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CONTRACTIONS OF GRAPHS: A THEOREM OF ORE AND AN EXTREMAL PROBLEM*

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The contractibility number (also known as the Hadwiger number) of a connected graph G, $Z(G)$, is defined as the maximum order of a connected graph onto which G is contractible. An elementary proof is given of a theorem of Ore about this invariant. Also, the extremal problem of finding the maximum $Z(G)$ over all graphs G of a given order and regularity degree is solved.

1. Introduction

Let G be a finite graph as in [3]. An *elementary contraction of G* is the identification of two adjacent points of G to a point z. Alternatively, it may be viewed as a function of graphs $f: G \to G'$ where for some pair of adjacent points u, $v \in V(G)$, we have $V(G') = (V(G) - \{u, v\}) \cup \{z\}$, and $E(G') = E(G - \{uv\}) \cup \{zx\colon xu \in E(G) \text{ or } xv \in E(G)\}$ with loops and multiple edges of G' suppressed. The graph G' will be denoted by $G(u = v)$. A *contraction* of G is a sequence of elementary contractions starting from G. If there exists a contraction $f: G \to H$, we say G is contractible to H. Strictly speaking, the order in which the elementary contractions are performed is not relevant. In practice, however, the contraction f will be often most easily described and understood when a specific order is used. An example of a contraction $f: G \to K_4$ is illustrated in Fig. 1.

Let G be a connected graph. We define $Z(G)$, the *contractibility number of G*, to be the maximum order of a complete graph onto which G is contractible. If G is disconnected, let $Z(G) = \max\{Z(G')\colon G' \text{ a connected component of } G\}$.

Some well-known theorems and conjectures in the literature of graph theory can be stated in terms of the contractibility number. For example, Hadwiger's conjecture says that every n-chromatic graph G satisfies $Z(G) \geq n$. Wagner [5] has shown that a graph G is planar if and only if it has no subgraph contractible to $K(3, 3)$ and $Z(G) \leq 4$. Hadwiger's conjecture for $n = 5$ thus implies the four color theorem, while the converse was also proved by Wagner [6].

In a contraction $f: G \to H$, the inverse image of a point in H must be a connected subgraph of G. Hence f defines a natural partition of G, $V(G) = \bigcup_{i=1}^{|H|} V_i$ such that $\langle V_i \rangle$ is a connected subgraph of G for $1 \leq i \leq |H|$. Indeed, the

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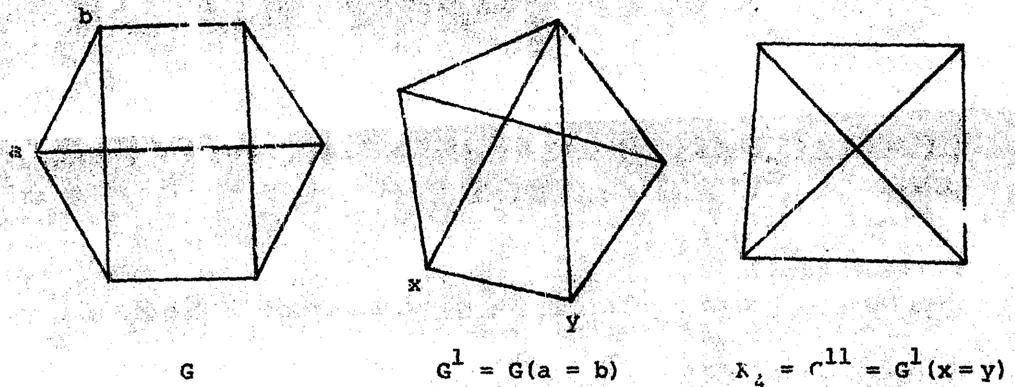


Fig. 1. A composition of two elementary contractions that yields the contraction $G \to K_4$.

sets V_i are identical with the sets $f^{-1}(v)$ $(v \in H)$. If H is a complete graph, then for each pair of distinct integers i j, $1 \leq i \leq j \leq |H|$, there exists an edge xy of G with $x \in V_i$ and $y \in V_j$. In this case, we will refer to the partition above as a complete $|H|$ partition of $V(G)$.

Any graph theoretic notation and terminology not defined here can be found in [3]. In particular, all graphs are finite with no loops or multiple edges.

2. Ore's theorem

In his book [4] Ore proves, using an elaborate method, that if the minimum degree of G satisfies $\delta(G) \geq 3$ with the possible exception of one point of degree 2, then $Z(G) \geq 4$. Under the modified hypothesis $\delta(G) \geq 3$, an elementary proof of this theorem is now given.

Theorem 2.1 (*Ore*). *If* $\delta(G) \geq 3$, *then* $Z(G) \geq 4$.

Proof: We proceed by induction on $p(G)$, the order of G. The smallest possible order for a graph satisfying the hypothesis is 4, in which case $G = K_4$ and the theorem is trivally true. Let G be a graph of order $p \geq 5$ with $\delta(G) \geq 3$, and assume the theorem true for all graphs of order $p-1$ or less.

Observe that it suffices to prove the theorem when $\delta(G) = 3$. For if this known, then in an arbitrary graph H with $\delta(H) \geq 4$, we may choose a point h of degree $\delta(H)$ and remove all but 3 edges incident with it, thereby getting a graph H' with a point of degree 3. We then know that $Z(H') \geq 4$, and since $H \supset H'$, we get $Z(H) \geq Z(H') \geq 4$.

