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A Nonlinear Elliptic Boundary Value Problem at Resonance

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1. INTRODUCTION

We are concerned here with existence theorems for nonlinear elliptic boundary value problems of the form $Lu = N(x, u, Du,..., D^{2m-1}u)$ for $x \in \Omega$, $Bu = 0$ on $\partial\Omega$. L is a uniformly elliptic linear partial differential operator of order $2m$ in a bounded domain Ω of $\mathbb{R}^n$, and $Bu = 0$ represents linear homogeneous boundary conditions (on u) with respect to which L is "coercive" but not necessarily self-adjoint. N is a real valued function of x in Ω, of u, and of all partial derivatives of u of order up to $2m - 1$, subject to certain conditions which will be specified later. We are interested in the difficult "resonance" case, i.e., the case in which there are nontrivial solutions θ to the homogeneous linear problem $L\theta = 0$ in Ω, $B\theta = 0$ in $\partial\Omega$. For $m = 1$, L self-adjoint with Fredholm indices (1, 1), $Bu = u$ (Dirichlet boundary conditions), N of the form $h(x) - g(u)$, $h \in L^2(\Omega)$, $g: \mathbb{R}^1 \to \mathbb{R}^1$ continuous and asymptotically constant, Landesman and Lazer [14] proved a remarkable theorem which has since been extended by Williams [22], Nirenberg [18], DeFigueiredo [9], and others.

In this paper we consider linear operators L (with equal Fredholm indices (M, M)) whose eigenfunctions share regions of positivity and negativity with their corresponding adjoint eigenfunctions. This class includes the self-adjoint operators, as well as several types of general second order operators, ordinary and partial. As in [14, 22] use is made of the alternative method of Cesari, a version of the Liapunov–Schmidt procedure based on functional analysis. Thus, the problem is split into two coupled equations, the auxiliary and the bifurcation equations. Here, the auxiliary equation is solved by the Schauder Fixed Point Theorem, and the bifurcation equation is solved by topological methods. Central to our analysis is a consideration of the connectivity properties of the fixed point set of a family of Schauder (fixed point) maps. In the case of indices (1, 1), these properties follow from point-set topology only, whereas differential topology (transversality) and topological degree are needed for indices (M, M), $M \geqslant 1$.

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2. Assumptions and Statement of Results

Let Ω be a bounded, connected open set in $\mathbb{R}^n$ with smooth (C^∞) boundary $\partial\Omega$; in fact, most of the following generalizes to a compact connected smooth Riemannian n-manifold with or without boundary. Let L be a (not necessarily self-adjoint) uniformly elliptic linear partial differential operator of order $2m$, i.e.,

$$L = \sum_{|\alpha| \leqslant 2m} a_\alpha(x)\, D^\alpha, \qquad \text{where} \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}},$$

and for some constant $\mu > 0$, $\mu^{-1} \mid \xi \mid^{2m} \leqslant \sum_{|\alpha|=2m} a_\alpha(x)\, \xi^\alpha \leqslant \mu \mid \xi \mid^{2m}$ for all $x \in \Omega$, $\xi \in \mathbb{R}^n$. The coefficients $a_\alpha(x)$ are real valued, and we assume, for convenience, that they are smooth (C^∞) functions on $\bar{\Omega}$. For s a nonnegative integer, we denote by H^s the Sobolev (Hilbert) space of square-integrable functions on Ω whose (distribution) derivatives of order $\leqslant s$ are also square integrable; $H^0 = L^2$. Let B be a system of m linear boundary operators of order $<2m$ defined on $\partial\Omega$ with (smooth) real coefficients. For $s \geqslant 2m$, we denote by $H_\partial{}^s$ the subspace of H^s of functions u satisfying $Bu = 0$ on $\partial\Omega$.

We will generally assume that L is coercive with respect to B, i.e., that $L\colon H_\partial^{2m} \to H^0$ has a finite-dimensional kernel, a closed range, and a finite-dimensional cokernel, and that a "coerciveness" inequality is satisfied. That is, there is a constant C (independent of u) such that $\| u \|_{H^{2m}} \leqslant C(\| Lu \|_{H^0} + \| u \|_{H^0})$ for all $u \in H_\partial^{2m}$. More specifically, we assume the existence of two finite families of functions, $\{\theta_i\}_{i=1}^M$ and $\{\psi_i\}_{i=1}^M$, all in $C^\infty(\bar{\Omega})$, each family orthonormal in the sense of $H^0 = L^2$, such that $\{\theta_i\}$ are a basis for the kernel of L, and such that $g \in \operatorname{Range} L$ iff $\int g\psi_i = 0$ for all i. (These assumptions are generally true for reasonable boundary conditions B; see [15, pp. 148–154, 111–113]). Notice that we assume that (L, B) has equal Fredholm indices (M, M), that L fails to be injective and surjective by the same number of dimensions M.

