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The Zariski–Lipman Conjecture in the Graded Case

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INTRODUCTION

Let K be a field of characteristic 0, let R be a finitely generated reduced K-algebra, and let P be a prime ideal of R. The Zariski–Lipman conjecture asserts that if $\mathrm{Der}_K(R_P, R_P)$ (which may be identified with $(\mathrm{Der}_K(R, R))_P$) is R_P-free, then R_P is regular. It is known that if $\mathrm{Der}_K(R_P, R_P)$ is R_P-free, then R_P is a normal domain [5], and in the case where either R is a hypersurface [7, 8] or else R is a homogeneous complete intersection and P is the irrelevant ideal [6] (also, [4]) the conjecture has been verified. Our main objective here is to prove the conjecture in the case $R = \bigoplus_{i=0}^{\infty} R_i$ is graded by the nonnegative integers N, $R_0 = K$, and $P = m$, where $m = \bigoplus_{i=1}^{\infty} R_i$ is the irrelevant maximal ideal. (We do not require that R be generated by its one-forms.)

The paper concludes with a section containing several remarks about the inhomogeneous case, including a criterion for the freeness of the module of derivations of a two-dimensional local complete intersection which we feel may lead to a counterexample.

1. THE GRADED CASE

In this section R denotes a finitely generated reduced K-algebra graded by N, where K is a field of characteristic 0, such that $R_0 = K$, and m denotes the maximal ideal $\bigoplus_{i=1}^{\infty} R_i$.

Let $\mathscr{D} = \mathrm{Der}_K(R, R)$. *We assume, for the rest of this section, that* $\mathscr{D}_m$ *is free.* We represent R as S/I, where $S = K[X_1, ..., X_n]$ is a polynomial ring in which the X_i have positive integral degrees d_i, where $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$, and $I \subset (X_1, ..., X_n)^2 S$ is homogeneous. Our main result is then:

THEOREM. *Under the hypotheses above,* $I = (0)$. *In other words* $R = S$ *is a polynomial ring.*

This theorem establishes the Zariski–Lipman conjecture in the graded case.

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Proof. We denote by $x_1, \ldots, x_n$ the images of $X_1, \ldots, X_n$, respectively, in R. Thus, $R = K[x_1, \ldots, x_n]$. We let $F_1, \ldots, F_m \in S$ be a minimal system of homogeneous generators for I. We may inject $\phi: \mathscr{D} \to R^n$ by $\phi(D) = (D(x_1), \ldots, D(x_n))$. Let $^-$ denote reduction modulo I (i.e., $X_i^- = x_i$). Then ϕ maps $\mathscr{D}$ isomorphically onto the R-relations on the columns of the matrix $J = ((\partial F_i/\partial X_j)^-)$. If we grade R^n by assigning degree $-d_j$ to the jth free generator (i.e., $R^n = R(d_1) \oplus \cdots \oplus R(d_n)$, where, if E is graded, $E(t)$ denotes the graded module such that $E(t)_i = E_{t+i}$), then $\mathscr{D} \simeq \phi(\mathscr{D}) \subset R^n$ may be regarded as a homogeneous submodule of $\bigoplus_i R(d_i)$ and thus has an inherited grading. Since $\mathscr{D}$ is graded and $\mathscr{D}_m$ is R-free, $\mathscr{D}$ itself is R-free.

Our hypothesis and desired conclusion are unaffected by tensoring, over K, with an algebraic closure of K. Thus, we may assume that K is algebraically closed.

Now, it is easy to check that if $F \in S$ is a form, $\sum_{j=1}^n (\partial F/\partial X_j)(d_j X_j) = (\deg F)F$, and it follows that there is a unique derivation $D_0 \in \mathscr{D}$ such that $D_0(u) = (\deg u)(u)$ for each form $u \in R$. Thus, $D_0 = \phi^{-1}(d_1 x_1, \ldots, d_n x_n)$.