For every $v \in V(G)$, let $N(v)$ refer to the set of all points of G adjacent to v, also called the neighborhood of v.

First we reduce to the case where for every $v \in V(G)$, there exists $s \in N(v)$ with $d(s) = 3$. Suppose to the contrary that there exists $v_0 \in V(G)$ such that $d(t) \geq 4$ for all $t \in N(v_0)$. Then $G - v_0$ satisfies the hypothesis, so $Z(G - v_0) \geq 4$ by induction. Hence $Z(G) \geq 4$.

Next we reduce to the case where $\langle N(v)\rangle \cong P_3$ for all points v of degree $d(v) = 3$. If $N(v) \cong K_3$, then $\langle N(v) \cup \{v\}\rangle \cong K_4$, so that $Z(G) \geq 4$. Hence we may assume $N(v) \not\cong K_3$. To finish the reduction, we need only consider the possibility that $\langle N(v)\rangle$ contains an isolated point y. Then the contraction $\lambda : G \to G(y = v)$ yields a graph $G(y = v)$ of order $p - 1$ and minimal degree at least 3. Hence by induction $Z(G(y = v)) \geq 4$, so that $Z(G) \geq 4$.

We may therefore now assume that $\langle N(v)\rangle \cong P_3$ for all points v of degree 3, and that for each such v there exists $s \in N(v)$ with $d(s) = 3$. It will now be shown that under these conditions the theorem is true.

Let v be a point of degree 3, and let $s \in N(v)$ satisfy $d(s) = 3$. We show that s may be taken to be an endpoint of $\langle N(v)\rangle \cong P_3$. For if it cannot, then both endpoints have degree 4 at least. Consider the contraction $\lambda : G \to G(v = x)$ where $x \in N(v)$ and $d(x, N(v)) = 2$. Either the graph $G(v = x)$ satisfies $\delta \geq 3$ or another elementary contraction (identifying the image of v and x with a neighbor) yields such a graph G'. By induction, we have $Z(G(v = x)) \geq 4$ or $Z(G') \geq 4$ and hence $Z(G) \geq 4$.

We are now able to construct the contraction of G onto a graph with contractibility number at least 4. Let $u \in N(s)$ be such that $u \notin N(v)$ and $u \neq v$. Such a point exists since $|N(s)| = 3$ and $N(v) \not\cong K_3$. Since $N(s) \cong P_3$, we must have $xu \in E(G)$ where $d(x, N(v)) = 2$ as above. But then the contraction $\lambda : G \to G(s = v)$ has image graph satisfying $Z(G(s = v)) \geq 4$ by induction, so that $Z(G) \geq 4$.

It has been pointed out by A. Blass that a slight modification of the above proof actually yields Ore's theorem as originally stated (that is, with the possible point of degree 2). This modification has been omitted, however, for reasons of space.

We now turn our attention to an extremal problem involving the contractibility number.

3. An extremal problem

All graphs discussed in this section are assumed to be connected.

We will investigate the problem of finding, given k and p, the maximum contractibility number among the class of all k regular graphs of order p.

We begin with a lemma which will be used later in the construction of an infinite class of extremal regular graphs. When x is a real number, $\{x\}$ will be the least integer greater than or equal to x.

Lemma 3.1. *Let G be k-regular with $\beta_0(G) \geq \{\frac{1}{2}k\}$. Then there exists a k regular graph G' satisfying $Z(G') \geq Z(G)$, with*

(a) $|G'| = |G| + \frac{1}{2}k$ *if k is even,*

(b) $|G'| = |G| + [\frac{1}{2}k]$ *if $k \equiv 1 \pmod 4$*

$\quad = |G| + \{\frac{1}{2}k\}$ *if $k \equiv 3 \pmod 4$,*

(c) $\beta_0(G') \geq \beta_0(G)$.

Proof. Suppose first k is even, and let G satisfy the hypothesis. Let $v^1 \in V(G)$, and let $N(v^1)$ be as above the neighborhood of v^1. We take a partition of $N(v^1)$, $N(v^1) = N_1(v^1) \cup N_2(v^1)$, such that $|N_1(v^1)| = |N_2(v^1)|$. We then define the graph $H(G; N_1, N_2)$ by $V(H) = V(G - v^1) \cup \{v_2^1, V_1^1\}$, and $E(H) = E(G - v^1) \cup \{v_1^1 x: x \in N_1(v^1)\} \cup \{v_2^1 y: y \in N_2(v^1)\} \cup \{v_1^1 v_2^1\}$. Now define the set of graphs $S^1(G; v^1)$ by

$$S^1(G; v^1) = \{H(G; N_1, N_2): N(v^1) = N_1 \cup N_2, |N_1| = |N_2|, N_1 \cap N_2 = \emptyset\}.$$

The operation of transforming G to a graph in the class $S^1(G; V^1)$ will be called a *splitting of G at v^1*. A graph G and two different splittings of it at a point are shown in Fig. 2.

We now iterate this procedure by defining sets $S^i(G; v^1, v^2, \ldots, v^i)$. Let $\{v^j\}_{j=1}^i$ be a set of i independent points of G. Having performed successive splittings of G at the points $\{v^j\}_{j=1}^{i-1}$ resulting in a graph H^{i-1} of the set $S^{i-1}(G; v^1, v^2, \ldots, v^{i-1})$, we perform a splitting of H^{i-1} at v^i to get a graph H^i. The set of all such graphs H^i will be denoted by $S^i(G; \{v^j\}_{j=1}^i)$. More precisely, we let $S^i(G; \{v^j\}_{j=1}^i) = \bigcup S^1(H^{i-1}, v^i)$ where the union is over all graphs H^{i-1} of the set $S^{i-1}(G; \{v^j\}_{j=1}^{i-1})$. A graph in $S^3(K_5; \{v^j\}_{j=1}^3)$ is shown in Fig. 3.

