant t \leqslant t_0\}.$$

If $U \in \Omega$, then $\| U(t) \|_B \leqslant 2 \| U^0 \|_B$, for $0 \leqslant t \leqslant t_0$, so by (2.4), we may choose a constant k, independent of t_0 such that for $U, V \in \Omega$,

$$\| F(U(t)) - F(V(t)) \|_{C([0,t_0]|B)} \leqslant k \| U(t) - V(t) \|_{C([0,t_0]|B)} . \tag{2.8}$$

Let $t_0 = (2k)^{-1}$, so that t_0 clearly depends only on F and $\| U^0 \|_B$. We define a map Γ from $C([0, t_0] | B)$ into itself by

$$\Gamma U(t) = G(t) * U^0 + \int_0^t G(t - s) * F(U(s))\, ds.$$

We first show that Γ maps the closed set Ω into itself. For an element $W = (w_1, \ldots, w_n)$ in B, let $| W | = (| w_1 |, \ldots, | w_n |) \in BC^0$. We also define a partial ordering on B by $W \leqslant \tilde{W}$ if the inequality holds on each component. Since for every a, $t \geqslant 0$ the distributions $g_a(t)$ are positive measures of total mass equal to 1, we have, for $U \in \Omega$,

$$\| \Gamma(U)(t) - G(t) * U^0 \|_B \leqslant k \int_0^t \| U(s) \|_B\, ds,$$

where we have used (2.8) with $V = 0$. Hence, for $0 \leqslant t \leqslant t_0$,

$$\| \Gamma(U)(t) - G(t) * U^0 \|_B \leqslant 2kt_0 \| U^0 \|_B = \| U^0 \|_B$$

so that Γ maps Ω into itself.

We can also show that Γ is a contraction mapping on Ω, for, if $U, V \in \Omega$, then

$$\begin{aligned} \| \Gamma(U)(t) - \Gamma(V)(t) \|_B &\leqslant \int_0^t \| G(t-s) * (F(U(s)) - F(V(s))) \|_B \, ds \\ &\leqslant \int_0^t \| F(U(s)) - F(V(s)) \|_B \, ds \\ &\leqslant k \int_0^t \| U(s) - V(s) \|_B \, ds \\ &\leqslant kt_0 \| U - V \|_{C([0,t_0]|B)} \\ &\leqslant \tfrac{1}{2} \| U - V \|_{C([0,t_0]|B)} . \end{aligned}$$

Banach's theorem implies that Γ has a unique fixed point in Ω.

Banach's theorem tells us that U is the only solution in Ω, but leaves open the possibility of solutions outside of Ω. However, uniqueness of $U \in C([0, t_0] \| B)$ is a special case of Theorem 2.3 of the next section.

To complete the proof we must extend the solution to an interval $0 \leqslant t \leqslant t_1$ which depends only on $\| U^0 \|_\infty$. Since $B \subset BC^0$ the above argument shows that there is a t_1 depending only on F and $\| U^0 \|_\infty$ and a solution $V \in C([1, t_1]/BC^0)$ with $\| V(t) \|_\infty \leqslant 2 \| U^0 \|$ for $t \in [0, t_1]$. By uniqueness in $C([0, t_0]/BC^0)$ we have $U = V$ for $0 \leqslant t \leqslant t_0$. To complete the proof it suffices to show that $V \in C([0, t_1]/B)$; then V provides the desired extension. To prove the regularity of V we will show that there is an $\eta > 0$ independent of $t_2 \in [0, t_1]$ with the property that if $V \in C([0, t_2]/B)$ then $V \in C([0, t_2 + \eta]/B)$. A finite number of applications of this result implies that $V \in C([0, t_1]/B)$. The main point is an estimate for $\| V \|_{C([0,t_2]/B)}$ which is independent of t_2. Now V is a solution of the integral equation (2.7) so taking norms of both sides yields

$$\| V(t) \|_B \leqslant \| U^0 \|_B + \int_0^t \| F(V(s)) \|_B \, ds.$$

By property (2.4) and the fact that $\| V(t) \|_\infty \leqslant 2 \| U^0 \|_\infty$ we may choose K_2 so that $\| F(V(s)) \|_B \leqslant K_2 \| V(s) \|_B$. An application of Gronwall's inequality yields a constant $c > 0$, independent of $t_2 \in [0, t_1]$, such that $\| V(t) \|_B \leqslant C$ for $0 \leqslant t \leqslant t_2$. Choose $\eta > 0$ so that the initial value problem for (2.1) with data of B norm at most C has a solution in $C([0, \eta]/B)$. Let $W \in C([0, \eta]/B)$ solve (2.1) with $W(0) = V(t_2)$ and then define $V(t_2 + s) = W(s)$ for $0 \leqslant s \leqslant \eta$. This provides the extension to $[0, t_2 + \eta]$ and completes the proof. □

The result of Theorem 2.1 can be strengthened in two ways if Eq. (2.1) is linear, that is, if $F(U) = FU$ for a constant matrix F. In that case inequality (2.8) holds for all $U(t)$, $V(t)$ in $C([0, t_0] | B)$ if k is chosen suitably. In addition we need not assume that B satisfies (2.2), (2.4); B need only be a translation-invariant Banach space of distributions on $\mathbb{R}$ satisfying (2.5). These remarks show that in the linear case we may choose t_0 so that a solution exists on $0 \leqslant t \leqslant t_0$ for any $U^0 \in B$ and $\| U \|_{C([0,t_0]|B)} \leqslant 2 \| U^0 \|_B$. The solution can then be continued to $t_0 \leqslant t \leqslant 2t_0$ by solving the initial value problem

$$(\partial_t - A\partial_{xx} - F)\tilde{U} = 0, \qquad t_0 \leqslant t \leqslant 2t_0,$$
$$\tilde{U}(t_0) = U(t_0).$$

The extended solution is then defined by

$$\begin{aligned} U(t) &= \text{original } U(t) \text{ if } 0 \leqslant t \leqslant t_0, \\ &= \tilde{U}(t) \text{ if } t_0 \leqslant t \leqslant 2t_0. \end{aligned}$$

Continuing in this manner we find a solution

$$U \in C([0, \infty) \mid B) \qquad \text{with} \quad \| U(t)\|_B \leqslant 2^{[t/t_0]+1} \| U^0 \|_B .$$

Summarizing these remarks we have

THEOREM 2.2. *If* (2.1) *is linear, that is,* $F(U) = FU$ *for a constant matrix* F, *and* B *is a Banach space of distributions satisfying only* (2.3) *and* (2.5), *then for any* $U^0 \in B$ *there is a unique* $U \in C([0, \infty)| B)$ *which satisfies the initial value problem for* (2.1) *with data* $U(0) = U^0$. *Furthermore, there are constants* k *and* c *independent of* U^0 *such that*

$$\| U(t)\|_B \leqslant ke^{ct} \| U^0 \|_B .$$

2.2. *Differentiable Dependence on Initial Conditions*

The integral equation (2.7) allows us to get some information on the dependence of solutions on the initial data, U^0. First observe that if U and $\tilde{U}$ are solutions belonging to $C([0, T]| B)$, then for any $t \in [0, T]$

$$U(t) - \tilde{U}(t) = G(t) * (U(0) - \tilde{U}(0)) + \int_0^t G(t - s) * (F(U(s)) - F(\tilde{U}(s)))\, ds.$$

Suppose that $\| U(t)\|_B \leqslant M$ and $\| \tilde{U}(t)\|_B \leqslant M$ for $0 \leqslant t \leqslant T$, and choose k so that (2.4) holds for all W, $\tilde{W}$ with $\| W \|_B$ and $\| \tilde{W} \|_B$ bounded by M. Then

$$\| U(t) - \tilde{U}(t)\|_B \leqslant \| U(0) - \tilde{U}(0)\|_B + k \int_0^t \| U(s) - \tilde{U}(s)\|_B \, ds,$$

so that Gronwall's inequality yields

$$\| U(t) - \tilde{U}(t)\|_B \leqslant e^{kt} \| U(0) - \tilde{U}(0)\|_B . \tag{2.9}$$

We state this formally as

THEOREM 2.3. *Let U and $\tilde{U}$ be elements of $C([0, T]| B)$ which satisfy both* (2.1) *and $\| U(t)\|_B$, $\| \tilde{U}(t)\|_B \leqslant M$ for $0 \leqslant t \leqslant T$. Then there is a k depending only on M such that* (2.9) *holds.*

This result asserts Lipschitz continuous dependence on initial data. Our next result strengthens this to differentiable dependence. Define a nonlinear map $S(t): B \to B$ by $S(t)\, U^0 = U(t)$, where $U(t)$ is the solution of (2.1), with $U(0) = U^0$. Let $S_L(t)$ be the analagous solution operator for the linearized equations

$$V_t = AV_{xx} + dF_0(V), \qquad V(0) = U^0, \tag{2.10}$$

where dF_0 is the Jacobian matrix of F evaluated at zero.

THEOREM 2.4. *Suppose that $T > 0$ is so small that the initial value problem for* (2.1) *has a solution in $C([0, T]| B)$ for any initial data $U(0) = U^0$ in a neighborhood, $\mathcal{O}$, $0 \in \mathcal{O}$. Then the map $S(t)$ is Frechet differentiable at zero and*

$$dS(T)_0 = S_L(T). \tag{2.11}$$

Proof. For $0 \leqslant t \leqslant T$ let $U(t) = S(t)(h\Phi)$ with $\| \Phi \|_B \leqslant 1$ and $| h | \leqslant \epsilon$ so that $h\Phi \in \mathcal{O}$. We must show that $\delta(t) \equiv h^{-1}U(t)$ converges to $S_L(t)\Phi$ as $h \to 0$ uniformly in Φ. For δ we have the equation

$$(\partial_t - A\partial_{xx})\, \delta = h^{-1}F(U(t)).$$

From the Taylor formula with remainder (see [7, Sect. 8.14]) and property (2.4) of B we have, for Ψ ranging over any bounded subset of B,

$$\|F(\Psi) - dF_0(\Psi)\|_B \leqslant c_1 \| \Psi \|_B^2 .$$

By (2.9) we have $\| U(t)\|_B \leqslant c_2 h$ for $0 \leqslant t \leqslant T$ with c_2 independent of Φ, so if $W = \delta(t) - S_L(t)\Phi$, we have

$$(\partial_t - A\partial_{xx} - dF_0)\, W = R(t),$$

where $\| R(t)\|_B \leqslant c_3 h$ for $t \in [0, T]$. The usual Gronwall estimate shows that $\| W(t)\|_B \leqslant c_4 h$ for $0 \leqslant t \leqslant T$ with c_4 independent of Φ, and the proof is complete. □

2.3. *Dependence on the Parameters* $a_1, \ldots, a_n$.