We next reduce, by induction on n (or on Krull dim R), to the case where the degree 0 form D_0 of $\mathscr{D}$ is part of a minimal homogeneous basis for $\mathscr{D}$. For assume that D_0 is not part of such a basis. Then it can be written $\sum_{t=1}^r u_t b_t$, where $u_1, \ldots, u_t$ are nonzero forms of positive degree and $b_1, \ldots, b_t$ is part of a minimal homogeneous basis for $\mathscr{D}$. Then $d_n x_n = D_0(x_n) = \sum_{t=1}^r u_t b_t(x_n)$ and since each $u_t \in m$ and $x_n \notin m^2$ (or else $X_n \in (X_1, \ldots, X_n)^2 + I = (X_1, \ldots, X_n)^2$), some $b_t(x_n) \notin m$. i,e., there is a homogeneous derivation $D \in \mathscr{D}$ such that $D(x_n) \in K - \{0\}$ (i.e., $D = b_t$); it follows that $\deg D = -d_n$. Suppose that $\deg x_m = \cdots = \deg x_n = d_n$ while $\deg x_j < d_n$ if $j < m$ (possibly, $m = n$). If $j < m$ we must have $\deg D(x_j) = d_j - d_n < 0$ or $D(x_j) = 0$, and the former is impossible. Thus $D(x_j) = 0$ for $j < m$ while for $m \leqslant j \leqslant n$, $D(x_j) \in K$. After a linear change of variables involving only $x_m, \ldots, x_n$ (the variables of biggest possible degree d_n), we can arrange that $D(x_j) = 0$ for $j < n$ while $D(x_n) = 1$. It follows that I is closed under the action of $\partial/\partial X_n$. Let $I_0 = I \cap K[X_1, \ldots, X_{n-1}]$. We claim that $I = I_0 S$. For if F were a form in $I - I_0 S$ of lowest possible degree c in X_n, then $\partial F/\partial X_n$ is of lower degree in X_n and in I, and hence in $I_0 S$, while $F - c^{-1} X_n(\partial F/\partial X_n)$ is also of lower degree in X_n and in I, and hence in $I_0 S$. Thus, $F = c^{-1} X_n(\partial F/\partial X_n) + (F - c^{-1} X_n(\partial F/\partial X_n)) \in I_0 S$.

But then $R = (K[X_1, \ldots, X_{n-1}]/I_0)[X_n]$, where X_n is an indeterminate over $R_0 = K[X_1, \ldots, X_{n-1}]/I_0$, and it easily follows that $\mathrm{Der}_K(R_0, R_0)$ is R_0-free: hence, by the induction hypothesis, $I_0 = (0)$, and then $I = (0)$.

Henceforth we assume that D_0 is part of a minimal homogeneous basis for $\mathscr{D}$, and since $\mathscr{D}$ is R-free, this basis is free, so that the exact sequence of graded R-modules and degree 0 maps given by

$$0 \longrightarrow R \xrightarrow{D_0} \mathscr{D} \longrightarrow T \longrightarrow 0, \qquad (*)$$

where $T = \mathscr{D}/RD_0$, is split.

It is convenient to assume from now on that the subsemigroup $\{i \in N: R_i \neq 0\}$ contains all sufficiently large positive integers: this is true after passing to a constant multiple of the original grading.

Let $X = \mathrm{Proj}(R)$. We know from the results of [5] that R is a normal domain, and so X is a normal variety. We regard X as the patching together of open affine subvarieties $X_u = \mathrm{Spec}([R_u]_0)$, where $u \neq 0$ is any form and $R_u = R[1/u]$. Then the $\{X_i\} = \{X_{x_i}\}$ are a cover. As usual, each graded module E of finite type over R gives rise to a coherent sheaf $E^\sim$ on X such that $\Gamma(X_u, E^\sim) = [E_u]_0$. A degree 0 map of graded modules induces a morphism of sheaves functorially, and so the exact sequence $(*)$ gives rise to a *split* exact sequence of sheaves:

$$0 \to \mathcal{O}_X \to \mathscr{D}^\sim \to T^\sim \to 0. \tag{$**$}$$

$\mathcal{O}_X$ is the structure sheaf on X. Let $R^{(t)}$ denote the graded K-algebra whose ith graded piece is R_{ti}, i.e., $R^{(t)} = \bigoplus_{i=0}^{\infty} R_{ti}$. Then we may choose q, a positive integer, such that $R^{(q)}$ is generated by $R_1^{(q)}$, and we may also regard X as $\mathrm{Proj}(R^{(q)})$. This gives an (arithmetically normal) projective embedding of X. The sheaf $L = R^{(q)}(1)^\sim$ is a very ample invertible sheaf on X.