The required graph G' may now be constructed. We find a set of $\frac{1}{2}k$ independent points $\{v^j\}_{j=1}^{k/2}$, and let $G^0 \in S^{k/2}(G; \{v^j\}_{j=1}^{k/2})$. The construction of G^0 produces a set of k points $\{v_m^j\}_{m=1}^2{}_{j=1}^{k/2}$, the "splittings" of the points $\{v^j\}_{j=1}^{k/2}$ of G. Let the two subsets $X_1(G^0)$ and $X_2(G^0)$ of $V(G^0)$ be defined as $X_1(G^0) = \{v_1^j\}_{j=1}^{k/2}$, $X_2 = \{v_2^j\}_{j=1}^{k/2}$. Each point in the $X_i(G^0)$ has degree $(\frac{1}{2}k) + 1$ by construction. Define the graph G' by $V(G') = V(G^0)$ and $E(G') = E(G^0) \cup \{v_2^i v_1^j: i \neq j\}$. Thus G^0 is a

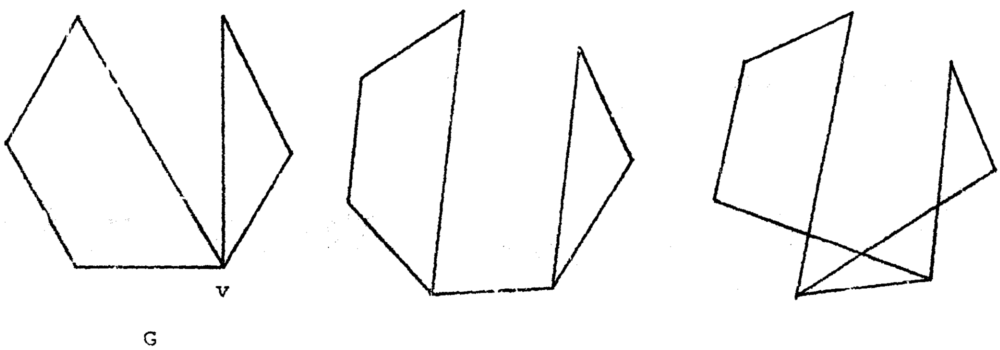


Fig. 2. Two different splittings of G at v.

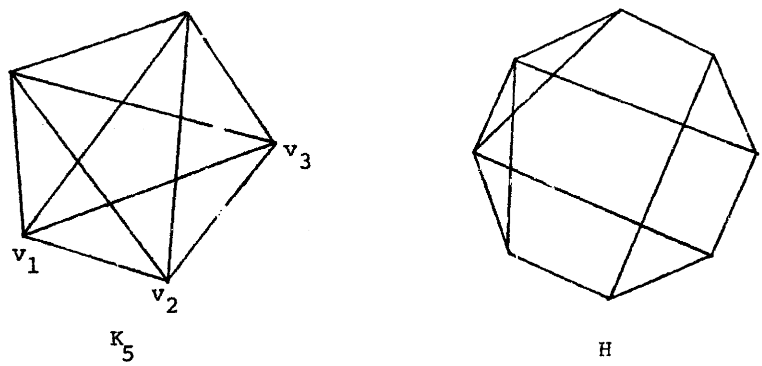


Fig. 3. A graph $H \in S^3(K_5; v_1, v_2, v_3)$.

spanning subgraph of G' containing all edges of G' except those joining points of X_1 with points of X_2.

It is now shown that G' has the required properties. Obviously $|G'| = |G^0| = |G| + \frac{1}{2}k$. To see that G' is k-regular, observe that the points of $G - \{v^i\}_{j=1}^{k/2}$ are naturally present in G' as the points of $G' - (X_1 \cup X_2)$. As such, they have the same degree in G' as they do in G. Thus all points of $G' - (X_1 \cup X_2)$ have degree k in G'. As for points $x \in X_1 \cup X_2$, we have $d(x, G') = d(X, G^0) + (\frac{1}{2}k) - 1 = k$. Hence G' is k-regular. We now show that $Z(G') \geq Z(G)$. If we contract each of the edges $\{v_1^i v_2^i\}$, $1 \leq i \leq \frac{1}{2}k$, of G', we arrive at a graph F having G as a (non-induced) subgraph. That is, F is just G with some extra edges, those joining v^i and v^j. This graph can be contracted to $K_{Z(G)}$ because G can. Composing the contractions $G' \to F$ and $F \to K_{Z(G)}$, we get the required contraction $G' \to K_{Z(G)}$. Finally, it remains to show that $\beta_0(G') \geq \beta_0(G)$. Let S be an independent set in G, and suppose without loss of generality that $S \cap \{v^i\}_{j=1}^{k/2} = \{v^i\}_{j=1}^{n}$. Then the set $S' = (S \cap (G - \{v^i\}_{j=1}^{k/2})) \cup \{v_1^i\}_1^n$ is independent in G' and satisfies $|S'| = |S|$. Hence $\beta_0(G') \geq \beta_0(G)$.

For k odd, we proceed similarly. Certain changes in the construction above will be necessary, however.