We now investigate the dependence of solutions to (2.1) on the parameters $a_1, \ldots, a_n$. The main question that we are concerned with is how the solutions behave when some of the a_i's tend to zero; i.e., when certain diffusion terms are allowed to shut off. We show that the limiting solution (i.e., the one where some $a_i = 0$) is indeed a limit of the solutions where the a_i are positive. In other words, the solutions depend continuously on the a_i's. We assume the following existence assertion.

(2.12) There is a function $\phi: \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ such that for any $a \in (\bar{\mathbb{R}}_+)^n$, and any $U^0 \in B$, Eq. (2.1) has a unique solution $U \in C([0, T] \| B)$ with $U(0) = U^0$, and $\sup_{0 \leqslant t \leqslant T} \| U(t) \|_B \leqslant \phi(\| U^0 \|_B)$.

Our local existence theorem, Theorem 2.1, shows that (2.12) holds if T is sufficiently small. In the next section we shall show that for the nerve equations, (2.12) is valid for all $T > 0$.

We now make an important observation; namely, if (2.12) holds for a space B, it also holds for any subspace $B' \subset B$, with the same ϕ. This is true since the a priori sup norm estimate in (2.12) allows the passage from a local solution to a solution defined for $0 \leqslant t \leqslant T$.

For any B and integer k define a new Banach space B_k by $B_k = \{W \in B : (d/dx)^i W \in B, \text{ for } 1 \leqslant i \leqslant k\}$,

$$\| W \|_{B_k} = \sum_{n=0}^{k} \left\| \frac{d^n}{dx^n} W \right\|_B$$

We then have

Lemma 2.5. *B_k is dense in B for any $k \geqslant 0$.*

Proof. Let j and j_ϵ be as in (2.6) then

$$\left(\frac{d}{dx}\right)^k (j_\epsilon * W) = \left(\frac{d^k}{dx^k} j_\epsilon\right) * W \in B,$$

so $j_\epsilon * W \in B_k$. However, (2.6) asserts that $j_\epsilon * W \to W$ as $t \to 0$, so B_k is dense in B. □

Observe that $B_k \subset BC^k$, so in particular, we see that $B \cap BC^k$ is dense in B for all k.

Now if we assume that (2.12) holds, we may define the solution operator $S_a : B \to C([0, T] \| B)$ by letting $S_a U^0$ be the solution of (2.1) with $U(0) = U^0$. We then have the main result of this section.

Theorem 2.5. *If* (2.12) *holds then for any* $U^0 \in B$, *the map* $a \to S_a U^0$ *is continuous from* $(\bar{\mathbb{R}}_+)^n$ *to* $C([0, T] \| B)$.

Proof. Let N be any positive number and let

$$B^N = \{U^0 \in B : \| U^0 \|_B \leqslant N\}.$$

We prove that $S_a U^0$ is a continuous function of a for $U^0 \in B^N$.

With $\phi(N)$ as in (2.12), we use (2.4) to find a k so that

$$\| F(W) - F(\tilde{W}) \|_B \leqslant k \| W - \tilde{W} \|_B \tag{2.13}$$

for all W, $\tilde{W} \in B^N$. Then from (2.9) we have

$$\| S_a U^0 - S_a \tilde{U}^0 \|_{C([0,T] | B)} \leqslant e^{kT} \| U^0 - \tilde{U}^0 \|_B \tag{2.14}$$

for all U^0, $\tilde{U}^0$ in B^N. Thus, as maps of B^N to $C([0, T] | B)$, the family $\{S_a : a \in (\mathbb{R}_+)^n\}$ is uniformly equicontinuous. Thus, if $a_k \to a$ in $(\bar{\mathbb{R}}_+)^n$, it suffices to prove that $S_{a_k} U^0 \to S_a U^0$ for U^0 in a dense subset of B^N. The dense set we choose is $B_2 \cap B^N$.

Since $B_2 \subset B$, our previous remarks show that both (2.12) and (2.14) hold for B replaced by B_2. In particular, taking $\tilde{U}_0 = 0$, we have that

$$\| S_a U^0 \|_{C([0,T] | B_2)} \leqslant e^{kT} \| U^0 \|_{B_2} . \tag{2.15}$$

For a, $a' \in (\bar{\mathbb{R}}_+)^n$, let $U = S_a U^0$, $U' = S_{a'} U^0$, and let

$$A = \operatorname{diag}(a), \qquad A' = \operatorname{diag}(a').$$

Then if $V = U - U'$, we obtain the following differential equation for V:

$$V_t - A V_{xx} = (A - A') \, U'_{xx} + F(U) - F(U'). \tag{2.16}$$

If $\epsilon = \max\{| a_i - a_i' | : 1 \leqslant i \leqslant n\}$, then (2.14), with B replaced by B_2, shows that the right-hand side of (2.16) is a continuous function H, with values in B, and (2.15) yields

$$\| H(t) \|_B \leqslant \epsilon e^{kT} \| U^0 \|_{B_2} + k \| V(t) \|_B . \tag{2.17}$$

If we write (2.16) in integral form,

$$V(t) = G(t) * V(0) + \int_0^t G(t - s) * H(s) \, ds,$$

and note that $V(0) = 0$, we get, from (2.17),

$$\| V(t) \|_B \leqslant \epsilon e^{kT} T \| U^0 \|_{B_2} + k \int_0^t \| V(s) \|_B \, ds.$$

Then Gronwall's inequality implies that for $0 \leqslant t \leqslant T$,

$$\| V(t)\|_B \leqslant \epsilon T e^{2kT} \| U^0 \|_{B_2} .$$

It follows that if $a_k \to a$ in $(\overline{\mathbb{R}}_+)^n$ then $S_{a_k} U^0 \to S_a U^0$ in $C([0, T] \| B)$, and the proof is complete. □

3. Contracting Rectangles for the FitzHugh–Nagumo Equations

3.1. *Contracting Rectangles*

In our proofs of theorems concerning global (in time) existence, stability, and asymptotic behavior of solutions of (1.1), a central role is played by rectangles which are contracting for the vector field $(f(v) - u, \sigma v - \gamma u)$ in the following sense.

Definition 3.1. A bounded convex set $R \subset \mathbb{R}^n$ is contracting for the vector field $F(U)$ if for every point $U \in \partial R$ and every outward unit normal n at U, $F(U) \cdot n < 0$.

The importance of such sets for deriving sup norm estimates was pointed out in [5, 13]. For systems of partial differential equations whose principal symbol is *not* scalar, a special role is played by rectangles (see [5]). In this section we gather together several results concerning the existence of such rectangles. We restrict attention to $U = (v, u) \in \mathbb{R}^2$.

Lemma 3.2. *Suppose $H(U)$ and $\tilde{H}(U)$ are vector fields with the same second component, and that R is a rectangle containing* 0 *which is contracting for $H(U)$. If* $(\operatorname{sgn} v)\, \tilde{h}_1(U) \leqslant (\operatorname{sgn} v)\, h_1(U)$, *on ∂R, then R is contracting for $\tilde{H}$.*

Proof. Since R is contracting for $H(U)$ and $h_2(U) = \tilde{h}_2(U)$, we see that $\tilde{h}_2(U)$ is negative on the "top" of R, and positive on the "bottom" of R. On the other hand, the inequality in the hypothesis implies that $\tilde{h}_1(U)$ is negative on the "right edge" of R and positive on the "left edge" of R. □

Lemma 3.3. *For the linear vector field $F_L(U) = (-\beta v - u,\ \sigma v - \gamma u)$, γ, $\beta > 0$, $\sigma \geqslant 0$, there is a contracting rectangle containing* $(0, 0)$ *if and only if* $\beta > \sigma/\gamma$.

Proof. Suppose that R is a contracting rectangle for F_L containing (0, 0). Then it is easy to see that the top and bottom of R must lie, respectively, above and below the line $\sigma v - \gamma u = 0$, while the left- and right-hand sides of R must lie, respectively, below and above the line $-\beta v - u = 0$; cf. Fig. 2.

Let $\alpha = \sigma/\gamma$; then if we let (v_0, u_0) denote the upper right-hand corner of R and (v_1, u_0), (v_1, u_1), (v_0, u_1) denote the other corners of R (oriented counter-clockwise from (v_0, u_0)), then there exist $\epsilon_i > 0$, $1 \leqslant i \leqslant 4$, such that $v_0 =$

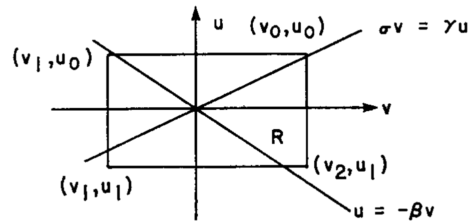

FIGURE 2

$(u_0/\alpha) - \epsilon_1$, $v_1 = (-u_0/\beta) - \epsilon_2$, $u_1 = \alpha v_1 - \epsilon_3$, $v_0 = v_2 = (u_1/\beta) + \epsilon_4$. Using the last three of these equations, we have

$$v_0 = \alpha u_0/\beta^2 + \epsilon_3/\beta + \alpha\epsilon_2/\beta + \epsilon_4,$$

so that together with the first equation, we get

$$(1/\alpha - \alpha/\beta^2)\, u_0 = \epsilon_1 + \epsilon_4 + \epsilon_3/\beta + \alpha\epsilon_2/\beta. \qquad (*)$$

Since the right-hand side of this last equation is positive, and u_0 must be positive, we have $\beta > \alpha$. Conversely, suppose $\beta > \alpha$. We let (v_0, u_0), $u_0 > 0$ be any point as in Fig. 2, and successively construct points (v_1, u_0), (v_1, u_1), (v_2, u_1) satisfying the above relations for some $\epsilon_i > 0$, $1 \leqslant i \leqslant 4$. We must show that we can choose these ϵ_i so as to satisfy $v_2 = v_0$. But this means that we must solve $(*)$ for the ϵ_i's. Since the left-hand side of $(*)$ is positive, this can obviously be done. □

Notice that for the linear vector field of Lemma 3.3, if a set R is contracting, then τR is contracting for all $\tau > 0$. We now investigate the existence of small contracting rectangles for the FitzHugh–Nagumo vector field.