Let N be a nonlinear Nemytsky operator of the form $Nu = h(x) + g(u) + f(x, u, Du,..., D^{2m-1}u)$, where h is a fixed element of H^0, g is a continuous real-valued function of one real variable with (finite) asymptotic limits $g(\infty)$ and $g(-\infty)$, and f is a bounded continuous real valued function on $\bar{\Omega} \times \mathbb{R}^k$ ($k = 1 + n + (n(n+1)/2) + \cdots$ is the number of different partial derivatives in $\mathbb{R}^n$ of order $\leqslant 2m - 1$). As defined, N is bounded and continuous from H^{2m-1} into H^0 (see [21, p. 155] or [13, p. 27]), and the range of N lies in a bounded subset of H^0.

We begin with the case of simple resonance ($M = 1$); let $\theta = \theta_1$, $\psi = \psi_1$. Define $\Omega_+ = \{x \mid \psi(x) > 0\}$ and $\Omega_- = \{x \mid \psi(x) < 0\}$, and we require that

$$g(-\infty) \int_{\Omega_+} \psi + g(\infty) \int_{\Omega_-} \psi + \int_\Omega h\psi + \alpha \int_\Omega | \psi | < 0,$$

and

$$g(\infty) \int_{\Omega_+} \psi + g(-\infty) \int_{\Omega_-} \psi + \int_\Omega h\psi - \alpha \int_\Omega |\psi| > 0, \tag{1}$$

where $\alpha = \sup\{|f|\}$.

THEOREM 1. *In addition to the previous assumptions, assume that L is self-adjoint with respect to B (so $\theta = \psi$). Then $Lu = Nu$ is solvable for u in H_∂^{2m}.*

One can relax the condition that L be self-adjoint somewhat; Theorem 1 is a special case of the following.

THEOREM 2. *In addition to the previous assumptions, assume only that θ and ψ share regions of positivity and negativity, i.e., that $\theta(x) > 0$ on Ω_+ and that $\theta(x) < 0$ on Ω_-. Then $Lu = Nu$ is solvable in H_∂^{2m}. This includes the following non-self-adjoint cases for which $m = 1$ (L is of second order), and $Bu = u$ (Dirichlet conditions on $\partial\Omega$):*

(a) *L is an ordinary differential operator ($n = 1$);*

(b) *L admits a "separation of variables," i.e., L is of the form $Lu = \sum a_i(x_i)\, u_{x_i x_i} + \sum b_i(x_i)\, u_{x_i} + c(x)\, u$, where the coefficients a_i, b_i depend only on the independent variable x_i;*

(c) *L has constant coefficients in the higher order terms, i.e., $Lu = \sum_{1 \leqslant |\alpha| \leqslant 2} a_\alpha D^\alpha u + c(x)\, u$;*

(d) *L is of the form $Lu = \Delta u + (\nabla q(x)) \cdot \nabla u + c(x)\, u$, where q is a smooth function on $\bar{\Omega}$;*

(e) *the spectrum of L (considered as an operator from the dense domain H_∂^2 of H^0 into H^0) lies in the half-plane $\{\operatorname{Re} z \leqslant 0\}$. This holds in particular if L is dissipative, i.e., $\int (u)(Lu) \leqslant 0$ for all $u \in H_\partial^2$.*

In the case of multiple resonance ($M > 1$), the analogous techniques are limited to a smaller class of nonlinear operators N. In particular, we assume that $Nu = h(x) + g(u)$, h and g as before, and also that g is continuously differentiable and g' vanishes at $\pm\infty$. The condition corresponding to (1) is

$$g(\infty) \int_{\Omega_+} e\psi + g(-\infty) \int_{\Omega_-} e\psi + \int_\Omega he\psi > 0 \tag{2}$$

for every $e\psi = \sum_{i=1}^M e_i\psi_i$, $\sum e_i^2 = 1$ on the unit sphere of span $\{\psi_i\}$; here, $\Omega_+ = \{x \mid e\psi > 0\}$ depends on e; similarly for Ω_-.

THEOREM 3. *In addition to the previous assumptions, we assume that L satisfies a "unique continuation" condition. That is, the only solution $u \in H_\partial^{2m}$ of $Lu = 0$ that vanishes on a set of positive measure is $u \equiv 0$. We assume that either L is* *self-adjoint or that $e\psi$ and $e\theta$ ($= \sum e_i\theta_i$) share regions of positivity and negativity. Then $Lu = Nu$ is solvable in H_∂^{2m}. As before, admissible L includes the second order Dirichlet cases* (b), (c), (d) *in Theorem* 2 (*cases* (a) *and* (e) *are impossible if $M > 1$*).