Suppose, then, that we are given a k-regular graph G satisfying the hypothesis of the lemma with k odd. We begin by slightly altering the definition of splitting. If $v \in V(G)$, we take a partition of $N(v)$, $N(v) = N_1(v) \cup N_2(v)$ such that $N_1(v) = [\frac{1}{2}k]$ and $N_2(v) = \{\frac{1}{2}k\}$. We then define the splitting $H(G; N_1, N_2)$, the set $S^1(G; v)$, and the iterated sets $S^n(G; \{v^j\}_{j=1}^n)$ as before.

Let us examine a typical graph $G^0 \in S^n(G; \{v^j\}_{j=1}^n)$. Each v^j is replaced by two points, v_1^j and v_2^j say, where without loss of generality we take $d(v_1^j, G^0) = \{\frac{1}{2}k\} + 1$ and $d(v_2^j, G^0) = [\frac{1}{2}k] + 1$ for $1 \leq j \leq n$. If we let $X_1(G^0) = \{v_1^j\}_{j=1}^n$ and $X_2(G^0) = \{v_2^j\}_{j=1}^n$, then $V(G^0)$ may be partitioned into the three sets $X_1(G^0)$, $X_2(G^0)$, and $(V(G^0) - (X_1 \cup X_2))$, having the properties

(i) $v \in X_1(G^0) \Rightarrow d(v, G^0) = \{\frac{1}{2}k\} + 1$

(ii) $v \in X_2(G^0) \Rightarrow d(v, G^0) = [\frac{1}{2}k] + 1$

(iii) $v \in V - (X_1 \cup X_2) \Rightarrow d(v, G^0) = k$.

We now proceed to construct the graph G' of the lemma. This will be done according to the cases $k\equiv 1 \pmod 4$ and $k\equiv 3 \pmod 4$.

If $k\equiv 1 \pmod 4$, we find $[\frac{1}{2}k]$ independent points $\{v^j\}_{j=1}^{[k/2]}$ and form a graph $G^0 \in S^{[k/2]}(G;\{v^j\}_{j=1}^{[k/2]})$. Now define the graph G' by $V(G')=V(G^0)$, and

$$E(G')=E(G^0)\cup \{v_2^i v_2^{i+1}\colon i \text{ odd},\ 1\leq i\leq [\tfrac{1}{2}k]-1\}\cup\{v_2^i v_1^j\colon i\neq j\}.$$

If $k\equiv 3 \pmod 4$, we begin with $\{\frac{1}{2}k\}$ independent points and form a graph $G^0\in S^{\{k/2\}}(G,\{v^j\}_{j=1}^{\{k/2\}})$. The graph G' is now defined by $V(G')=V(G^0)$ and

$$E(G')=E(G^0)\cup\{v_2^i v_2^{i+1}\colon i \text{ odd},\ 1\leq i\leq \{\tfrac{1}{2}k\}-1\}\cup\{v_2^i v_1^j\colon j\neq i,\, i+1\}$$

where i and j are read mod$\{\frac{1}{2}k\}$.

These constructions clearly yield k-regular graphs G'. The proof that these graphs have the properties requred in the lemma is almost identical to the proof given in the case k even.

In order to state and prove the main result, we introduce some relevant terminology. Let G be a graph, and $X\subseteq V(G)$. We let ∂X denote the set of all vertices in X adjacent to vertices in $V(G)-X$. We now define the *external degree of* X by exd $(X)=|\partial(V(G)-X)|$. Given positive integers p and k which are not both odd, we let $M(p,k)=\max\{Z(G)\colon |G|=p,\ G \text{ is } k \text{ regular}\}$, and $Q(p,k)=\max\{\lambda\in\mathbf{Z}^+\colon \lambda\{(\lambda-3)/(k-2)\}\leq p\}$.

Theorem 3.2. *The integers $M(p,k)$ and $Q(p,k)$ are related as follows.*

(i) *$M(p,k)\leq Q(p,k)$.*

(ii) *For each $k\geq 3$, $M(p,k)=Q(p,k)$ for all but finitely many p.*

(iii) *$M(p,4)=Q(p,4)$ for $p\neq 6$, and $M(p,3)=Q(p,3)$ for all even p.*

Proof. We begin by proving (i). Let G have the properties of being k-regular, having p points, and satisfying $Z(G)=M(p,k)$. Set $M=M(p,k)$. In a complete M-partition of $V(G)$, let C be a class of minimum cardinality r. Since $\langle C\rangle$ contains at least $r-1$ edges, we have exd $(C)\leq rk-2(r-1)=r(k-2)+2$. As there are $M-1$ classes besides C, we have $M-1\leq$ exd $(C)\leq r(k-2)+2$, and hence $r\geq \{(M-3)/(k-2)\}$. Since C has minimum cardinality, $Mr\leq p$. Combining with the previous inequality we get $M\{(M-3)/(k-2)\}\leq p$, as required.

To establish (ii), we begin by constructing for each $k\geq 3$ an infinite set $S^{(k)}=\{G^{(k)}(\lambda)\}_{\lambda=k+1}^{\infty}$ of k regular graphs. Each graph $G^{(k)}(\lambda)$ will have order $p^{(k)}(\lambda)=\lambda\{(\lambda-3)/(k-2)\}$ and contractibility number λ. Thus by definition of $Q(p,k)$ we will have $\lambda=Q(p^{(k)}(\lambda),k)$. This will serve to show that for each $k\geq 3$, the equality $M(p,k)=Q(p,k)$ holds for infinitely many p, namely, the subset of the integers $P(k)=\{p^{(k)}(\lambda)\}_{\lambda=k+1}^{\infty}$. For by the definitions of λ and the inequality (i), we have $Q(p^{(k)}(\lambda), k) = \lambda = Z(G^{(k)}(\lambda)) \leq M(p^{(k)}(\lambda), k) \leq Q(p^{(k)}(\lambda), k)$. It follows that $M(p^{(k)}(\lambda), k) = Q(p^{(k)}(\lambda), k)$ for the set $P(k)$.