LEMMA 3.4. *If $f'(0) = -\beta$, and $\beta > \sigma/\gamma$, then there is a rectangle R containing* 0 *such that τR is contracting for $F(U) = (f(v) - u, \sigma v - \gamma u)$ for all sufficiently small τ.*

Proof. For v small, $f(v) = -\beta v + O(v^2)$. Thus we may choose $\epsilon > 0$ so small that $\beta - \epsilon > \sigma/\gamma$ and $(\operatorname{sgn} v) f(v) \leqslant (\operatorname{sgn} v)(-\beta + \epsilon)v$ for v small. The result now follows from the last two lemmas since we may choose R contracting for the field $((-\beta + \epsilon)\, v - u, \sigma v - \gamma u)$. Then τR is contracting for this field for all $\tau > 0$. If τ is sufficiently small, the above inequality, together with Lemma 3.2 shows that τR is contracting for F. □

If we examine the proof of Lemma 3.4 we find that we can show more; namely, we can show that there is a critical rectangle R_c, which, roughly speaking, is an upper limit of "small" contracting rectangles containing 0. For small $v > 0$ (resp. $v < 0$), the graph $u = f(v)$ lies below (resp. above) the line $u - (\sigma/\gamma)v$. For $f(v)$ a cubic, we see that even if $|f'(0)| > \sigma/\gamma$, the curve $u = f(v)$

crosses this line for large v. Assuming that $-f'(0) > \sigma/\gamma$, and that $\{v: f(v) = -(\sigma/\gamma)\, v, v \neq 0\}$ is nonvoid, let

$$v_C = \min\{|\, v\,|: f(v) = -(\sigma/\gamma)\, v, v \neq 0\}.$$

Let R_C be the rectangle symmetric in the u and v axes with upper right-hand corner at the point $(v_C, \sigma/\gamma v_C)$, see Fig. 3. We then have the following result which will be used in Section 4.2.

LEMMA 3.5. *Suppose that* $F(U) = (f(v) - u,\ \ \sigma v - \gamma u)$, $-f'(0) > \sigma/\gamma$, *and* R_C *is the rectangle described above. For any compact set* Q *in the interior of* R_C, *there is a rectangle* R *and a constant* $k > 0$ *such* $Q \subset R \subset R_C$ *and* $F(U) \cdot n < -k\tau$ *for all* $\tau \in (0, 1]$, $U \in \partial(\tau R)$, *and outward unit normals* n *at* U.

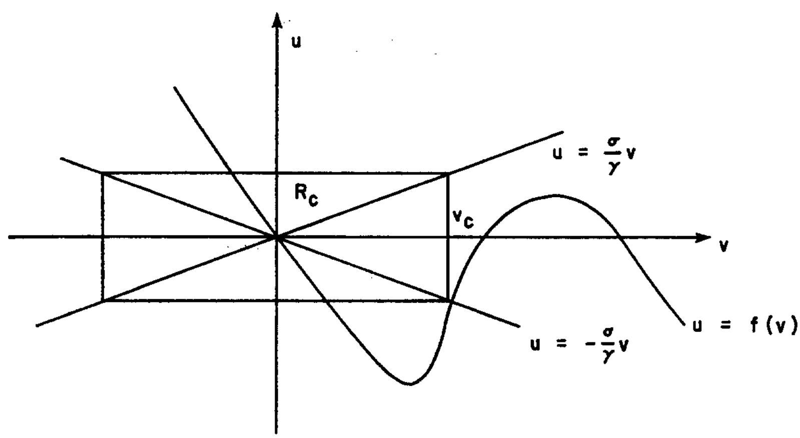

FIGURE 3

Proof. Choose λ, $0 < \lambda < 1$ so that $Q \subset \lambda R_C$ and let $R(\epsilon)$ be the rectangle symmetric with respect to the u and v axes, with upper right-hand corner $(\lambda v_C, (\sigma/\gamma + \epsilon)\, \lambda v_C)$. Choose $\epsilon_0 > 0$ so small that $Q \subset R(\epsilon_0) \subset \text{int } R_C$; then $R(\epsilon_0)$ is the desired rectangle. □

Next we consider the question of the existence of large contracting rectangles for the vector field $F(U)$. Notice that for the biological models of the nerve conduction equations, the function $f(v)$ is a cubic polynomial. For such functions, we have

$$\lim_{|v| \to \infty} f(v)/v = +\infty.$$

In particular the following growth condition is satisfied

$$\liminf_{|v| \to \infty} |\, f(v)/v\,| > \sigma/\gamma. \tag{3.1}$$

LEMMA 3.6. *Suppose that $f(v)$ satisfies* (3.1). *Then there is a rectangle R containing* 0, *and a real number $\tau_0 > 0$ such that τR is contracting for $F(U) = (f(v) - u, \sigma v - \gamma u)$ if $\tau \geqslant \tau_0$.*

Proof. Choose $\epsilon > 0$ so small that

$$|f(v)| < (-\sigma/\gamma - \epsilon)v$$

for v sufficiently large, say $|v| > M$. Let R be a contracting rectangle for the linear field $((-\sigma/\gamma - \epsilon)v - u, \sigma v - \gamma u)$. Then τR is contracting for F provided that $(\pm M, 0) \in \tau R$, in particular, for τ large. □

Remark. Observe that the rectangle R of Lemma 3.6 is contracting for all the fields $\tilde{F}(U) = (f(v) - u, \tilde{\sigma} v - \gamma v)$ with $0 \leqslant \sigma \leqslant \tilde{\sigma}$. Reducing the value of σ makes the field point more toward the interior of R. This fact, which is analogous to Lemma 3.2, will be needed later.

Just as we have a critical "small rectangle" R_c, there is a lower limit R^c to "large" contracting rectangles. Thus, suppose f satisfies (3.1) and set

$$v^c = \max\{|v| : f(v) = -(\sigma/\gamma)v\}.$$

Let R^c be the rectangle symmetric in the u and v axes with upper right-hand corner $(v^c, \sigma v_c/\gamma)$, see Fig. 4. We then have the following refinement of Lemma 3.6 whose proof proceeds in a manner analogous to Lemma 3.5.

LEMMA 3.7. *Suppose f and F are as in Lemma* 3.6. *Then for any compact set Q in the exterior of R^c, there is a rectangle R such that $R^c \subset R$, Q is in the exterior of R, and τR is contracting for F for $1 \leqslant \tau < \infty$.*

3.2. *The Basic Lemma*

The basic fact about contracting rectangles is that they allow us to define non-linear functionals, which are decreasing functions of time for solutions of the differential equations (2.1). This result is applied in Section 3.3 to prove global existence and in Sections 4.2, 4.4, and 5.2 to study asymptotic behavior as $t \to \infty$. For our applications the a_i are not equal, so a special role is played by rectangles (though for $A = cI$ the results of this section can be extended to arbitrary convex R, cf. [13]).

The functionals we consider are associated with rectangles R such that the origin is in the interior of R. Let $|\cdot|_R$ be the norm on $\mathbb{R}^n$ defined by such a rectangle in the usual way:

$$|U|_R = \inf\{t \geqslant 0: U \in tR\}. \tag{3.2}$$

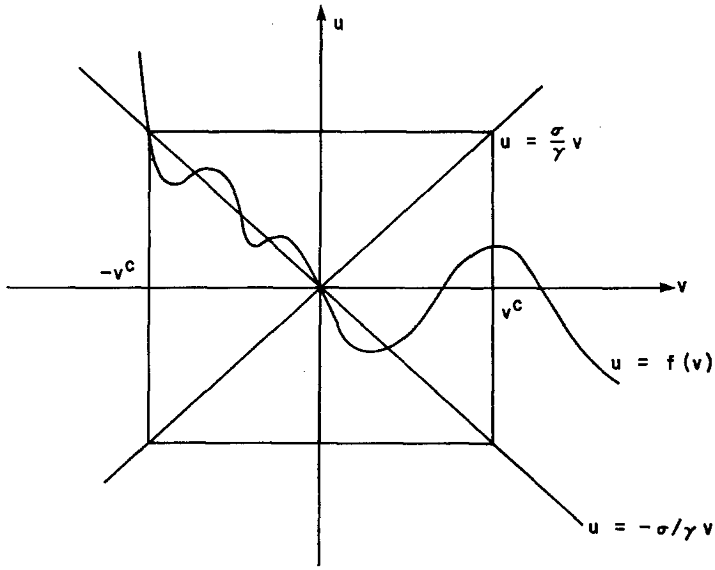

FIGURE 4

Thus $| U |_R$ is the smallest multiple of R containing U. We define a continuous function $\mathscr{V}_R : BC \to \mathbb{R}$ by

$$\mathscr{V}_R(W) = \sup_{x \in R} | W(x)|_R . \tag{3.3}$$

The basic fact, which is a quantitative version of the maximum principle of Chueh, Conley, and Smoller [5], is given in the next lemma. Recall that for a function $\Psi: \mathbb{R} \to \mathbb{R}$, the upper Dini derivate is defined by

$$\bar{D}\Psi(T) = \limsup_{h \to 0} \frac{\Psi(T+h) - \Psi(T)}{h} .$$

LEMMA 3.8. *Let $F(U)$ be a vector field on $\mathbb{R}^n$, and let R be a rectangle with $0 \in \operatorname{int}(R)$. Suppose that $U \in C((T - \delta, T + \delta), C_0(\mathbb{R}))$ is a smooth solution of (2.1) for $| t - T | < \delta$ and that $\mathscr{V}_R(U(T)) = s$. If there is an $\eta > 0$ such that for any $W \in \partial(sR)$ and $n(W)$ normal to $\partial(sR)$ at W we have $F(W) \cdot n(W) < -\eta$, then*

$$\bar{D}\mathscr{V}_R(U(T)) \leqslant -(2\eta/sL)\, \mathscr{V}_R(U(T)), \tag{3.4}$$

where L is the length of the shortest side of R.