The rest of the argument is devoted to establishing the following facts: $T^\sim$ is the tangent sheaf θ_X (the sheaf of germs of K-derivations) on X and is locally free. Let Ω_X be the cotangent sheaf on X (germs of Kähler differentials) and let $\hat{} = \mathbf{Hom}_{\mathcal{O}_X}(\ , \mathcal{O}_X)$. Then we may identify

$$\mathrm{Ext}^1_{\mathcal{O}_X}(T^\sim, \mathcal{O}_X) \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\theta_X, \mathcal{O}_X) \cong H^1(X, \theta_X\hat{}) \cong H^1(X, \Omega_X\hat{}\hat{}).$$

Let $\mathcal{O}_X{}^*$ be the sheaf of germs of units of $\mathcal{O}_X$. There is a map of sheaves $\mathcal{O}_X{}^* \to \Omega_X$ given locally by logarithmic differentiation ($\alpha \mapsto \alpha^{-1}\, d\alpha$, where α is a local section of $\mathcal{O}_X{}^*$), and this map induces a composite map

$$f\colon \mathrm{Pic}(X) = H^1(X, \mathcal{O}_X{}^*) \to H^1(X, \Omega_X) \to H^1(X, \Omega_X\hat{}\hat{}).$$

Now L corresponds to an element of $\mathrm{Pic}(X)$ and we show that the element of $\mathrm{Ext}^1_{\mathcal{O}_X}(T^\sim, \mathcal{O}_X) \cong H^1(X, \Omega_X\hat{}\hat{})$ represented by $(**)$ is $q^{-1}f(L)$. Since $(**)$ is split, it follows that $f(L) = 0$. But it is quite easy to show that when X is normal $f\colon \mathrm{Pic}(X) \to H^1(X, \Omega_X\hat{}\hat{})$ cannot kill an ample sheaf. We give a short proof of this fact below by reducing to the well-known classical case where X is a nonsingular projective curve.

It remains to verify these assertions. We first note that there is a natural map $\rho\colon \mathscr{D}^\sim \to \theta_X$, induced by restriction. On the open affine X_u corresponding to a form u, $\Gamma(X_u, \mathscr{D}^\sim) = [\mathrm{Der}_K(R, R))_u]_0 \cong [\mathrm{Der}_K(R_u, R_u)]_0$, and the grading is such that derivations of degree δ shift degrees by δ. If $\Delta \in [\mathrm{Der}_K(R_u, R_u)]_0$, then since Δ shifts degrees by 0, $\Delta \mid [R_u]_0 \in \mathrm{Der}_K([R_u]_0, [R_u]_0) = \Gamma(X_u, \theta_X)$. These maps patch to give the map $\rho\colon \mathscr{D}^\sim \to \theta_X$. We compute $\mathrm{Ker}\,\rho$. From the definition of ρ, on X_u we have $\Gamma(X_u, \mathrm{Ker}\,\rho) =$

$[\mathrm{Der}_{[R_u]_0}(R_u, R_u)]_0$. Now, an element of the module of derivations $\mathrm{Der}_{[R_u]_0}(R_u, R_u)$ is completely determined by how it maps $[R_u]_q = \Gamma(X_u, L)$, and if it has degree 0 it restricts to an $[R_u]_0$-linear map of $[R_u]_q$ to itself, i.e., to an element of $\mathrm{Hom}_{\Gamma(X_u, \mathcal{O}_X)}(\Gamma(X_u, L), \Gamma(X_u, L))$. Thus, patching, we have an injection

$$\mathrm{Ker}\,\rho \hookrightarrow \mathbf{Hom}(L, L) \cong \mathcal{O}_X,$$

where the last isomorphism identifies the global section 1 of $\mathcal{O}_X$ with the identity map id_L on L (we get this isomorphism because L is invertible). Moreover, $q^{-1}D_0$ is a global section of $\mathrm{Ker}\,\rho$ and, in fact, for each u its restriction to X_u induces the identity map on L. Thus, the element of $\mathrm{Hom}(\mathcal{O}_X, \mathrm{Ker}\,\rho) \cong \Gamma(X, \mathrm{Ker}\,\rho)$ represented by $q^{-1}D_0$ is an inverse for $\mathrm{Ker}\,\rho \to \mathcal{O}_X$, and we have the following commutative diagram of maps of sheaves with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{D_0} & \mathcal{D}^\sim & \longrightarrow & T^\sim & \longrightarrow & 0 \qquad (**)\\
 & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \zeta} & & \\
0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{q^{-1}D_0} & \mathcal{D}^\sim & \xrightarrow{\rho} & \theta_X. & & \qquad (\#)
\end{array}$$