As will be evident from the proofs, these theorems are true if we reverse both inequalities and the sign of α in (1) or if we reverse the inequality in (2). It should also be noted that if we assume that $g(-\infty) < g(s) < g(\infty)$ and that $\alpha = 0$ in (1), then (1) (or (2)) is necessary (as well as sufficient) for the existence of a solution to $Lu = Nu$ (see [14] for a proof). The proofs of these theorems may also be modified to handle the case in which g has slow growth, i.e., $|g(s)| \leqslant C(1 + s^\alpha)$ where $0 < \alpha < 1$.

3. Splitting the Equation

In this section, we split the equation $Lu = Nu$ into two simpler equations according to the procedure of Liapunov–Schmidt. We view these equations from the standpoint of functional analysis as developed by Cesari and others [2, 5–8].

We first define two projection operators P and Q on the spaces H_∂^{2m} and H^0 by $Pu = \sum (\int_\Omega \theta_i u)\, \theta_i$, $Qu = \sum (\int_\Omega \psi_i u)\, \psi_i$, $i = 1,..., M$. Since $\{\theta_i\}, \{\psi_i\} \subset C^\infty(\bar{\Omega})$, it follows that P and Q are bounded linear projections on their respective spaces. Moreover, $Pu = u$ iff $u \in \operatorname{Kernel} L$ and $Qu = 0$ iff $u \in \operatorname{Range} L$. Neither injective nor surjective as yet, L becomes injective if we restrict its domain to $\operatorname{Kernel} P \subset H_\partial^{2m}$ and surjective if we consider its range to be $\operatorname{Kernel} Q \subset H^0$. L is continuous in this context, and so has a continuous inverse K: Kernel $Q \to$ Kernel P by the open mapping theorem.

We wish to solve the equation $Lu = Nu$. For $u \in H_\partial^{2m}$, write $u = v + Pu = v + c\theta$, where we abbreviate $Pu = \sum c_i\theta_i$ as $c\theta$, and $v = u - Pu$. Our equation then is of the form $L(v + c\theta) = N(v + c\theta)$ or

$$Lv = N(v + c\theta). \tag{3}$$

For any solution $u = v + c\theta$, it is clear that

$$QN(v + c\theta) = 0, \tag{4}$$

since $N(v + c\theta) \in \operatorname{Range} L = \operatorname{Kernel} Q$, and consequently

$$Lv = (I - Q)\, N(v + c\theta). \tag{5}$$

Conversely, for any pair $(v, c\theta) \in \operatorname{Kernel} P \times \operatorname{Range} P = H_\partial^{2m}$ which simultaneously solves (4) and (5), $u = v + c\theta$ is easily seen to be a solution to (3). Applying K to (5), we write (5) in the equivalent form

$$v = K(I - Q)\,N(v + c\theta). \tag{6}$$

This is the auxiliary equation; we will think of $c\theta$ as a fixed element of Range P and prove that (6) has a solution $v \in \text{Kernel}\,P \subset H_\partial^{2m}$ for that fixed $c\theta$.

The map $v \to K(I - Q)\,N(v + c\theta)$ is compact and continuous on Kernel P (with respect to the topology inherited from H_∂^{2m}). We begin with the compact inclusion map $H^{2m} \to H^{2m-1}$, add $c\theta$ (a fixed element of H^{2m-1}), apply the Nemytsky operator N (which brings us to H^0), and finally return to Kernel P via $K(I - Q)$. The form of N guarantees an a priori bound on $\| N(v + c\theta)\|_{H^0}$; it follows that $\| K(I - Q)\,N(v + c\theta)\|_{H^{2m}} \leqslant M_1$ for some $M_1 > 0$. It is a consequence of the Schauder fixed point theorem that a (not necessarily unique) solution $v = v(c\theta)$ to (6) exists in $B(M_1) = \{w \in H_\partial^{2m} \mid \| w \|_{H^{2m}} \leqslant M_1\}$.

To prove Theorems 1 and 2, it only remains to show that under the existing assumptions, there is a solution $c\theta \in \text{Range}\,P = \text{Kernel}\,L$ of the bifurcation equation

$$QN(v(c\theta) + c\theta) = 0, \tag{4}$$

where $v(c\theta)$ is one of the solutions to (6) corresponding to that choice of $c\theta$. Theorem 3 will be somewhat more difficult; we will perturb (6) slightly and solve the perturbed equation, then solve (4) for $v = v(c\theta)$ a solution to the perturbed equation, and finally apply a limit argument.