Proof. Let $U = (u_1, ..., u_n)$, $F = (f_1, ..., f_n)$, and let R be defined by the inequalities $-l_i \leqslant u_i \leqslant r_i$, $i = 1, 2, ..., n$. Multiplying R by a scalar if necessary we may assume that $\mathscr{V}_R(U(T)) = 1$ so $-l_i \leqslant u_i(T, x) \leqslant r_i$ for all x. We say that $U(t, x)$ is in the jth right-hand face if $u_j(t, x) = r_j$, with an analogous definition for the left-hand face. If now $U(T, x) \in \partial R$, then there is a subset $J \subset \{1, 2,..., n\}$ such that $U(T, x)$ is on one of the jth faces if and only if $j \in J$. If $U(T, \bar{x})$ is in the jth right-hand face, then $u_j(T, x) \leqslant r_j$ for all x with equality at $x = \bar{x}$ so $a_j \partial_{xx} u_j(T, \bar{x}) \leqslant 0$. Thus

$$\partial_t u_j = a_j \partial_{xx} u_j + f_j(U) < -\eta$$

so that $u_j(T + h, \bar{x}) < r_j - \eta h$ for small h. By continuity, this holds for all x in a neighborhood of $\bar{x}$. A similar result holds for left-hand edges.

Let $X = \{x: U(T, x) \in \partial R\}$; then X is a compact set in $\mathbb{R}$ and by the above computation, there is a neighborhood $\mathcal{O} \supset X$ such that for $\theta \in \mathcal{O}$, we have

$$U(T + h, \theta) \in (1 - h\eta(\min_{1 \leqslant j \leqslant n} (r_j, l_j))^{-1})R$$

for small h.

For $x \in \mathbb{R} \backslash \mathcal{O}$ we have $U(t, x) \subset \operatorname{int}(R)$, so since $U \in C((T - \delta, T + \delta) | C_0(\mathbb{R}))$ there is an $h_0 > 0$ and a compact set $K \subset \operatorname{int}(R)$ for which $U(T + h, x) \subset K$ for all $|h| < h_0$. Thus for sufficiently small h, $U(T + h, x) \subset (1 - 2h\eta/L)R$ for all $x \in \mathbb{R}$. Hence $\mathscr{V}_R(U(T + h)) \leqslant 1 - (2h\eta/L)$, so that $(\mathscr{V}_R(U(T + h)) - \mathscr{V}_R(U(T)))/h \leqslant -2\eta/L$, and the proof is complete. □

Remark. If $A = cI$ and R is a arbitrary convex set contracting for $F(U)$ and $0 \in \operatorname{int} R$ then we would have $\bar{D}\mathscr{V}_R(U(T)) \leqslant -(2\eta/sl)\, \mathscr{V}_R(U(T))$ where

$$l = \sup\{x \cdot r \mid |x| = 1, r \in R\},$$

The proof of this result is similar to the proof of Lemma 3.8 and is omitted.

3.3. *Global Existence for the FitzHugh–Nagumo Equations*

The basic lemma of the last section allows us to prove global (in time) existence theorems for the FitzHugh–Nagumo equations for $U = (v, u)$ (where $\epsilon \geqslant 0$),

$$\begin{aligned} v_t &= v_{xx} + f(v) - u, \\ u_t &= \epsilon u_{xx} + \sigma v - \gamma u. \end{aligned} \tag{3.5}$$

THEOREM 3.9. *If $B \subset C_0$ and growth condition* (3.1) *holds, then for any $U^0 \in B$ there is a unique solution $U \in C([0, \infty) | B)$ of* (3.5) *with $U(0) = U^0$.*

Proof. Suppose $U^0 \in B$. By Lemma 3.7 we may choose a rectangle R such that R is contracting for $F(U)$ and $\mathscr{V}_R(U^0) < 1$. Let $U \in C([0, t_0] | B)$ be the solution of (3.5) with $U(0) = U^0$ where t_0 is as in Theorem 2.1. Then we must have $\mathscr{V}_R(U(t)) < 1$ for $0 < t < t_0$. For if this were not true, let

$$\mathrm{t} = \inf\{t \in (0, t_0) | \mathscr{V}_R(U(t)) = 1\}.$$

Then, $\bar{t} > 0$ by the continuity of $\mathscr{V}_R(U(t))$ and by Lemma 3.8 $\bar{D}\mathscr{V}_R(U(\bar{t})) < 0$. Thus we must have $\mathscr{V}_R(U(t)) > 1$ for $t \in (\bar{t} - \epsilon, \bar{t}]$, contradicting the definition of $\bar{t}$. The estimate $\mathscr{V}_R(U(t)) < 1$ for $t \in [0, t_0)$ is the sup norm estimate we need to extend U to a global solution with $\mathscr{V}_R(U(t)) < 1$ for all $t > 0$. The proof is complete. □

The sup norm estimate of the theorem can be made more quantitative as follows. Using Lemma 3.7 we can show that there is a constant c such that for any r there is a contracting rectangle R whose largest side has length less than $c(1 + r)$ and such that $\{(v, u) | \max(|v|, |u|) \leqslant r\} \subset R$. Then for solutions of the FN equations, we have that

$$\| U(t)\|_\infty \leqslant c(1 + \| U(0)\|_\infty) \tag{3.6}$$

for all $t \geqslant 0$.

4. The Threshold Problem

4.1. *Stability via Linearization*

In this section the decay of solutions with small initial data is proved by studying the linearized equations. This line of attack going back to Poincare yields results which are very general in the sense that the convergence is in the topology of any space $H_s(\mathbb{R})$ with $s > \frac{1}{2}$. In addition the results of this section apply when $\gamma = 0$.

The linearized (at zero) FitzHugh–Nagumo system for $U = (v, u)$ is

$$\partial_t U = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} U_{xx} + \begin{pmatrix} -\beta & -1 \\ \sigma & -\gamma \end{pmatrix} U \tag{4.1}$$

where $\beta = -f'(0)$. We assume that $\beta > 0$, $\sigma \geqslant 0$, $\epsilon \geqslant 0$, $\gamma \geqslant 0$, and $\gamma + \sigma > 0$. These equations can be solved explicitly by using the Fourier transform. Let

$$\hat{U}(t, \xi) = \frac{1}{(2\pi)^{1/2}} \int e^{-ix\xi} U(t, x)\, dx$$

be the partial Fourier transform and let

$$\Phi(\xi) = \begin{pmatrix} -\beta - \xi^2 & -1 \\ \sigma & -\gamma - \epsilon\xi^2 \end{pmatrix}.$$

For $U^0 \in \mathscr{S}$, the Schwartz class, the solution of (4.1) with $U(0) = U^0$ is given by the formula

$$\hat{U}(t, \xi) = e^{t\Phi(\xi)} \hat{U}^0(\xi).$$

Recall that $H_s = \{u \in \mathscr{S}'(\mathbb{R}) | (1 + |\xi|^2)^{s/2} \mathscr{F}u \in L_2\}$, where $\mathscr{F}u$ is the Fourier transform of u. For any $s \in \mathbb{R}$, H_s is a translation invariant Hilbert space of distributions and for $s > \frac{1}{2}$, H_s is an admissible space B.

THEOREM 4.1. *There are positive constants K and c such that for any $s \in \mathbb{R}$ and any solution U of the linearized FitzHugh–Nagumo equations* $\| U(t)\|_{H_s} \leqslant Ke^{-ct} \| U(0)\|_{H_s}$.

Proof. To prove the theorem it suffices to show that $\sup_{\xi \in \mathbb{R}} \| e^{t\Phi(\xi)} \| \leqslant Ke^{-ct}$. First we treat the case $\epsilon > 0$. Let $b = -\xi^2 - \beta$, $d = -\epsilon\xi^2 - \gamma$; then the eigenvalues of Φ are

$$\lambda_{\pm} = (b + d) \pm ((b - d)^2 - 4\sigma)^{1/2},$$

the solutions of $\lambda^2 - (b + d)\lambda + (bd + \sigma) = 0$. Since $b + d < 0$ and $bd + \sigma > 0$ the roots $\lambda_{\pm}$ have negative real part for all $\xi \in \mathbb{R}$. In addition, for $|\xi|$ large, we have

$$\lambda_- = -\xi^2 + O(|\xi|),$$
$$\lambda_+ = -\epsilon\xi^2 + O(|\xi|).$$

The result[2] follows from the following inequality for $n \times n$ matrices, M,

$$\| e^M \| \leqslant c_n(1 + \| M \|)^{n-1} \exp[\sup\{\operatorname{Re} \lambda : \lambda \in \text{spectrum } (M)\}]. \tag{4.2}$$

For $M = t\Phi(\xi)$, there is a $\delta > 0$ such that

$$\sup\{\operatorname{Re} t\lambda_{\pm}\} \leqslant -\delta\xi^2 t \qquad \forall t \geqslant 0, \quad \xi \in \mathbb{R},$$

so $\| e^{t\Phi(\xi)} \| \leqslant c(1 + t\xi^2)\, e^{-\delta\xi^2 t}$; this implies the desired inequality.

When $\epsilon = 0$ the analysis is a little trickier. In that case

$$((b - d)^2 - 4\sigma)^{\frac{1}{2}} = (\xi^4 + 2\xi^2(\gamma - \beta) + O(1))^{\frac{1}{2}}$$
$$= \xi^2 + (\gamma - \beta) + O(\xi^{-2}),$$

so

$$\lambda_+ = -\beta + O(\xi^{-2}),$$
$$\lambda_- = -\xi^2 - \gamma + O(\xi^{-2}),$$

[2] A long proof of (4.2) can be found in [9]. We sketch a short proof. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of M, let $\Omega = \{z \in \mathbb{C} \mid |z - \lambda_j| < 1 \text{ for some } j\}$ and let $\Gamma = \partial\Omega$. The curve Γ consists of a finite number of circular arcs. In addition, length of $\Gamma \leqslant 2\pi n$ and $z \in \Gamma$ implies $|z| \leqslant 1 + \| M \|$, $|z - \lambda_j| \geqslant 1$ for all j, and $\operatorname{Re} z \leqslant 1 + \sup_j \operatorname{Re} \lambda_j$. It follows that for $z \in \Gamma$, $|\det(zI - M)| = \prod |z - \lambda_j| \geqslant 1$, and by Cramer's rule that $\|(zI - M)^{-1}\| \leqslant C_n(1 + \| M \|)^{n-1}$. The desired inequality follows from the formula $e^M = (2\pi i)^{-1} \oint_\Gamma (zI - M)^{-1} e^z\, dz$.

so the crude inequality (4.2) does not suffice, and we must make a more detailed analysis of Φ. The eigenvectors $Y_\pm$ corresponding to eigenvalues $\lambda_\pm$ are given by $Y_\pm = (1, -\beta - \xi^2 - \lambda_\pm)$ so

$$Y_+ = (0, -\xi^2) + O(1),$$
$$Y_- = (1, -\beta + \gamma) + O(\xi^{-2}).$$

Thus the angle between the normalized eigenvectors $E_\pm = Y_\pm/\| Y_\pm \|$ is bounded away from zero uniformly in ξ. Therefore, the matrices $D(\xi)$ with columns $E_+(\xi)$ and $E_-(\xi)$ satisfies

$$D^{-1}\Phi D = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix},$$

and $\| D \|$, $\| D^{-1} \|$ are uniformly bounded in ξ. Since

$$e^{t\Phi} = D \begin{pmatrix} e^{t\lambda_+} & 0 \\ 0 & e^{t\lambda_-} \end{pmatrix} D^{-1},$$

we have $\| e^{t\Phi} \| \leqslant Ke^{-ct}$ and the proof is complete. $\square$

This result for the linearization allows us to prove asymptotic stability of the zero solution of the FitzHugh–Nagumo equations.