Once we have established (ii) for $P(k)$ (given k), we will establish it for all but finitely many of the remaining integers $\mathbf{Z}^+ - P(k)$ by means of a method of subdivisions (soon to be defined) and the lemma.

Fix $k \geq 3$. The graph $G^{(k)}(\lambda)$ is constructed as follows. Suppose first $\{(\lambda - 3)/(k-2)\} = (\lambda - 3)/(k-2)$, so that $p^{(k)}(\lambda) = \lambda(\lambda - 3)/(k-2)$.

We will begin with a cycle having $p^{(k)}(\lambda)$ points. In going once around the cycle, number the points in order of their traversal $x_1, x_2, \ldots, x_{\lambda(\lambda-3)/(k-2)}$. Now partition them into λ subsets $\{V_i\}_{i=1}^{\lambda}$ where $V_i = \{x_n\colon (i-1)(\lambda-3)/(k-2) + 1 \leq n \leq i(\lambda-3)/(k-2)\}$. For later use, we will relabel our points with double indices according to the rule $x(i, j) = x_n$ if and only if $x \in V_i$ and j is the congruence class of n mode $(\lambda - 3)/(k-2)$. Our hamiltonian cycle H therefore consists of the edges

$$\{\{x(i, j)x(i, j+1)\} \cup \{x(i, (\lambda-3)/(k-2))x(i+1, 1)\}\colon$$
$$1 \leq j \leq (\lambda-3)/(k-2) - 1, 1 \leq i \leq \lambda\}$$

with $x(1, 1) \equiv x(\lambda + 1, 1)$.

The edges of $G^{(k)}(\lambda)$ not in H will now be defined recursively. First, form the edges $\{x(1, 1)x(i, (\lambda-3)/(k-2))\colon 3 \leq i \leq k\}$. The point $x(1, 1)$ now has degree k and no more edges incident with it will be formed. To describe the other edges, we will define a total ordering of the points of $G^{(k)}(\lambda)$. For points $x(i, j)$, $x(i, l)$ in the same class V_i, we let $x(i, j) < x(i, l)$ if and only if $j < l$. If $x(i, j)$, $x(l, m)$ are points in distinct classes V_i, V_j, then let $x(i, j) < x(l, m)$ if and only if $i < l$. According to this ordering, then, $x < y$ if and only if x comes before y in the orginal traversal $x_1, x_2, \ldots, x_{\lambda(\lambda-3)/(k-2)}$ of the cycle H. Now suppose that a graph $G(i, j)$ has been constructed in which $d(x, G(i, j)) = k$ for all $x < x(i, j)$ and $d(x(i, j), G(i, j)) = t < k$. In each class V_n, $n \geq i+2$, having a point of degree less than k in $G(i, j)$, let $x(n) \in V_n$ be the point with this property that is maximal with respect to the ordering. Also, let $X = \{x(n)\colon n \geq i+2\}$. The remaining edges incident with $x(i, j)$ will now be defined according to whether $j = 1$ or $j > 1$. If $j = 1$, join $x(i, j)$ to the first $k - t$ points of X (with respect to the ordering induced on X). If $j > 1$, let n_0 be the maximum index such that $x(i, j-1)$ is adjacent in $G(i, j)$ to some point of V_{n_0}. Then join $x(i, j)$ to the first $k - t$ points of the subset $X(n_0) = \{x(n)\colon n \geq n_0 + 1\}$ of X. In either case, let us call the resulting graph $G(i, j)'$. There are now two possibilities, either $G(i, j)'$ is k-regular or it is not. If it is not, let

$$\begin{aligned} G(i, j)' &= G(i, j+1) \quad \text{if } j \leq (\lambda-3)/(k-2) - 1 \\ &= G(i+1, 1) \quad \text{if } j = (\lambda-3)/(k-2), \end{aligned}$$

and repeat the above procedure. A k-regular graph $G(i_0, j_0)'$ must eventually be reached, and we let $G^{(k)}(\lambda) = G(i_0, j_0)'$. Thus $V(G^{(k)}(\lambda))$ is composed of λ classes

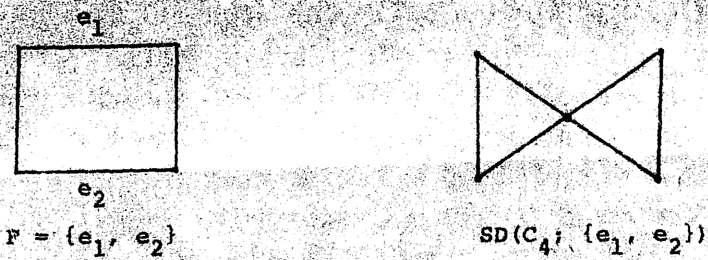


Fig. 4. The graphs C_4 and $SD(C_4; \{e_1, e_2\})$.