4. The Bifurcation Equation—Simple Resonance

Proof of Theorems 1 *and* 2. We first prove that for $c > 0$ sufficiently large, $I(v, c) \equiv \int_\Omega \psi QN(v + c\theta) > 0$, i.e.,

$$\int_\Omega \psi\{h + g(v + c\theta) + f(D^\alpha(v + c\theta))\} > 0.$$

This follows directly from the second inequality of (1), and the following.

Lemma 1. *We have assumed that g is continuous with limits $g(\infty)$, $g(-\infty)$, we have a priori bounds $\| v \|_{H^{2m}} \leqslant M_1$ 1 $\| \psi \|_{L^\infty} \leqslant M_2$, $\| g \|_{L^\infty} \leqslant M_3$, and we have that θ and ψ share regions of positivity (Ω_+) and negativity (Ω_-). Then $\int_\Omega \psi g(v + c\theta) \to g(\infty) \int_{\Omega_+} \psi + g(-\infty) \int_{\Omega_-} \psi$ as $c \to \infty$, uniformly over such v.*

Similarly, for sufficiently large negative c, $I(v, c) < 0$; this uses the first inequality of (1). Let $S = \{(v, c) \in \text{Kernel}\,P \times \mathbb{R} \mid v = K(I - Q)\,N(v + c\theta)\}$ be the set of all solutions to the auxiliary equation (6); we assert the existence of a connected subset S_0 of S that "stretches across" from large positive to large negative values of c. I is continuous on Kernel $P \times \mathbb{R} \simeq H_\partial^{2m}$, hence on S_0, and so $I(v, c) = 0$ for some $(v, c) \in S_0$. This is the bifurcation equation (4), and $(v, c) \in S_0$ implies that the auxiliary equation (6) is also solved.

The existence of the connected component S_0 follows from:

LEMMA 2. *Let $B(M_1)$ be a closed bounded convex subset of a Banach space, and let $[p, q]$ be a closed bounded interval in $\mathbb{R}$. Assume that F: $B(M_1) \times [p, q] \to B(M_1)$ is compact and continuous. (The usual Schauder theorem asserts that for each $c \in [p, q]$, $F(v, c) = v$ for some $v \in B(M_1)$.) Then there exists a connected set $S_0 \subset B(M_1) \times [p, q]$ of fixed points of F, i.e., $F(v, c) = v$ for $(v, c) \in S_0$. Moreover, this set S_0 meets both $B(M_1) \times \{p\}$ and $B(M_1) \times \{q\}$ (the "end-discs" of $B(M_1) \times [p, q]$).*

This lemma is a consequence of the Leray–Schauder fixed point theory; for a proof, see [4] or [16]. In our application, $F(v, c) = K(I - Q)\, N(v + c\theta)$, and $B(M_1)$ is the ball of radius M_1 around the origin in H_∂^{2m}. To finish the proofs of Theorems 1 and 2, it only remains to prove Lemma 1 and to demonstrate that in cases (a)–(e) of Theorem 2, the hypothesis of "common regions of positivity and negativity" holds for θ and ψ.

Proof of Lemma 1. Fix $\epsilon > 0$, and let $\delta < (\epsilon/6)(M_2 M_3)^{-1}$. $\| v \|_{H^0} \leqslant \| v \|_{H^{2m}} \leqslant M_1$, and so $| v(x)| \leqslant M_1' = 2^{1/2} M_1 \delta^{-1/2}$ except on a set of measure $\leqslant \delta/2$. Since the measure of $\{x \mid | \theta(x)| > \rho\}$ tends to the measure of $\{x \mid | \theta(x)| > 0\}$ *as* $\rho \to 0^+$, there is a $\rho > 0$ with $0 < | \theta(x)| \leqslant \rho$ only for those x in a set of measure $\leqslant \delta/2$. Let Ω_δ be the union of these two sets; the measure of $\Omega_\delta \leqslant \delta$. Pick a so large that $| g(\infty) - g(s)| < (\epsilon/6) \| \psi \|_{L^1}^{-1}$ for $s > a$ and that $| g(s) - g(-\infty)| < (\epsilon/6) \| \psi \|_{L^1}^{-1}$ for $s < -a$. Take $c > (a + M_1')\, \rho^{-1}$. For such c, we have:

$$
\begin{aligned}
&\left| \int_\Omega \psi g(v + c\theta) - g(\infty) \int_{\Omega_+} \psi - g(-\infty) \int_{\Omega_-} \psi \right| \\
&\quad \leqslant \left| \int_{\Omega \sim \Omega_\delta} \psi g(v + c\theta) - g(\infty) \int_{\Omega_+ \sim \Omega_\delta} \psi - g(-\infty) \int_{\Omega_- \sim \Omega_\delta} \psi \right| \\
&\qquad + \left| \int_{\Omega_\delta} \psi g(v + c\theta) - g(\infty) \int_{\Omega_+ \cap \Omega_\delta} \psi - g(-\infty) \int_{\Omega_- \cap \Omega_\delta} \psi \right| \\
&\quad \leqslant \Big(\sup_{\Omega_+ \sim \Omega_\delta} | g(v + c\theta) - g(\infty)| \Big) \left(\int_\Omega | \psi | \right) \\
&\qquad + \Big(\sup_{\Omega_- \sim \Omega_\delta} | g(v + c\theta) - g(-\infty)| \Big) \left(\int_\Omega | \psi | \right) + 3\delta \, \| \psi \|_{L^\infty} \| g \|_{L^\infty} \\
&\quad \leqslant (\epsilon/6) + (\epsilon/6) + (\epsilon/2) < \epsilon.
\end{aligned}
$$

This proves Lemma 1. Notice that we only assume that θ and ψ share regions of positivity and negativity, not that $\theta = \psi$.

We now complete the proof of Theorem 2. In case (a), the linear operator is of the form $Lu = a(x)\, u_{xx} + b(x)\, u_x + c(x)\, u$, which can be made self-adjoint with respect to a different (equivalent) inner product. Specifically, let $\rho(x) = a(x)^{-1} \operatorname{Exp}(\int^x b(t)\, a(t)^{-1}\, dt)$. L is uniformly elliptic hence $\mu \geqslant a(x) \geqslant \mu^{-1} > 0$; therefore ρ is smooth and bounded (below) from zero. $\{f, g\} = \int_\Omega fg\rho$ defines an equivalent inner product on $H^0 = H^0(\Omega)$, and $\{Lf, g\} = \{f, Lg\}$ for $f, g \in H_\partial^2$. It is not hard to show that θ and ψ are related by $\psi = \rho\theta$, and so θ and ψ share regions of positivity and negativity. A similar argument can be applied in case (b), for which $\rho(x) = (\prod a_i(x_i))^{-1} \operatorname{Exp}(\sum \int^{x_i} b_i(t)\, a_i^{-1}(t)\, dt)$. In the case of constant coefficients (c), we can pick a new basis for $\mathbb{R}^n$ with respect to which the second order part of L is Δ (see [19, p. 60], or [23, pp. 60–62]). The first order part of L again has constant coefficients, which reduces this case to (b). In case (d), the "weight function" $\rho(x) = \operatorname{Exp} q(x)$ makes L self-adjoint, and we proceed as before. (A general framework which includes (d) may be found in [17, pp. 687–690].)

It is well known that (with respect to the eigenvalue problem $Lu = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$) a linear operator of the form of L has a simple real eigenvalue λ_1, that the corresponding eigenfunction θ_1 is strictly positive in Ω, and that all other eigenvalues λ_i satisfy $\operatorname{Re} \lambda_i \leqslant \lambda_1$ [19, pp. 89–91; 20].). The adjoint operator is of the same form and consequently ψ_1 is strictly positive in Ω as well. The hypotheses of Theorem 2(e) imply that $\lambda_1 = 0$, and so θ and ψ share regions of positivity ($\Omega_+ = \Omega$) and negativity ($\Omega_- = \phi$).

5. The Bifurcation Equation—Multiple Resonance

Proof of Theorem 3. The main idea is first to modify the auxiliary equation (6) slightly so as to improve the structure of the solution set, then to solve the bifurcation equation (4) exactly, and finally to solve the original auxiliary equation by a limit process. Fix $\epsilon > 0$; we will perturb (6) by less than ϵ.

We first write (6) in the form

$$v = K(I - Q)\, N(v + \lambda e\theta) \tag{6}$$

where $\lambda e\theta = c\theta = \sum c_i\theta_i$, $\lambda > 0$, and e is a unit vector in $\mathbb{R}^M$ ($\sum e_i^2 = 1$). We assert that for each such e, the map $v \to K(I - Q)\, N(v + \lambda e\theta)$ becomes a contraction for sufficiently large λ. This follows directly from the following lemma, to be proved later.