THEOREM 4.2. *For any $s > \frac{1}{2}$ there is a ball $\mathscr{B} = \{W \in H_s \mid \| W \|_{H_s} \leqslant \epsilon\}$ and positive constants K and c, such that if $U^0 \in \mathscr{B}$ then the solution U of the FitzHugh–Nagumo system with $U(0) = U^0$ exists for all $t \geqslant 0$ and*

$$\| U(t) \|_{H_s} \leqslant Ke^{-ct} \| U^0 \|_{H_s}, \qquad \forall t \geqslant 0.$$

Remark. When $\gamma = 0$ the global existence part of this assertion is not contained in our Theorem 3.9.

Proof. Let $S_L(t)$ be the linear map from H_s into itself sending $U(0)$ to $U(t)$, where U is the solution of the linearized equations (4.1), and let $S(t)$ be the analogous operator for the nonlinear system (2.1). From Theorem 4.1, we can find $T > 0$ such that $\| S_L(T) \| < \frac{1}{2}$.

We first show that there is an $\epsilon_1 < 0$ such that if $\| U^0 \|_{H_s} < \epsilon_1$, then there is a solution of (2.1) defined for $0 \leqslant t \leqslant T$ with $U(0) = U^0$. To see this, observe that from the proof of Theorem 2.1 there is a $t_0 > 0$ such that if $\| U^0 \|_B \leqslant 1$, then the system (2.1) has a solution defined on $0 \leqslant t \leqslant t_0$ satisfying $\| U(t) \|_{H_s} \leqslant 2 \| U^0 \|_{H_s}$ on this interval. Let N be an integer such that $Nt_0 \geqslant T$, and let ϵ_1 be chosen so that $2^N \epsilon_1 < 1$. If $\| U^0 \|_{H_s} < \epsilon_1$, then as in the proof of Theorem 2.2, the solution $U(t)$ can be continued from $[0, t_0]$ to $[t_0, 2t_0]$. Continuing in this manner for K steps, we can define $U(t)$ on $0 \leqslant t \leqslant Kt_0$ and $\| U(t) \|_B \leqslant$

$2^K \| U^0 \|_B \leqslant 2^K \epsilon_1$. Thus, if $K \leqslant N$, $2^K \epsilon_1 \leqslant 1$ and this process can be repeated. In this way $U(t)$ is constructed on $0 \leqslant t \leqslant Nt_0$.

Next, we note that by Theorem 2.4, $S(T)$ is differentiable at 0 and $dS(T)_0 = S_L(T)$. Since $S_L(T)$ is a linear map of norm less than $\frac{1}{2}$, it follows that there is an ϵ in $(0, \epsilon_1]$ such that

$$\| S(T)\, W \|_{H_s} < \tfrac{1}{2} \| W \|_{H_s} \qquad \text{if} \quad \| W \|_{H_s} < \epsilon.$$

Thus if $\mathscr{B} = \{w \in H_s : \| W \|_{H_s} < \epsilon\}$, $S(T)$ maps $\mathscr{B}$ into itself. It follows that for $U^0 \in \mathscr{B}$, the solution $S(t)\, U^0$ can be extended as a solution of the nonlinear system for $T \leqslant t \leqslant 2T$, since the "initial data" $S(T)\, U^0 \in \mathscr{B}$. Continuing in this way we get a solution $U \in C([0, \infty) | \, H_s)$, with $\| U(nT) \|_{H_s} \leqslant (\frac{1}{2})^n \| U^0 \|_{H_s}$. But then from (2.9)

$$\| U(t) \|_{H_s} \leqslant K(\tfrac{1}{2})^{[t/T]} \| U^0 \|_{H_s},$$

where $[t/T]$ is the greatest integer in t/T. This completes the proof. □

4.2. *Stability by Contracting Rectangles*

In this section stability of the zero solution is investigated by constructing a Lyapunov function with the aid of Lemma 3.8. The most important aspect of the result obtained is that an explicit and simple estimate is given on how small U^0 must be in order to imply $U(t) \to 0$; namely, $U^0(x) \in R_c$ for all $x \in \mathbb{R}$. Precisely, we prove

THEOREM 4.3. *For the FitzHugh–Nagumo equations, suppose $-f'(0) > \sigma/\gamma$, and let R_c be the critical rectangle described in Lemma 3.5. If $U^0 \in C_0(\mathbb{R})$ and $U^0(x) \in \operatorname{int}(R_c)$ for all $x \in \mathbb{R}$, then there are nonnegative constants c, K such that $\| U(t) \|_\infty \leqslant K e^{-ct}$, for all $t \geqslant 0$.*

Proof. Since $U^0 \in C_0$, there is a compact set $Q \subset \operatorname{int}(R_c)$ such that $U^0(x) \in Q$ for all $x \in \mathbb{R}$. Choose a rectangle R with $Q \subset R \subset R_c$, as in Lemma 3.5, and let $\mathscr{V}_R$ be the associated function. If L is the length of the shortest side of R and k is as in the above quoted lemma, we may apply Lemma 3.8 with $\eta = k\mathscr{V}_R(v)$ to conclude that

$$\bar{D}\mathscr{V}_R(U) \leqslant (-2k/L)\, \mathscr{V}_R(U), \qquad \mathscr{V}_R(U^0) \leqslant 1.$$

Thus $\mathscr{V}_R(U(t)) \leqslant e^{-2kt/L}$ and the proof is complete. □

The authors feel that the restriction range $U^0 \subset \operatorname{int} R_c$ is much stronger than is necessary to ensure that $U(t)$ decays to zero. In particular we feel that the smallest positive zero of $f(v)$ is a critical parameter which does not enter the description of R_c. The importance of this zero is illuminated by the results of the next sections.

4.3. *An a Posteriori Decay Theorem; the Energy Method.*

The standard asymptotic stability theorems assert that if U^0 is suitably restricted then $U(t)$ tends to zero as $t \to +\infty$. In this section we present a decay result with a different flavor; namely, if $U(t)$ satisfies a suitable condition for all t sufficiently large, then $U(t) \to 0$. Our result asserts that if v is always less than the smallest positive root, α, of $f(v) = 0$, then $U(t)$ decays exponentially in $L^2 \cap L^\infty$. This result is applied in the next section.

THEOREM 4.4. *Suppose the function $f(v)$ in the FitzHugh–Nagumo equations* (1.1) *satisfies the growth condition* (3.1), $f'(0) < 0$, *and in addition $f(v) > 0$ for $v < 0$. Let $\alpha = \inf\{v > 0 \mid f(v) = 0\}$ be the smallest positive zero of f and suppose that $(v, u) \equiv U \in C([0, \infty) \mid L_2 \cap BC^0)$ is a solution with* $\sup_{t>0, x\in\mathbb{R}} v(t, x) < \alpha$. *Then there are positive constants K and c depending only on $\| U(0)\|_{L_2 \cap BC^0}$ such that $\| U(t)\|_\infty + \| U(t)\|_{L_2} \leqslant Ke^{-ct}$.*

Proof. It suffices to prove the theorem assuming $U(0) \in H_2(\mathbb{R})$ for if $U(0) \in L_2 \cap BC^0$, we may choose $U_n(0) \in H_2(\mathbb{R})$ with $U_n(0) \to U(0)$ in $L_2 \cap BC^0$. Then

$$\| U_n(t)\|_\infty + \| U_n(t)\|_{L_2} \leqslant Ke^{-ct}$$

with K and c independent of n. Since $U_n \to U$ in $C([0, \infty) \mid L_2 \cap B)$ the desired inequality for U follows.

If $U(0) \in H_2$ then $U \in C([0, \infty) \mid H_2(\mathbb{R}))$. Multiply the equation $v_t = v_{xx} + f(v) - u$ by σv, the equation $u_t = \epsilon u_{xx} + \sigma v - \gamma u$ by u, and add to get

$$\tfrac{1}{2}(d/dt)(\sigma v^2 + u^2) = \sigma v v_{xx} + \epsilon u u_{xx} + \sigma v f(v) - \gamma u^2.$$

Integrating this expression over $\mathbb{R}$ we obtain, after an integration by parts,

$$\tfrac{1}{2}(d/dt) \int_{-\infty}^{\infty} (\sigma v^2 + u^2)\, dx = -\int_{-\infty}^{\infty} \sigma v_x^2 + \epsilon u_x^2 + \sigma v f(v) + \gamma u^2\, dx. \quad (4.3)$$

Since $v < \alpha$ we have $-vf(v) \geqslant \delta v^2$ for some $\delta > 0$ so

$$\tfrac{1}{2}(d/dt) \int_{-\infty}^{\infty} \sigma v^2 + u^2\, dx \leqslant -\int_{-\infty}^{\infty} \sigma\, \delta v^2 + \gamma u^2\, dx,$$

and it follows that $\| U(t)\|_2 \leqslant Ke^{-Ct}$.

Assume now that $\epsilon > 0$ and observe that

$$U(t) = \int_{t-1}^{t} G(t-s) * F(U(s))\, ds + G(1) * U(t-1). \quad (4.4)$$

In Section 3.3 we showed $\| U(t)\|_\infty$ is bounded independent of $t > 0$, so,

$\|F(U(s))\|_2 \leqslant c_2 \| U(s)\|_2$ with c_2 independent of s. Applying the Schwarz inequality to (4.4) yields

$$\| U(t)\|_\infty \leqslant c_3 \| G(t, x)\|_{L^2([0,1]\times\mathbb{R})} \| U \|_{L^2([t-1,t]\times\mathbb{R})} + \| G(1)\|_{L^2(0)} \| U(t-1)\|_{L^2(\mathbb{R})} .$$

Thus the decay of sup norm follows from that of the L^2 norm. This completes the proof in case $\epsilon > 0$.