It follows that there is an induced injection $\zeta: T^\sim \to \theta_X$. Since T is R-free, $T^\sim$ is a locally free sheaf on X. $T^\sim$ and θ_X are, moreover, both torsion-free of torsion-free rank equal to $\dim X = \dim R - 1$. It now follows that ζ is an isomorphism. To see this, we note that Coker ζ, if nonzero, is supported at a height one prime P of $\Gamma(X_u, \mathcal{O}_X) = [R_u]_0$ for some open affine X_u, since $T^\sim$ is locally free, $\theta_X/T^\sim$ is torsion, and X is normal. But if V is the stalk of $\mathcal{O}_X$ at P, V is a discrete valuation ring, and $(R_u)_P = V[t, t^{-1}]$, where t is any element of $[R_u]_1 - \{0\}$. But then, passing to stalks at P, we can see easily that ρ_P is surjective, which implies at once that ζ_P is an isomorphism.

Thus, the diagram above yields an isomorphism ζ of T with θ_X, and so we have that θ_X is locally free and that the sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{D_0} \mathcal{D}^\sim \xrightarrow{\rho} \theta_X \longrightarrow 0 \qquad (\#)$$

is a split exact sequence of locally free sheaves. This sequence represents an element of $\mathrm{Ext}^1_{\mathcal{O}_X}(\theta_X, \mathcal{O}_X)$, and since θ_X is locally free

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\theta_X, \mathcal{O}_X) \cong H^1(X, \mathbf{Hom}_{\mathcal{O}_X}(\theta_X, \mathcal{O}_X)).$$

By tracing back definitions we next make an explicit computation of Cech 1-cocycle in $H^1(X, \theta_X{}^\wedge)$ which represents the extension (#): this computation is made from our knowledge of ρ. (Then we use the "fact" that the extension is trivial.)

First, choose forms $u_0, \ldots, u_s \in R_q - \{0\}$ such that $X = \bigcup_i X_{u_i}$. Let $X_i = X_{u_i}$

Recalling the definition of L, we have that u_i spans $\Gamma(X_i, L) \simeq R_i$, and we can choose unique elements $\alpha_{ij} \in [R_{u_i u_j}]_0^*$ (D^* denotes the invertible elements of D) such that

$$u_j = \alpha_{ij} u_i \qquad \text{on} \quad X_i \cap X_j,\ 0 \leqslant i \leqslant s,\ 0 \leqslant j \leqslant s,$$

i.e., $c_L = ((i, j) \mapsto \alpha_{ij})$ is a Cech 1-cocycle which represents L. Consider the map $f\colon \operatorname{Pic}(X) \to H^1(X, \Omega_X^{\wedge\wedge})$ $(= H^1(X, \mathbf{Hom}_{\mathcal{O}_X}(\theta_X, \mathcal{O}_X))$ described earlier, induced by logarithmic differentiation. We establish that the Cech 1-cocycle $q^{-1}f(c)$ represents the element of $H^1(X, \mathbf{Hom}_{\mathcal{O}_X}(\theta_X, \mathcal{O}_X)) \simeq \operatorname{Ext}^1_{\mathcal{O}_X}(\theta_X, \mathcal{O}_X)$ which corresponds to the exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{D_0} \mathcal{D}^\sim \longrightarrow \theta_X \longrightarrow 0. \tag{\#}$$

First note that $q^{-1}f(c)$, by definition, is given by

$$(i, j) \mapsto q^{-1}(D \mapsto D(\alpha_{ij})/\alpha_{ij}),$$

where D represents an element of

$$\Gamma(X_i \cap X_j, \theta_X) = \operatorname{Der}_K([R_{u_i u_j}]_0, [R_{u_i u_j}]_0).$$

On the other hand, we can obtain a Cech 1-cocycle which represents (#) by first applying $\mathbf{Hom}_{\mathcal{O}_X}(\theta_X, \ )$ to (#), second, on each X_i choosing a lifting of the identity map in $\mathbf{Hom}_{\Gamma(X_i)}(\theta_{X_i}, \theta_{X_i})$ to $\mathbf{Hom}_{\Gamma(X_i)}(\theta_X, \mathcal{D}^\sim|_{X_i})$, and then considering the Cech 1-cocycle

$$(i, j) \mapsto \Delta_{ij} = (\text{the lifting on } X_i)|_{X_i \cap X_j} - (\text{the lifting on } X_j)|_{X_i \cap X_j};$$

this is just a matter of tracing definitions and identifications.