$\{V_i\}_{i=1}^{\lambda}$, each class inducing the path $P_{(\lambda-3)/(k-2)}$. These classes are then positioned in increasing order on the hamiltonian cycle H. The 4-regular graph $G^{(4)}(7)$ of order $p^{(4)}(7)=14$ and contractibility number 7 is shown in Fig. 5a. In this example we have $G^4(7)=G(4;2)'$.

The workability of the algorithm given above, and hence the formation of the k-regular graph $G(i_0, j_0)'=G^{(k)}(\lambda)$, depends upon the existence of at least $k-t$ points in the set X of the graph $G(i,j)$. This condition is in turn equivalent to having at least $k-t$ classes V_n, $n \geq i+2$, each containing a point of degree less than k in $G(i,j)$. The verification that this holds requires a careful count of the number of edges $e=(a,b)$ in $G(i,j)$ with the property $a<x(i,j)$ and $b \geq x(i+2,1)$. This count will here be omitted as it is straightforward and lengthy.

The construction above has the property that for each pair of distinct classes, V_i, V_j, there exists a unique edge xy with $x \in V_i$ and $y \in V_j$. Hence $G^{(k)}(\lambda)$ has the complete λ-partition $V(G^{(k)}(\lambda))=\bigcup_{i=1}^{\lambda} V_i(\lambda)$, so that $\lambda \leq Z(G^{(k)}(\lambda))$. On the other hand, we have $\lambda = Q(p^{(k)}(\lambda), k) \geq M(p^{(k)}(\lambda), k) \geq Z(G^{(k)}(\lambda))$. The equality $Q(p^{(k)}(\lambda), k)=\lambda=Z(G^{(k)}(\lambda))$ follows. Note also that the uniqueness of the edge xy is the beginning of the proof that the $G^{(k)}(\lambda)$ are Z-critical. That is, the removal of any edge of $G^{(k)}(\lambda)$ results in a graph with smaller contractibility number.

The construction of the graphs $G^{(k)}(\lambda)$ for which $\{(\lambda-3)/(k-2)\} \neq (\lambda-3)/(k-2)$ is similar, and the details are omitted here. An example, $G^{(4)}(6)$ is shown in Fig. 5b.

We have constructed the graphs $\{G^{(k)}(\lambda)\}_{\lambda=1}^{\infty}$, and thus the equality (ii) is proved for the subset of the integers $P(k)$. To finish the proof, it remains to show $M(p,k)=Q(p,k)$ for all but finitely many of the integers in the set $\Omega = \mathbf{Z}^+ - P(k)$. As in the proof for $P(k)$, the inequality (i) reduces our problem to the construction of a k-regular graph of order p and contractibility number $Q(p,k)$ for all but finitely many $p \in \Omega$. This will only be done for k even below, as the construction for k odd is similar.

Let λ_0 be the least integer λ satisfying the inequality

$$\{(\lambda-3)/(k-2)\} \geq k. \tag{1}$$

We will show that the construction of the required graph of order p is possible for all $p \in \Omega$ satisfying $p \geq p^{(k)}(\lambda_0)$. That is, $M(p, k) = Q(p, k)$ holds for $p \geq \lambda_0\{(\lambda_0 - 3)/(k-2)\} > k(k^2 - 3k + 5)$.

Fix k, and write $p(\lambda)$ for $p^{(k)}(\lambda)$. Let us begin by observing that it suffices to construct, for each λ satisfying (1), a sequence $\{G_i(\lambda): 1 \leq i \leq p(\lambda+1) - p(\lambda) - 1\}$ of k-regular graphs which satisfy $|G_i(\lambda)| = |G^{(k)}(\lambda)| + i$ and $Z(G_i(\lambda)) = Z(G^{(k)}(\lambda))$. For by definition, $Q(p, k) = Q(p(\lambda), k)$ holds for $p(\lambda) \leq p < p(\lambda+1)$. Hence we have $Z(G_i(\lambda)) = Z(G^{(k)}(\lambda), k) = M(p(\lambda), k) = Q(p(\lambda), k) = Q(p(\lambda)+i, k)$. As the inequalities $Z(G_i(\lambda)) \leq M(p(\lambda)+i, k) \leq Q(p(\lambda)+i, k)$ hold by definition and by (i), we get $M(p(\lambda)+i, k) = Q(p(\lambda)+i, k)$ for $1 \leq i \leq p(\lambda+1) - p(\lambda) - 1$. This being true for each λ satisfying (1), it follows that (ii) is proved for all $p \in \Omega$ with $p \geq p(\lambda_0)$.

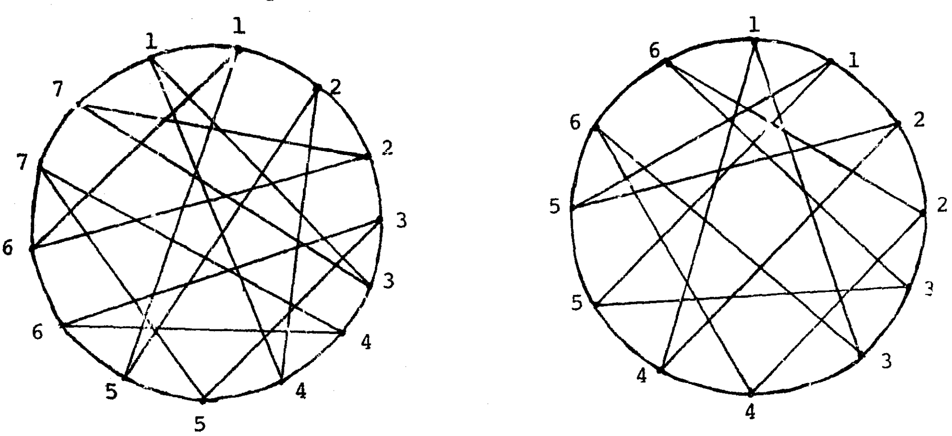
For the constructions to follow, we need the following definitions. Let $F = \{e_i\}_{i=1}^n$ be a set of independent edges of G, with $e_i = a_i b_i$. Define the *F subdivision graph of G*, SD($(G; F)$ as follows. Let $V(\mathrm{SD}(G; F)) = V(G) \cup \{z\}$, and $E(\mathrm{SD}(G; F)) = (E(G) - F) \cup \{za_i\}_{i=1}^n \cup \{zb_i\}_{i=1}^n$. An F subdivision graph of C_4 is shown in Fig. 4.