Lemma 3. *Let e_0 be fixed and $\eta > 0$ be arbitrary. Then there exists $\lambda_0 > 0$ such that for $\lambda \geqslant \lambda_0$, the map $v \to g(v + \lambda e_0\theta)$ is Lipschitzian of constant $\leqslant \eta$*

as a map from $B(M_1) = \{w \in H_\partial^{2m} \mid \| w \|_{H^{2m}} \leqslant M_1\}$ *into* H^0. *Moreover,* λ_0 *can be chosen so that it works for all unit vectors* e *in a neighborhood of* e_0.

The unit sphere $\{e \mid \sum e_i^2 = 1\}$ in $\mathbb{R}^M$ is compact and hence is covered by a finite number of these neighborhoods. It follows that there is a $\lambda_1 > 0$ making the map $v \to K(I - Q)\, N(v + \lambda e\theta)$ a uniform contraction for all unit vectors e and all $\lambda \geqslant \lambda_1$. For any such fixed e and λ, this map has a unique fixed point $v_1(\lambda e)$, and $v_1(\lambda e)$ depends smoothly (C^1) on λ and e (see [12, p. 9]). We will later fix $\lambda \geqslant \lambda_1$, and view v_1 as a diffeomorphism from the unit sphere in $\mathbb{R}^M$.

We observe that for any fixed e on the sphere, there is a $\lambda_2 \geqslant \lambda_1$ such that for $\lambda \geqslant \lambda_2$,

$$I(v_1, \lambda e) \equiv \int \lambda e\psi\{QN(v_1 + \lambda e\theta)\} > 0. \tag{7}$$

This follows from (2) and Lemma 1, if $\lambda e\psi$ and $\lambda e\theta$ share regions of positivity and negativity. "Sharing regions" is the case if L is self-adjoint or if L satisfies the alternative hypotheses in Theorem 3, as in the proof of Theorem 2. λ_2 may be chosen to work for all unit vectors in a neighborhood of e, and so by compactness (as before), we may take λ_2 so that (7) holds over the entire unit sphere. A large ball $D = \{\lambda e \in \mathbb{R}^M \mid \lambda \leqslant \lambda_2\}$ will be the parameter space for a family of Schauder maps, as was $[p, q]$ in Lemma 2.

We now define two mappings E and F, $E\colon B(M_1) \times D \to B(M_1) \times B(M_1)$ and $F\colon B(M_1) \times D \to \mathbb{R}^M$. Let $E(v, \lambda e) = (v, K(I - Q)\, N(v + \lambda e\theta))$ and $F(v, \lambda e) = \{d_i\}$ where the d_i are the scalar components of $QN(v + \lambda e\theta)$ with respect to the basis $\{\psi_i\}$. We wish to solve $F(v, \lambda e) = \mathbf{0}$ (4) for some $(v, \lambda e) \in E^{-1}(\Delta)$ (6); Δ is the diagonal set in $B(M_1) \times B(M_1)$. The range of $K(I - Q)\, N$ is a compact subset of $B(M_1)$ in the H_∂^{2m} topology, so there is a finite set in H_∂^{2m} that is close ($(\epsilon/2)$-dense) to this range. Let R be an H_∂^{2m} orthogonal projection onto the linear span of this finite set, hence $\| Rv - v \|_{H^{2m}} < \epsilon/2$ for all v in Range $K(I - Q)\, N$. We define $E_1\colon RB(M_1) \times D \to RB(M_1) \times RB(M_1)$ by $E_1(v, \lambda e) = (v, RK(I - Q)\, N(v + \lambda e\theta))$. The map $v \to RK(I - Q)\, N(v + \lambda e\theta)$ is also a uniform contraction for $\lambda = \lambda_2$ (since $\| R \| = 1$), so it has a unique fixed point $v_2 = v_2(\lambda_2 e)$ depending smoothly on $\lambda_2 e \in \partial D$. On this fixed point set $S_1 = \{(v_2, \lambda_2 e) \mid E_1(v_2, \lambda_2 e) \in \Delta\}$ (a smooth $M - 1$ sphere), the map $F(F(v_2, \lambda_2 e) = \mathbf{d}$, where $d_i = \int_\Omega \psi_i QN(v_2 + \lambda_2 e\theta))$, which maps ∂D into $\mathbb{R}^M \sim \{0\}$, maps S_1 with winding number 1 (with respect to $\mathbf{0} \in \mathbb{R}^M$). This last statement follows from topological degree theory; we define $H\colon [0, 1] \times S_0 \to \mathbb{R}^M$ by $H(t, (v_2, \lambda_2 e)) = (t)(\lambda_2 e) + (1 - t)\, F(v_2, \lambda_2 e)$, where $S_0 = \{(v_2, \lambda_2 e) \mid E(v_2, \lambda_2 e) \in \Delta\}$. H is a homotopy connecting the identity and $F|_{S_0}$, and H is nonvanishing by (7), and so the degree of $F|_{S_0}$ is one. Topological degree is preserved under small deformations; we can ensure that S_1 is close to S_0 by (if necessary) choosing a "denser" subspace of Range $K(I - Q)\, N$ upon which to project with R.