We pick an element h_i of

$$\mathbf{Hom}_{\Gamma(X_i)}(\theta_{X_i}, \mathcal{D}^\sim|_{X_i}) = \operatorname{Hom}_{[R_{u_i}]_0}(\operatorname{Der}([R_{u_i}]_0, [(R_{u_i})]_0), [\operatorname{Der}(R_{u_i}, R_{u_i})]_0)$$

which lifts the identity as follows: For convenience, let $u = u_i$. Given $D \in \operatorname{Der}([R_u]_0, [R_u]_0)$, there is a unique element $h_i(D) \in [\operatorname{Der}(R_u, R_u)]_0$ which extends D and vanishes on $u = u_i$. [To see that a derivation exists, first pick $D_1 \in [\operatorname{Der}(R_u, R_u)]_0$ such that D_1 extends D. This is possible, since the map of sheaves $\mathcal{D}^\sim \to \theta$ is already known to be surjective and X_i is affine. Then $D_1(u) \in R_u$ has degree q, and we can write $D_1(u) = r_0 u$ where $r_0 \in [R_u]_0$. Then $D_1 - (1/q) r_0 D_0$ extends D and kills u. If D_1, D_2 are two derivations which extend D and kill u, then $D_3 = D_1 - D_2$ kills $[R_u]_0[u, 1/u]$, and each form of R_u has its qth power in this ring. Since R_u is a domain and q is invertible, D_3 kills R_u.] Clearly, the map h_i taking D to $h_i(D)$ lifts the identity.

Thus, the cocycle

$$(i, j) \mapsto \Delta_{ij} = h_i|_{X_i \cap X_j} - h_j|_{X_i \cap X_j}$$

corresponds to the exact sequence (#).

We compute Δ_{ij} on $D \in \mathrm{Der}([R_{u_i u_j}]_0, [R_{u_i u_j}]_0)$: we know that $\Delta_{ij}(D)$ has the form $\lambda_D(D_0|_{X_i \cap X_j})$, where $\lambda_D \in \Gamma(X_i \cap X_j, \mathcal{O}_X)$, and then the required cocycle has the form

$$(i, j) \mapsto (D \mapsto \lambda_D)$$

(where $D \mapsto \lambda_D \in \mathbf{Hom}_{\mathcal{O}_X}(\theta_X, \mathcal{O}_X)$). The derivation $\Delta_{ij}(D)$ is completely determined by its value on $u_i|_{X_i \cap X_j}$. Now, with everything restricted to $X_i \cap X_j$, as necessary, we have, on $X_i \cap X_j$,

$$\begin{aligned}
\Delta_{ij}(D)(u_i) &= h_i(D)(u_i) - h_j(D)(u_i) \\
&= 0 - h_j(D)(\alpha_{ij}^{-1}u_j) \qquad \text{(by definition of } h_i\text{)} \\
&= -h_j(D)(\alpha_{ij}^{-1})u_j \\
&= -D(\alpha_{ij}^{-1})u_j \qquad \text{(by definition of } h_j\text{)} \\
&= -(-\alpha_{ij}^{-2})\, D(\alpha_{ij})u_j = \alpha_{ij}^{-1}D(\alpha_{ij})(\alpha_{ij}^{-1}u_j) \\
&= (D(\alpha_{ij})/\alpha_{ij})u_i,
\end{aligned}$$

while on $X_i \cap X_j$, $D_0(u_i) = qu_i$.

It follows that $\lambda_D = q^{-1}D(\alpha_{ij})/\alpha_{ij}$ and the cocycle is

$$(i, j) \mapsto (D \mapsto q^{-1}D(\alpha_{ij})/\alpha_{ij})$$

which is $q^{-1}c_L$, precisely as claimed.

Now, on the one hand, we have already shown, using the hypothesis of the Zariski–Lipman conjecture, that $q^{-1}c_L$ represents 0 in $H^1(X, \Omega_X^{\wedge\wedge})$, and hence so does c_L.