We now construct the first $\frac{1}{2}k$ of the graphs $G_i(\lambda)$, using the graph $G^{(k)}(\lambda)$ as our beginning. Recall the complete λ-partition of $G^{(k)}(\lambda)$ given by $V(G^{(k)}(\lambda)) = \bigcup_{i=1}^{\lambda} V_i$. By assumption (1), we have $|V_i| = \{(\lambda-3)/(k-2)\} > k$, and by construction of $G^{(k)}(\lambda)$ each $\langle V_i \rangle$ is a path. In particular, $\langle V_1 \rangle$ is a path of order greater than $k > \frac{1}{2}(k-2)$. Hence there exists an edge $e \in E(\langle V_1 \rangle)$ and a set $B^1 = \{e_i = g_i h_i : 1 \leq i \leq \frac{1}{2}(k-2)\}$ of $\frac{1}{2}(k-2)$ independent edges of $G^{(k)}(\lambda)$ with the properties

(a) $h_i \in V_1$, $g_i \notin V_1$;

(b) no edge of B^1 is incident on e.

Since $\lambda > k$, we may actually choose these edges so that the points g_i are among the sets V_n, $n \geq \frac{1}{2}k + 1$. Let $F^1 = B^1 \cup \{e\}$. We then define $G_1(\lambda)$ as the F^1-subdivision graph $\mathrm{SD}(G^{(k)}(\lambda); F^1)$. Clearly $G_1(\lambda)$ is k regular and $|G_1(\lambda)| = p(\lambda)+1$. Write z^1 for the point z in the definition of F-subdivision. Then with z^1



(a) $G^{(4)}(7)$ (b) $G^4(6)$

Fig. 5. The graphs $G^4(7)$ and $G^4(6)$.

adjoined to V_1, it follows that $Z(G_1(\lambda)) = Z(G^{(k)}(\lambda)) = \lambda$ with complete λ-partition

$$V(G_1(\lambda)) = (V_1 \cup \{z^1\}) \cup (\bigcup_{j=2}^{\lambda} V_j).$$

We will now repeat this procedure $(\frac{1}{2}k)-1$ times. That is, having constructed $G_i(\lambda)$, $i<\frac{1}{2}k$, we carry out the same construction as above with the class V_{i+1} playing the role of V_1. Thus we let $x \in E(\langle V_{i+1}\rangle)$ and let $B^{i+1} = \{h_j g_j\}_{j=1}^{(k-2)/2}$ be a set of $\frac{1}{2}(k-2)$ independent edges with the property

(a) $h_j \in V_{i+1}$, $g_j \notin V_{i+1} (1 \le j \le \frac{1}{2}(k-2))$;

(b) no edge of B^{i+1} is incident on x.

Let $F^{i+1} = B^{i+1} \cup \{x\}$. We then define $G_{i+1}(\lambda)$ to be the F^{i+1} subdivision graph $\mathrm{SD}(G_i(\lambda); F^{i+1})$. Again we observe that the points g_i of the edges in F^{i+1} can be taken to be among the sets V_n, $n \ge (k/2)+1$. This convention on the g_i's insures that the needed independent edges exist at each step. For in each graph $G_i(\lambda)$, $i < \frac{1}{2}k$, no point of the set V_{i+1} has played the role of some point g_j in any of the previous subdivisions. Hence all edges with a point in V_{i+1} that were independent in the original graph $G^{(k)}(\lambda)$ are still independent in $G_i(\lambda)$. Since there were at least k such edges in $G^{(k)}(\lambda)$, there are certainly $\frac{1}{2}(k-2)$ of them in $G_i(\lambda)$.

The graph $G_{i+1}(\lambda)$ is easily seen to have the required properties. Clearly $|G_{i+1}(\lambda)| = |G_i(\lambda)| + 1 = p(\lambda) + i + 1$ holds. Letting z^{i+1} be the point z in the definition of $\mathrm{SD}(G_i(\lambda); F^{i+1})$, the complete λ-partition of $G_{i+1}(\lambda)$ is gotten by adjoining z^{i+1} to V_{i+1} and using the partition

$$V(G_{i+1}(\lambda)) = \bigcup_{j=1}^{i+1} (V_j(\lambda) \cup \{z^j\}) \cup \left(\bigcup_{j=i+2}^{\lambda} V_j(\lambda) \right).$$

In this way, we construct the first $\frac{1}{2}k$ graphs $\{G_i(\lambda): 1 \le i \le \frac{1}{2}k\}$.