Next suppose that $\epsilon = 0$. As above, the decay of $\| v(t)\|_\infty$ follows from the decay of $\| U(t)\|_{L_2}$ and the representation

$$v(t) = g(1) * v(t-1) + \int_{t-1}^{t} g(t-s) * F_1(U(s))\, ds,$$

where $F = (F_1, F_2)$ and g is the fundamental solution of the heat equation. To estimate u we solve the equation $u_t = -\gamma u + \sigma v$ obtaining

$$u(t) = e^{-\gamma t}u(0) + \sigma \int_0^t e^{-\gamma(t-s)}\, v(s)\, ds. \tag{4.5}$$

In Section 3.3 we showed that there is an $M > 0$ such that $\| U(t)\|_\infty \leqslant M$ for all $t \geqslant 0$ and we have shown that $\| v(t)\|_\infty \leqslant Ke^{-ct}$. We may assume that $\gamma > c > 0$; then

$$\| u(t)\|_\infty \leqslant Me^{-\gamma t} + (\sigma K/(\gamma - c))[e^{-ct} - e^{-\gamma t}].$$

This proves the exponential decay of $\| u(t)\|_\infty$ and the proof is complete. □

Unfortunately there are data U^0 with $v^0(x) < \alpha$ for all x and for which it is not true that $v(t, x) \leqslant \alpha$ for all t, x. To see this we suppose that $f'(\alpha) > 0$ and sketch, in Fig. 5, the integral curves of the vector field $F(v, u) = (f(v) - u, \sigma v - \gamma u)$. Let W_0 be a point as in the diagram so that the integral curve $W(t) = (w_1(t), w_2(t))$ of F with $W(0) = W_0$ passes from $v < \alpha$ to $v > \alpha$, say $w_2(1) > \alpha$. Choose $\psi \in C_0^\infty(\mathbb{R})$ with $0 \leqslant \psi \leqslant 1$, and $\psi(0) = 1$. Then it is not hard to show that if $U_\epsilon(t, x)$ is the solution to the FN equations with $U_\epsilon(0, x) = \psi(\epsilon x) W_0$, then for $0 < \epsilon \ll 1$ the derivatives of $U_\epsilon(t, x)$ with respect to x will be very small for $0 \leqslant t \leqslant T$. Thus $U_\epsilon(t, x) \approx \psi(\epsilon x) W(t)$ for $0 \leqslant t \leqslant 1$. In total, if ϵ is sufficiently small then $U_\epsilon(1, 0)$ is to the right of the line $v = \alpha$ even though $U_\epsilon(0, x)$ was to the left of this line for all x.

4.4. *A Subthreshold Result for Small* σ

The threshold result that we are interested in is the following. If the initial data $U_0(x) = (v_0(x), u_0(x))$ satisfies $v_0(x) < \alpha$ and $u_0 \geqslant 0$, then the solution of the FN system decays to 0. In this section we prove a result of this type provided

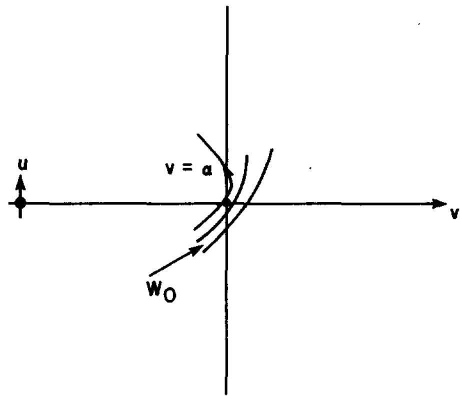

FIGURE 5

σ is sufficiently small. The basic idea is to perturb away from the FN system with $\sigma = 0$.

Thus, consider the FN system (1.1) with $\sigma = 0$:

$$u_t = \epsilon u_{xx} - \gamma u, \qquad v_t = v_{xx} + f(v) - u, \qquad \epsilon \geqslant 0.$$

Suppose $f'(0) < 0$ and let α be the smallest positive root of f. We suppose that $f(v) > 0$ for $v < 0$ that f satisfies the growth condition (3.1). It follows that

$$\lim_{v<0} \sup(f(v)/v) < 0. \tag{4.6}$$

For any $0 < \theta < 1$, let Q_θ be the quarter space defined by (see Fig. 6)

$$Q_\theta = \{(v, u) | \, v \leqslant \alpha\theta, u \geqslant f(\alpha\theta)\}.$$

The basic observation is that for $\sigma = 0$ our contracting rectangle construction can be appreciably strengthened.

LEMMA 4.5. *Suppose that $f'(0) < 0$ and f satisfies (4.6), and let $F(v, u) = (f(v) - u, -\gamma u)$. Then for any $\theta \in (0, 1)$ and any compact subset $K \subset Q_\theta$ there is a rectangle R such that $K \subset R$, and τR is contracting for $0 < \tau \leqslant 1$. In fact there is a constant c with $F(U) \cdot n(U) < -c\tau$ for all $U \in \partial(\tau R)$, where $n(U)$ is an outward normal to $\partial(\tau R)$ at U, and $0 < \tau \leqslant 1$.*

Proof. Let $\bar{u} = \max[\sup\{u \,|(v, u) \in K\}, 1]$, $R = Q_\theta \cap \{u \leqslant \bar{u}\} \cap \{v \geqslant -k\}$, where $k > 0$ is chosen so large that $k^{-1}\bar{u} < \underline{\lim}_{v<0} f(v)/|\, v \,|$ and $K \subset R$. (Geometrically $\bar{u}$ fixes the "height" of R and then k forces the rectangle far enough to the left so that the left-hand edge of τR lies below the graph of f for all $0 < \tau < \infty$.) This R does the trick. □

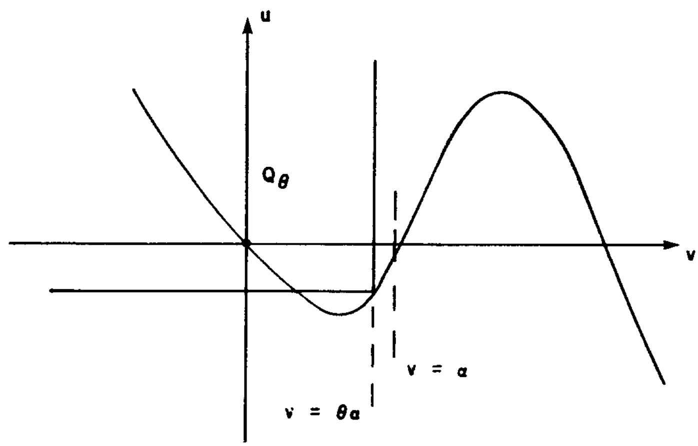

FIGURE 6

As in the proof of Theorem 4.3, the method of contracting rectangles can be applied to show that if $U^0 \in C_0$ with $U^0(x)$ in Q_θ for all $x \in \mathbb{R}$ then, for the solution of (4.6) with initial data $U(0) = U^0$, $\| U(t)\|_\infty$ decays exponentially. By a perturbation argument we show that this property is inherited by the full FN system, (1.1), provided σ is small.

THEOREM 4.7. *Let $B = C_0$ or $B = C_0 \cap L_2$, and suppose that $U^0 \in B$ with values in Q_θ for some $0 < \theta < 1$. If f satisfies* (4.6) *and $f'(0) < 0$, then there are positive constants c, K, σ_0 such that if $0 \leqslant \sigma \leqslant \sigma_0$ and U_σ is the solution of the FitzHugh–Nagumo equations* (1.1) *with $U(0) = U^0$ then $\| U(t)\|_B \leqslant Ke^{-ct}$.*

Proof. First we treat $B = C_0$. Let $K = \{U^0(x) \mid x \in \mathbb{R}\}$ so K is a compact subset of Q_θ. For any $\sigma \geqslant 0$ let U_σ be the solution of the FN equations with $U_\sigma(0) = U^0$. By Theorem 3.9 and the remark following the proof of Lemma 3.6, $U_\sigma \in C([0, \infty) \mid C_0)$, and there is a constant c such that for $0 \leqslant \sigma \leqslant 1$, $t \geqslant 0$ we have

$$\| U_\sigma(t)\|_\infty \leqslant c. \tag{4.7}$$

Now let $\Delta(t) = U_\sigma(t) - U_0(t)$; then

$$\Delta(t) = \int_0^t G(t-s) * [F_\sigma(U_\sigma(s)) - F_0(U_0(s))]\, ds$$

where $F_\sigma(v, u) = (f(v) - u, \sigma v - \gamma u)$.

Using (4.7) it follows that there is a constant K such that $\| F_\sigma(U_\sigma(s)) - F_0(U_0(s))\|_\infty \leqslant \sigma c + K \| \Delta(s)\|_\infty$. Thus

$$\| \Delta(t)\|_\infty \leqslant \sigma ct + K \int_0^t \| \Delta(s)\|_\infty\, ds.$$

By Gronwall's method (see [7, 10.5.1.3]) we have

$$\| \varDelta(t)\|_\infty \leqslant \sigma ct(1 + Ke^{tK}). \tag{4.8}$$

Choose $\sigma_1 \leqslant 1$ so that $-f'(0) > \sigma_1/\gamma$ and let R_c be the critical rectangle of Lemma 3.5. Choose $T > 0$ so large that $U_0(T, x) \in \frac{1}{2} R_c$ for all $x \in \mathbb{R}$. By (4.8) there is a $\sigma_0 \leqslant \sigma_1$ so that for $0 \leqslant \sigma \leqslant \sigma_0$ we have $\varDelta(T, x) \in \frac{1}{4} R_c$ for all $x \in \mathbb{R}$. Then since $U_\sigma(T, x) = \varDelta(T, x) + U_0(T, x)$ we have $U_\sigma(T, x) \in \frac{3}{4} R_c$ for all $x \in \mathbb{R}$. Theorem 4.3 then applies to show that $\| U_\sigma(t)\|_\infty$ decays exponentially; this proves our result for $B = C_0$.