We now perturb (6) one more time. If E_1 is transversal to Δ, $E_1^{-1}(\Delta)$ is a manifold. If not, there is a $\xi \in \text{Range } R$ with small norm (specifically with $\| \xi \|_{H^{2m}} \leqslant \epsilon/2$) with E_1 transversal to $\Delta_\xi = \{(v, w) \in RB(M_1) \times RB(M_1) \mid v = w + \xi\}$ (see [1, pp. 45–48], or [11, pp. 60, 68–69]), and $E_1 |_{RB(M_1)\times \partial D}$ as well. It follows that $S = E_1^{-1}(\Delta_\xi)$ is a manifold in $RB(M_1) \times D$ of codimension equal to the codimension of Δ_ξ in $RB(M_1) \times RB(M_1)$; hence S has dimension M. The map $v \to RK(I - Q) N(v + \lambda_2 e\theta) + \xi$ is a uniform contraction as before for $\lambda_2 e \in \partial D$, and so there is a unique fixed point v (for which $E_1(v, \lambda_2 e) \in \Delta_\xi$). This fixed point set $S_2 = \{(v, \lambda_2 e) \in B(M_1) \times D \mid E_1(v, \lambda_2 e) \in \Delta_\xi, \lambda_2 e \in \partial D\}$ is smoothly (C^1) diffeomorphic to the sphere ∂D as before, and S_2 is the boundary of the M-manifold $E_1^{-1}(\Delta_\xi)$ by transversality. Moreover, F maps S_2 into $\mathbb{R}^M \sim \{\mathbf{0}\}$ with winding number 1 (about $\mathbf{0}$) since S_2 is close to S_1 (for ξ sufficiently small).

We now use the following lemma to solve (4) on $E_1^{-1}(\Delta_\xi)$:

LEMMA 4. *Let T be a compact topological M-manifold with boundary ∂T homeomorphic to ∂D (D is the M-ball, ∂D is the $(M - 1)$-sphere). Assume that F is a continuous map from T to $\mathbb{R}^M$ with $\mathbf{0} \notin F(\partial T)$ and such that $F\colon \partial T \to \mathbb{R}^M \sim \{\mathbf{0}\}$ with winding number (topological degree) 1 with respect to $\mathbf{0} \in \mathbb{R}^M$. Then $\mathbf{0} \in F(T)$.*

This lemma may be viewed as a partial result on the M-connectivity of the fixed point set of an M-parameter family of Schauder maps. This raises an interesting question; what stronger statements (more closely analogous to Lemma 2) hold about the unperturbed fixed point set S_0 ? (Is the previous transversality argument necessary?)

In our application, $T = E_1^{-1}(\Delta_\xi)$. Hence $F(v, \lambda e) = \mathbf{0}$ (4) for some $(v, \lambda e) \in E_1^{-1}(\Delta_\xi) \subset B(M_1) \times D$. For such $(v, \lambda e)$, $v = RK(I - Q) N(v + \lambda e\theta) + \xi$, and so (6) is approximately solved:

$$\begin{aligned}
&\| v - K(I - Q) N(v + \lambda e\theta)\|_{H^{2m}} \\
&\quad \leqslant \| v - RK(I - Q) N(v + \lambda e\theta)\|_{H^{2m}} \\
&\qquad + \| RK(I - Q) N(v + \lambda e\theta) - K(I - Q) N(v + \lambda e\theta)\|_{H^{2m}} \\
&\quad \leqslant \| \xi \|_{H^{2m}} + (\epsilon/2) \leqslant \epsilon.
\end{aligned}$$

This can be done for any small $\epsilon > 0$; choose a sequence $\{\epsilon_j\}_{j=1}^{\infty}$, $\epsilon_j \to 0^+$. For each ϵ_j, the associated $(v_j, (\lambda e)_j)$ solves (4) exactly and (6) approximately (to within ϵ_j). This sequence $\{(v_j, (\lambda e)_j)\}$ has a convergent subsequence, since the distance from $(v_j, (\lambda e_j))$ to the compact set $C = (\text{Range } K(I - Q) N) \times D$ tends to zero. The limit $(v_\infty, (\lambda e)_\infty)$ solves (4) and (6) exactly, and this completes the proof of Theorem 3.