But, on the other hand, the following lemma asserts that this is *not* the case, and completes the proof of the Zariski–Lipman conjecture in the graded case.

LEMMA. *Let X be a normal reduced and irreducible projective variety over an algebraically closed field K of characteristic 0, and let L be an ample sheaf on X. Then the image of L under the map*

$$\mathrm{Pic}(X) \to H^1(X, \Omega_X^{\wedge\wedge})$$

induced by logarithmic differentiation is not 0.

Proof. If X is a nonsingular curve, i.e., a Riemann surface, this is truly a classical fact: in fact,

$$H^1(X, \Omega_X^{\wedge\wedge}) \cong H^1(X, \Omega_X) \cong H^0(X, \mathcal{O}_X) \cong K,$$

and the map described under a suitable identification of $H^1(X, \Omega_X^{\wedge\wedge})$ with $K \supset Z$, maps each line bundle to its Chern class or degree. In this case, L is ample if and only if it has positive degree, and the result is clear. [See the Remark following this proof.]

But the general case can be reduced easily to the case of a nonsingular curve. Let S be the singular locus of X and let $U = X - S$. We can choose a closed reduced and irreducible curve $Z \subset X$ such that $Z \cap S = \varnothing$, i.e., $Z \subset U$. (X is normal and so if $X = \mathrm{Proj}(R)$, S is defined by a homogeneous ideal I of R height 2 or more. Hence, there exists a proper ideal J generated by $(\dim R - 2)$ or fewer forms such that $I + J$ is primary to the irrelevant ideal, and we may take Z to be the curve defined by any homogeneous prime of coheight 2 which contains J.) Let Y be the normalization of Z. Thus, Y is a nonsingular curve and we have a finite morphism $Y \to X$ (the composite $Y \twoheadrightarrow Z \to X$), where $\mathrm{Im}(Y) = Z \subset U$; i.e., we have

$$Y \to U \to X,$$

where the second map is an open immersion. Since Y, U are nonsingular, we have canonical isomorphisms $\Omega_Y \cong \Omega_Y^{\wedge\wedge}$ and $\Omega_U \cong \Omega_U^{\wedge\wedge}$. We thus obtain a commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{Pic}(Y) & = & H^1(Y, \mathcal{O}_Y^*) & \longrightarrow & H^1(Y, \Omega_Y) & \xrightarrow{\cong} & H^1(Y, \Omega_Y^{\wedge\wedge}) \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} \\
\mathrm{Pic}(U) & = & H^1(U, \mathcal{O}_U^*) & \longrightarrow & H^1(U, \Omega_U) & \xrightarrow{\cong} & H^1(U, \Omega_U^{\wedge\wedge}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow{\scriptstyle\gamma} \\
\mathrm{Pic}(X) & = & H^1(X, \mathcal{O}_X^*) & \longrightarrow & H^1(X, \Omega_X) & \longrightarrow & H^1(X, \Omega_X^{\wedge\wedge}).
\end{array}$$

The arrow β is induced from α by the isomorphism, while γ is induced by the open immersion $U \to X$. (Note that if we have a morphism $Y \to X$, we do not get an induced map $H^1(X, \Omega_X^{\wedge\wedge}) \to H^1(Y, \Omega_Y^{\wedge\wedge})$ in general, although we do if Y, X are nonsingular or if the map is an open immersion. This is why we must be careful in choosing $Y \to X$ so that $(\mathrm{Im}\ Y) \subset U$.) Thus, we get a commutative diagram:

$$\begin{array}{ccc}
\mathrm{Pic}(Y) & \xrightarrow{f_Y} & H^1(Y, \Omega_Y^{\wedge\wedge}) \\
\uparrow & & \uparrow \\
\mathrm{Pic}(X) & \xrightarrow{f_X} & H^1(X, \Omega_X^{\wedge\wedge})
\end{array}$$

where f_Y, f_X are induced by logarithmic differentiation and the left vertical arrow by pullback. If L is ample on X, its pullback to Y will be ample ($Y \to X$ is finite, and Y is a smooth curve), and hence the pullback of L maps to a nonzero element of $H^1(Y, \Omega_Y^{\wedge\wedge})$. It follows that $f_X(L) \neq 0$. Q.E.D. for both the Lemma and the graded case of the Zariski–Lipman conjecture.