Observe that the set of points $\{x(1,j): j \text{ even } 2 \le j \le k\}$ is an independent set in $G_{k/2}(\lambda)$. Thus $\beta_0(G_{k/2}^{(\lambda)}) > \frac{1}{2}k$ holds and we may form the remaining graphs $\{G_i(\lambda): (k/2)+1 \le i \le p(\lambda+1) - p(\lambda) - 1\}$ by appealing to the lemma.

For (iii), we specialize the general construction.

When $k=3$, we must construct the cubic graphs $\{G_i(\lambda): 1 \le i \le \frac{1}{2}(p(\lambda+1) - p(\lambda)) - 1\}$ such that $|G_i(\lambda)| = |G^3(\lambda)| + 2i$, and $Z(G_i(\lambda)) = p(\lambda)$. Define $G_1(\lambda)$ by

$$V(G_1(\lambda)) = V(G^3(\lambda)) \cup \{z^{11}, z^{12}\},$$

and

$$\begin{aligned} E(G_1(\lambda)) = E(G^3(\lambda)) & \\ & - \{x(\lambda,(\lambda-3)/(k-2))x(1,1),\ x(1,(\lambda-3)/(k-2))x(2,1)\} \\ & \cup \{z^{11}x(\lambda,(\lambda-3)/(k-2)),\ z^{11}x(1,1),\ z^{12}x(1,(\lambda-3)/ \\ & \qquad (k-2)),\ z^{12}x(2,1),\ z^{11}z^{12}\}. \end{aligned}$$

Having defined $G_i(\lambda)$, form $G_{i+1}(\lambda)$ by letting

$$V(G_{i+1}(\lambda)) = V(G_i(\lambda)) \cup \{z^{(i+1)1}, z^{(i+1)2}\},$$

and

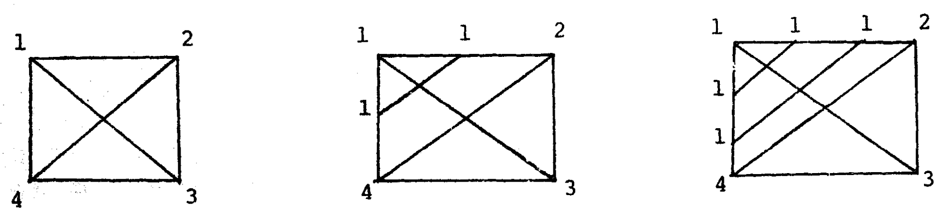
$$E(G_{i+1}(\lambda)) = E(G_i(\lambda)) - \{x(\lambda,(\lambda-3)/(k-2))z^{11}, x(2,1)z^{12}\} \cup \{x(\lambda,(\lambda-3)/(k-2))z^{(i+1)1}, z^{(i+1)}z^{i1}, x(2,1)z^{(i+1)2}, z^{(i+1)2}z^{i2}, z^{(i+1)1}z^{(i+1)2}\}.$$

In short, the graphs $G_i(\lambda)$ (for $k=3$) are formed by successively "inserting" a point of degree 2 in each of two edges, and then joining these two points of degree 2 by an edge.

For $k=4$, we carry out the F-subdivisions (with $|F|=2$) unencumbered by the requirement (1). The exception occurs with the unique quartic graph of order 6, $K(2,2,2)$. We have $Q(6,3)=5$, but $Z(K(2,2,2))=4$. Hence in the case $k=4$, we begin the inductive construction with a quartic graph of order 7 and contractibility number $Q(7,4)=5$.

The cubic graphs $G^3(4)$, $G_1(4)$, $G_2(4)$ are shown in Fig. 6a. A quartic graph of order 7 and contractibility number 5 is shown in Fig. 6b. It may be used as the first graph in the inductive construction of the graphs $\{G_i(5)\colon\ 2 \leq i \leq p(6)-p(5)-1\}$.

Corollary 3.3. *The maximum of $Z(G)$ over all cubic graphs of order p is given by* $M(p,3) = [\frac{1}{2}(3+\sqrt{9+4p})]$.



(a) The graphs $G^3(4)$, $G_1(4)$, $G_2(4)$

(b) A quartic graph of order 7 and contractibility number 5.

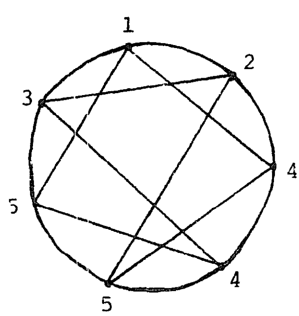


Fig. 6. Extremal cubic and quartic graphs.

4. Open problems

(i) Let $q(G)$ be the number of edges in a graph G, and let $f(n, p) = \max\{q(G): |G| = p, Z(G) = n\}$. We conjecture that $f(n, p) = \binom{n}{2} + (p-n)(n-1)$. The graphs $K_{n-1} + \bar{K}_{p-n+1}$ show that $f(n, p) \geq \binom{n}{2} + (p-n)(n-1)$.

(ii) The behavior of $Z(G)$ under operations on graphs remains undetermined. Some observations can be made, however.

(a) Let G_1 and G_2 be connected. By the theorem of Wagner [5] quoted earlier, the following are equivalent.

(I) $G_1 \times G_2$ is not contractible to $K(3, 3)$ and satisfies $Z(G_1 \times G_2) \leq 4$.

(II) $G_1 \times G_2$ is planar.

(II) is in turn equivalent to both G_i being paths, or one being a path and the other a cycle [3, problem 9.4].

(b) We have shown that $Z(C_n \times K_2) = 4 (n \geq 3)$, and that $Z(T \times K_n) = n + 1$ for all trees T.

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