Proof of Lemma 3. Since g' is continuous and $g'(s) \to 0$ as $|s| \to \infty$, there are positive constants M_2 and r such that $|g'(s)| < M_2$ (for all s) and $|g'(s)| < 2^{-1/2}\eta$ (for all s with $|s| \geqslant r$. By the Sobolev embedding theorem

[3, p. 221], there is a $\nu > 0$ depending on n with $\| u \|_{L^{2+\nu}} \leqslant M_3 \| u \|_{H^{2m}}$ (for some $M_3 > 0$, independent of $u \in H_\partial^{2m}$). Choose $\delta < (\frac{1}{2}\eta^2 M_2^{-2} M_3^{-2})^{(2+\nu)/\nu}$. Since meas$\{x \in \Omega \mid | e_0\theta(x)| \geqslant \alpha\}$ tends to meas Ω as $\alpha \to 0^+$ (using the unique continuation hypothesis), we can find an $\alpha > 0$ with meas $\Omega_1 < \delta/2$, where $\Omega_1 = \{x \in \Omega \mid | e_0\theta(x)| < \alpha\}$. Such an α may be chosen to do this for all e in a neighborhood of e_0, because $e\theta(x)$ depends continuously on e.

Now pick $\lambda_0 > \alpha^{-1}(r + 2M_1\delta^{-1/2})$. For $v, w \in B(M_1)$, $\| v \|_{H^0} \leqslant \| v \|_{H^{2m}} \leqslant M_1$, and if we define $\Omega_2 = \{x \in \Omega \mid | v(x)| \geqslant 2M_1\delta^{-1/2}\}$, then meas $\Omega_2 \leqslant \delta/4$. Similarly, $\Omega_3 = \{x \in \Omega \mid | w(x)| \geqslant 2M_1\delta^{-1/2}\}$ and meas $\Omega_3 \leqslant \delta/4$. Let $\Omega_\delta = \Omega_1 \cup \Omega_2 \cup \Omega_3$, so that meas $\Omega_\delta < \delta$. We compute:

$$\begin{aligned}
&\| g(v + \lambda e\theta) - g(w + \lambda e\theta)\|_{H^0}^2 \\
&\quad = \int_\Omega | g(v + \lambda e\theta) - g(w + \lambda e\theta)|^2 \\
&\quad = \int_{\Omega \sim \Omega_\delta} | g(v + \lambda e\theta) - g(w + \lambda e\theta)|^2 + \int_{\Omega_\delta} | g(v + \lambda e\theta) - g(w + \lambda e\theta)|^2 \\
&\quad \leqslant \int_{\Omega \sim \Omega_\delta} | g'(\beta)|^2 \, | v - w |^2 + \int_{\Omega_\delta} \| g' \|_{L^\infty}^2 \, | v - w |^2.
\end{aligned}$$

Here β lies between $v + \lambda e\theta$ and $w + \lambda e\theta$. On $\Omega \sim \Omega_\delta$, $| \beta | \geqslant r$, we continue:

$$\leqslant (\eta^2/2) \int_{\Omega \sim \Omega_\delta} | v - w |^2 + M_2^2 \int_{\Omega_\delta} | v - w |^2 \cdot 1$$

(using Holder's inequality on the second integral; $p = (2 + \nu)/2$, $q = (2 + \nu)/\nu$)

$$\begin{aligned}
&\leqslant (\eta^2/2) \int_\Omega | v - w |^2 + M_2^2 \left(\int_{\Omega_\delta} | v - w |^{2+\nu} \right)^{2/(2+\nu)} \left(\int_{\Omega_\delta} 1^{(2+\nu)/\nu} \right)^{\nu/(2+\nu)} \\
&\leqslant (\eta^2/2) \| v - w \|_{H^0}^2 + M_2^2 \| v - w \|_{L^{2+\nu}}^2 (\delta)^{\nu/(2+\nu)} \\
&\leqslant (\eta^2/2) \| v - w \|_{H^{2m}}^2 + M_2^2 M_3^2 \| v - w \|_{H^{2m}}^2 (\tfrac{1}{2}\eta^2 M_2^{-2} M_3^{-2}) \\
&\leqslant \eta^2 \| v - w \|_{H^{2m}}^2 .
\end{aligned}$$

This proves Lemma 3.

Proof of Lemma 4. If $0 \notin F(T)$, we may smoothly retract $F(T)$ to the $(M - 1)$-sphere in R^M. Composing this retraction with F, we may regard F as a smooth retraction of T onto its boundary ∂T (which is identified with the $(M - 1)$-sphere). This contradicts the no-retraction theorem of differential topology (see [11, p. 65]). This lemma may also be proved by writing the above composition as a commutative diagram, passing to another commutative diagram on the $M - 1$ cohomology level, and applying Lefschetz duality.

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