For $B = C_0 \cap L_2$ observe that $\| U(t)\|_\infty$ decays exponentially so there is a $t_0 > 0$ such that $v(t) \leqslant \frac{1}{2} \alpha$ for $t \geqslant t_0$. Then Theorem 4.4 applies to show that $\| U(t)\|_{L_2}$ decays exponentially, and the proof is complete. □

4.5. *Implications for the Mixed Problem*

Biological nerves are not infinitely long, and signals are not begun along the entire length of the nerve, but arise from stimulation at one end. A model which takes one-end stimulation into account is the following:

$$v_t = v_{xx} + f(v) + u, \qquad x \geqslant 0, \quad t \geqslant 0, \tag{4.9}$$

$$u_t = \sigma v - \gamma u, \qquad x \geqslant 0, \quad t \geqslant 0, \tag{4.10}$$

$$v(0, x) = u(0, x) = 0, \qquad x \geqslant 0, \tag{4.11}$$

$$v(t, 0) = h(t), \qquad t \geqslant 0, \tag{4.12}$$

$$(v(t, x), u(t, x)) \to (0, 0) \quad \text{as} \quad x \to \infty, \quad t > 0. \tag{4.13}$$

In this model, the nerve is initially at rest, and is stimulated at the endpoint $x = 0$. A reasonable assumption is that the stimulus $h(t)$ lasts only a finite time, so that we require $h(t) = 0$ for $t > T$.

The threshold problem here is to show that if h is sufficiently small then the solution to (4.9)–(4.13) tends to zero as $t \to +\infty$. Both numerical and biological experiments seem to indicate that a strong stimulus which lasts a short time and a weak stimulus which lasts a long time are subthreshold. Our main result is a mathematical result of this sort.

For the sake of brevity we will not discuss in detail the existence and regularity problem for (4.9)–(4.13), merely mentioning that if $h \in C(\overline{\mathbb{R}}_+)$, $h(0) = 0$ then this problem has a unique solution $U \in C([0, \infty)| L_2 \cap C_0)^3$ provided growth condition (3.1) is satisfied. As usual, the main problem is to get an estimate for $\| U(t)\|_\infty$, and this is provided by the method of contracting rectangles which

[3] In this section, spaces like L_2 and C_0, etc., mean $L_2(\mathbb{R}_+)$ and $C_0(\mathbb{R}_+) = \{u \in C(\mathbb{R}_+) \mid \lim_{x\to\infty} u(x) = 0\}$, etc.

works for mixed problems (see [13] for the case where $A = \text{const } I$). It is important to notice that since (4.12) gives the value of v at $x = 0$, Eq. (4.10) can be solved to yield

$$u(t, 0) = \sigma \int_0^t e^{-\gamma(t-s)} h(s) \, ds,$$

so $\| u(t, 0)\|_\infty \leqslant \gamma^{-1}\sigma \sup_{0 \leqslant s \leqslant t} h(s)$, and we have a sup norm estimate for U on $x = 0$. As in Eq. (3.6), the method of contracting rectangles provides a constant M such that

$$\| U(t)\|_\infty \leqslant M(1 + \sup_{0 \leqslant s \leqslant t} | h(s)|), \tag{4.14}$$

and global solvability follows (see [14], [16] for more details).

THEOREM 4.7. *Suppose that f satisfies growth condition* (3.1), *and* $-f'(0) > \sigma/\gamma$. *For $h \in C(\overline{\mathbb{R}}_+)$, with $h = 0$ for $t = 0$ and for $t \geqslant T$, let $U = (v, u)$ be the solution of* (4.9)–(4.13). *Then*

(i) *For any $T > 0$ there is a constant c such that if $\| h \|_\infty < c_T$ then $\| U(t)\|_{L_2 \cap C_0} \to 0$ exponentially as $t \to \infty$.*

(ii) *For any $r \in \mathbb{R}_+$ there is a T_r such that if $\| h \|_\infty \leqslant r$ and $T \leqslant T_r$, then $\| U(t)\|_{L_2 \cap C_0} \to 0$ exponentially as $t \to \infty$.*

The precise dependence of c on T and T_r on r are of interest but the results we get are quite crude (see [3] for numerical evidence and [15] for more precise mathematical results.)

Proof. In both cases (i) and (ii) the proof is given in two steps. First one shows that $U(T)$ is small in $L_2 \cap C_0$ and then proves decay of U for $t \geqslant T$ using the fact that the boundary condition is homogeneous in that region. Define the odd extension of U by

$$\begin{aligned} U_0(t, x) &= U(t, x) \qquad \text{if} \quad x \geqslant 0, \\ &= U(t, -x) \qquad \text{if} \quad x \leqslant 0. \end{aligned}$$

For $t \geqslant T$, U_0 satisfies the FN equations (4.9), (4.10) on $-\infty < x < \infty$.

To estimate U for $0 \leqslant t \leqslant T$ we use the basic integral equations for U, namely,

$$\begin{aligned} v(t) &= \int_0^t g(t - s) * F_1(U_0(s)) \, ds + w(t, x), \\ u(t) &= \gamma \int_0^t e^{-\gamma(t-s)} v(s) \, ds, \end{aligned} \tag{4.15}$$

where $F_1(v, u) = f(v) - u$, and w is the solution of the mixed problem

$$w_t - w_{xx} = 0 \qquad \text{for} \quad x > 0, \tag{4.16}$$

$$w(0, x) = 0 \qquad \text{for} \quad x \geqslant 0, \qquad w(t, x) \to 0 \quad \text{as} \quad x \to \infty, t > 0, \tag{4.17}$$

$$w(t, 0) = h(t). \tag{4.18}$$

To estimate U by Gronwall's method we must estimate w. For this we use the comparison function $\tilde{w}$ which satisfies (4.16), (4.17), and the boundary condition $\tilde{w}(t, 0) = 1$. It is easy to see that if $\tilde{w}$ is a solution of this problem, then so is $\tilde{w}(\rho t, \rho^{1/2}x)$ for any $\rho > 0$. Thus by uniqueness $\tilde{w}(t, x) = \phi(x/t^{1/2})$, where $\phi(s) = \tilde{w}(1, s)$. The equations for $\tilde{w}$ imply that ϕ satisfies the equation

$$-\tfrac{1}{2} s\phi' = \phi'',$$

and the boundary conditions $\phi(0) = 1$, $\phi(s) \to 0$ as $s \to +\infty$. This problem can be solved explicitly as

$$\phi(s) = 1 - (1/\pi^{1/2}) \int_0^s e^{-\xi^2/4}\, d\xi, \qquad s \geqslant 0.$$

Observe that for $t \geqslant 0$, $\| \tilde{w}(t)\|_\infty = 1$ and $\| \tilde{w}(t)\|_{L_2} = t^{\frac{1}{4}} \| \phi \|_{L_2}$.

We now prove assertion (i) of the theorem. Fix $t > 0$; then for $0 \leqslant t \leqslant T$ we have $| w(t)| \leqslant \| h \|_\infty \tilde{w}(t)$, and for $t \geqslant T$ $\| w(t)\|_{L_2} \leqslant \| w(T)\|_{L_2}$. Thus for all $t \geqslant 0$,

$$\| w(t)\|_{L_2} \leqslant \| \phi \|_{L_2} T^{1/4} \| h \|_\infty \overset{\text{def}}{=} \delta.$$

Suppose $\| h \|_\infty \leqslant 1$, then by (4.14), $\| U \|_\infty \leqslant 2M$, so there is a constant K depending only on M such that

$$\| U(t)\|_{L_2} \leqslant \delta + K \int_0^t \| U(s)\|_{L_2}\, ds.$$

Gronwall's inequality implies

$$\| U(t)\|_{L_2} \leqslant \| h \|_\infty \| \phi \|_{L_2} T^{1/4} e^{Kt}, \tag{4.19}$$

For the sup norm estimate, observe that $\| w(t)\|_\infty \leqslant \| h \|_\infty$ for all $t \geqslant 0$, so as above

$$\| U(t)\|_\infty \leqslant \| h \|_\infty + K' \int_0^t \| U(s)\|_\infty\, ds,$$

$$\| U(t)\|_\infty \leqslant \| h \|_\infty e^{K't}. \tag{4.20}$$

From Eqs. (4.19) and (4.20), it follows that for any $\epsilon > 0$ there is a $c > 0$ such that if $\| h \|_\infty \leqslant c$ then $\| U_0(T)\|_{L_2 \cap C_0} \leqslant \epsilon$.

Let R_c be the critical small rectangle of Lemma 3.5 and choose ϵ so small that $\| U(T)\|_\infty < \epsilon$ implies that the range of $U(T)$ lies in a compact subset Q of R_c. Choose R and k as in Lemma 3.5, and decreasing k if necessary we may suppose $0 < k < \gamma$. For $x = 0$ and $t \geqslant T$, $U(t, 0) = (0, U(T)\, e^{-\gamma(t-T)})$ so $| U(t, 0)|_R < e^{-\gamma(T-t)}$. With this information to control the boundary values, a straightforward extension of Lemma 3.8 (see [14] for details) yields $\bar{D}\gamma_R(U(t)) \leqslant -k\gamma_R(U(t))$ for $t \geqslant T$, so $\| U(t)\|_\infty \leqslant Ke^{-k(t-T)}$ for $t \geqslant T$. In particular, for t large, we have $v(t, x) < \alpha$, the first zeros of f. The energy method of Theorem 4.4 yields the exponential decay of $\| U(t)\|_{L_2}$ and the proof of (i) is complete.

To prove (ii) we observe as before that $\| w(t)\|_{L_2} \leqslant r \| \phi \|_{L_2} T^{1/4}$ and then by Gronwall's inequality,

$$\| U(t)\|_{L_2} \leqslant r \| \phi \|_{L_2} T^{1/4} e^{Kt} \tag{4.21}$$

for all $t \geqslant 0$. The derivation of sup norm estimates for U is trickier for this case. We analyze w for $t \geqslant T$ by considering the odd extension $w_0(t, x)$ which satisfies

$$w_0(t) = g(t - T) * w_0(T).$$

Then since $\| g(t)\|_{L_2} = ct^{-1/4}$, we have for $T \leqslant 1$

$$\| w_0(t)\|_\infty \leqslant crT^{1/4}(t - T)^{-1/4}.$$

Thus assuming $T \leqslant 1$ we have

$$\| w(t)\|_\infty \begin{cases} \leqslant crT^{1/8}, & t \geqslant T + T^{1/2}, \\ \leqslant r, & 0 \leqslant t \leqslant T + T^{1/2}. \end{cases} \tag{4.22}$$

By (4.14), we have an a priori bound on $\| U(t)\|_\infty$ which depends only on r, so the integral equation (4.15) yields

$$\| U(t)\|_\infty \leqslant \| w(t)\|_\infty + K \int_0^t \| U(s)\|_\infty \, ds$$

with K depending only on r. The Gronwall method (see Dieudonne [7, assertion 10.5.1.3]) implies

$$\| U(t)\|_\infty \leqslant \| w(t)\|_\infty + e^{Kt} \int_0^t \| w(t)\|_\infty \, dt,$$

so using (4.22) we have

$$\| U(2)\|_\infty \leqslant \tilde{c}r[T^{1/8} + T + T^{1/3}] \tag{4.23}$$

where $\tilde{c}$ depends only on r.

From (4.21) and (4.23), it is clear that for any $\epsilon > 0$ we may choose T sufficiently small to ensure $\| U(2)\|_{L_2 \cap C_0} < \epsilon$ and then decay of $U(t)$ follows as for (i). □

Using Theorem 4.7 instead of Theorems 4.3 and 4.4 in the analysis for $t \geqslant T$ the condition $-f'(0) > \sigma/\gamma$ can be eliminated at the expense of having to assume that σ is small. Furthermore, stability theorems in the spaces H_s can be proved using Theorem 4.2 in the endgame. The advantage of these spaces is that fewer hypotheses need to be placed on f. However, in this case more than just the supremum of h comes into play; the derivatives of h enter also. For the sake of brevity we will not present the details of these results.

5. Asymptotic Behavior of Large Solutions

5.1. *An Attractor for the Flow*

The method of contracting rectangles can be used to show that as $t \to \infty$ the values of $U(t, x)$ converge to R^c uniformly in x. The basic idea is to use Lyapunov's second method with the functions $\mathscr{V}_R$ (see Section 3.2) for appropriate rectangles R.

THEOREM 5.1. *For the FitzHugh–Nagumo equations satisfying growth condition* (3.1), *let R^c be the critical rectangle described in Lemma* 3.7. *If $U^0 \in C_0(\mathbb{R})$, we have* $\overline{\lim}_{t\to+\infty} \mathscr{V}_{R^c}(U) \leqslant 1$; *that is, the values of $U(t, x)$ lie inside* $(1 + \epsilon) R^c$ *for large* t.

Proof. For any real numbers $M, \epsilon > 0$, we shall show that there is a $T > 0$ such that if $\| U^0 \| < M$, then $\mathscr{V}_{R^c}(U(t)) \leqslant 1 + \epsilon$ if $t \geqslant T$. We first prove this assertion for $U^0 \in \mathscr{S}$, the Schwartz class; the general case will then follow easily. Thus, suppose $U^0 \in \mathscr{S}$, then by Theorem 2.1, $U \in C^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap C([0, \infty)| C_0(\mathbb{R}))$.

By Lemma 3.6, we may choose a rectangle $R_1 = R_1(M)$ contracting for $F(U) = (f(v) - u, \ \sigma v - \gamma u)$ so that $R_1 \supset \{w \in \mathbb{R}^2\colon \max[| w_1 |, | w_2 |] \leqslant M\}$. Then if U is the solution of (2.1) with data U^0, $| U(t, x)|_{R_1} \leqslant 1$ for all $t \geqslant 0$, $x \in \mathbb{R}$, by the results of Section 3.3. Let $Q \subset \mathbb{R}^2$ be the compact set defined by

$$Q = \{w \in \mathbb{R}^2 : w \notin (1 + \epsilon) R^c \text{ and } | w |_{R_1} \leqslant 1\}.$$

According to Lemma 3.7, we may choose a rectangle R such that $R^c \subset R$, $Q \subset \mathbb{R}^n \backslash R$ and τR is contracting for the vector field $F(U)$ for $1 \leqslant \tau < \infty$. We choose $\tau_0 = \tau_0(M)$ so that $\tau_0 R \supset R_1$. By continuity, there is an $\eta > 0$ such that $F(w) \cdot n < -\eta$, for any $w \in \partial(\tau R)$ for $1 \leqslant \tau \leqslant \tau_0$, and n an outward unit normal to $\partial(\tau R)$ at w. Let $\mathscr{V}_R$ be the associated functional on $C_0(\mathbb{R})$. If there is a $t_0 > 0$ such that $\mathscr{V}_R(U(t_0)) \leqslant 1$, then since R is contracting, we have $U(t) \in R \subset (1 + \epsilon) R^c$ for $t \geqslant t_0$, so we are done. If $\mathscr{V}_R(U(t)) > 1$ for all $t > 0$, then by Lemma 3.8, $\bar{D}\mathscr{V}_R(U(t)) \leqslant -2\eta/L$, for all $t > 0$. Since $\mathscr{V}_R(U_0) \leqslant \tau_0$, we have $\mathscr{V}_R(U(t)) \leqslant \tau_0 - 2\eta t/L$. But, if $T = L(\tau_0 - 1)(2\eta)^{-1}$, then $t \geqslant T$ implies $\mathscr{V}_R(U(t)) \leqslant 1$, a contradiction.

Suppose now $U^0 \in C_0$. For any $\delta > 0$, (2.9) implies that if $U^0 \in \mathscr{S}$ and $\| U^0 - \tilde{U}^0 \|_\infty < \delta$ then $\| U(T) - \tilde{U}(T) \|_\infty < \delta e^{KT}$, where T is defined above. Thus, we can choose δ so small that $U(T)$ lies in $R \subset (1 + \epsilon) R^c$, and since the former rectangle is contracting, we have $U(t) \in (1 + \epsilon) R^c$ for $t \geqslant T$. $\square$

As an application of this result, observe that if $U(t, x) = \Phi(x + \theta t)$ is a traveling wave solution of the FitzHugh–Nagumo equations with $\Phi \in C_0(\mathbb{R})$, then we must have $\Phi(z) \in R^c$ for all $z \in \mathbb{R}$.

5.2. *Global Stability of Zero for Short Nerves*

A real nerve has only finite extent, and the correct problem is a mixed initial-boundary value problem with zero initial conditions, and an inhomogeneous boundary condition at one end, corresponding to stimulation of the nerve at that end. A model based on the FN equations is

$$v_t = v_{xx} + f(v) - u, \qquad 0 \leqslant x \leqslant L, \tag{5.1}$$

$$u_t = \sigma v - \gamma u, \qquad 0 \leqslant x \leqslant L, \tag{5.2}$$

$$(v(0, x), u(0, x)) = U^0(x), \qquad 0 \leqslant x \leqslant L, \tag{5.3}$$

$$v(t, 0) = v_0(t), \qquad t \geqslant 0, \tag{5.4}$$

where $v_0(t)$ and $U^0(x)$ are prescribed functions. For this system we must also give another boundary condition at $x = L$. We shall suppose that one of the following homogeneous boundary conditions is given at $x = L$:

$$v(t, L) = 0 \qquad t \geqslant 0, \tag{5.5}$$

or

$$v_x(t, L) - av(t, L) = 0, \qquad t \geqslant 0, \quad a \leqslant 0. \tag{5.6}$$

In order to analyze the solutions, we suppose that the stimulus $v_0(t)$ is nonzero only over a finite time interval, $0 \leqslant t \leqslant T$. Then for $t \geqslant T$, (v, u) satisfies a mixed problem with homogeneous boundary conditions. For brevity, we again will not treat in detail the existence and regularity problem for these mixed problems (though this is not difficult), but we merely consider an interesting qualitative phenomenon. Namely, if the length of the nerve is small, then all stimuli decay exponentially in time. In order to make this precise, we must introduce the following parameter s (see Fig. 7)

$$s = \sup_{v \in \mathbb{R}} f(v)/v. \tag{5.7}$$

THEOREM 5.2. *Suppose that* (v, u) *is a classical solution of* (5.1)–(5.4), *and one*

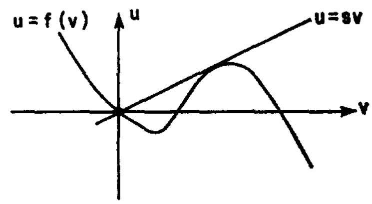

FIGURE 7

of (5.5) *or* (5.6), *and that* $v_0(t) = 0$ *for* $t \geqslant T$. *Let s be defined by* (5.7), *then if* $L^2 < \pi^2/4s$,

$$\| U(t)\|_\infty + \| U(t)\|_{L_2([0,L])} \leqslant Ke^{-\alpha t}$$

for some constants K, $\alpha > 0$.

Proof. Multiply (5.1) by v, (5.2) by $\sigma^{-1}u$, add the results and integrate over $0 \leqslant x \leqslant L$ to get

$$\int_0^L [vv_t + (1/\sigma)\, uu_t]\, dx = \int_0^L [vv_{xx} + vf(v) - (\gamma/\sigma)\, u^2]\, dx.$$

For $t \geqslant T$, we integrate the term vv_{xx} by parts and observe that the boundary contribution is nonpositive. Thus we get

$$\tfrac{1}{2}(\partial/\partial t) \int_0^L [v^2 + (1/\sigma)\, u^2]\, dx \leqslant \int_0^L [-(\gamma/\sigma)\, u^2 + sv^2 - v_x^2]\, dx. \quad (5.8)$$

Since $v(t, 0) = 0$ for $t \geqslant T$, we have

$$\int_0^L v_x^2\, dx \geqslant \lambda \int_0^L v^2\, dx,$$

where $\lambda = (\pi/2L)^2$ is the smallest eigenvalue of the operator $-(\partial/\partial x)^2$, with Dirichlet boundary conditions at $x = 0$, and Neumann boundary conditions at $x = L$ (see [6, VI.1]). Thus, if $s < (\pi/2L)^2$, we get, from (5.8),

$$\tfrac{1}{2}(\partial/\partial t) \int_0^L [v^2 + (1/\sigma)\, u^2] \leqslant \int_0^L [-(\gamma/\sigma)\, u^2 + (s - \pi^2/4L^2)\, v^2]\, dx$$

$$\leqslant -c \int_0^L [v^2 + (1/\sigma)\, u^2]\, dx,$$

where $c > 0$, and it follows that $\int_0^L [v^2 + (1/\sigma)\, u^2]\, dx$ decays exponentially. Sup norm decay follows as in the proof of Theorem 4.4. □

Finally we remark that if a diffusion term ϵu_{xx} is added to the right-hand side of (5.2) and a suitable boundary condition on u at the endpoints is imposed, then a similar result is true.

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