Remark. The following proof of the Lemma in the classical case was supplied by Lipman, who remarks that the steps are justified in [9, Chap. 2]:

> A divisor on a curve C over K is given by a family of "local equations" (i.e. a "repartition") $(f_P)_{P\in C}$, where $f_P \neq 0$ is in the function field $K(C)$, and $f_P \in \mathcal{O}_{C,P}$ for almost all P. Similarly, an element of $H^1(C, \Omega_C{}^1)$ can be specified by a family of differentials $(\omega_P)_{P\in C}$ with $\omega_P \in \Omega^1_{K(C)}$ and $\omega_P \in \Omega^1_{\mathcal{O}_{C,P}}$ for almost all P. Now the d.log map takes a divisor [given by] $(f_P)_{P\in C}$ to the element of $H^1(C, \Omega_C{}^1)$ given by $(df_P/f_P)_{P\in C}$. Moreover, the standard identification $H^1(X, \Omega_X{}^1) \to K$ is given by "sum of residues". But $\text{res}_P(df_P/f_P)$ is just the order of the zero of f_P at P (<0 if f_P has a pole). Hence $\sum_P \text{res}_P(df_P/f_P)$ is nothing but the degree of the divisor (f_P). Q.E.D.
>
> (Thus in char. p, the d.log image of an ample divisor is zero if p divides the degree.)

2. Remarks on the Nongraded Case

Remark 1. The graded case of the conjecture is not as special as it seems, since it has the following:

Corollary. *Let* (R, m) *be a complete reduced local ring with residue class field* $K \subset R$, *and suppose* char $K = 0$. *Then* R *is regular if and only if*:

(1) $\text{Der}_K(R, R)$ *is free and*

(2) *there exists a derivation* $D: R \to R$ *such that* $D(m) \subset m$ *and the induced map* $m/m^2 \to m/m^2$ *is the identity.*

Proof. The key point is that (2) is equivalent to assuming that R is the completion of a finitely generated graded K-algebra R' generated by its one-forms. But then, since $\text{Der}_K(R, R)$ is the completion of $\text{Der}_K(R', R')$, $\text{Der}_K(R', R')$ is free, and R' is a polynomial ring.

To see that (2) is equivalent to assuming that R is the completion of a graded ring generated by its one-forms, first suppose $R = R'$, where $R' = \bigoplus_i R_i'$. Define D by $D'(\sum_i f_i) = \sum_i if_i$ (where $f_i \in R_i'$).

Now suppose D is as described in (2). Let $R' = gr_m R = \bigoplus_{i=0}^{\infty} m^i/m^{i+1}$. It is easy to see that D induces a map $R_i' \to R_i'$ for all i and that this map is multiplication by i. We show that for every i and $u \in R_i'$ there is a *unique* element $h_i(u) \in m^i$ such that $h_i(u) \equiv u$ modulo m^{i+1} and $D(h_i(u)) = ih_i(u)$. We first define $T_i : m^i \to m^i$ as follows:

Given $v_1 \in m^i$, let v_t be defined recursively by

$$v_{t+1} = v_t - (1/t)\,(Dv_t - iv_t), \qquad t \geqslant 1. \tag{*}$$

Then the v_t satisfy

$$v_{t+1} \equiv v_t \bmod m^{i+t}, \tag{1_t}$$

$$v_t \equiv v_1 \bmod m^{i+1}, \tag{2_t}$$

$$D(v_t) \equiv iv_t \bmod m^{i+t}, \tag{3_t}$$

for all t, as is readily established by induction. The hardest part is to deduce (3_{t+1}) from (1_t), (3_t), and $(*)$. Let $w = Dv_t - iv_t$. By (3_t), $w \in m^{i+t} \Rightarrow Dw - (i + t)\, w \in m^{i+t+1}$. But $v_{t+1} = v_t - (1/t)w$ so that

$$\begin{aligned} D(v_{t+1}) - iv_{t+1} &= D(v_t - (1/t)w) - i(v_t - (1/t)w) \\ &= Dv_t - (1/t)\, Dw - iv_t + (i/t)w \\ &= (Dv_t - iv_t) - (1/t)\, Dw + (i/t)w \\ &= w - (1/t)\, Dw + (i/t)w = -(1/t)\,(Dw - (i + t)w) \in m^{i+t+1}, \end{aligned}$$

as required.

Thus, $\{v_t\}$ is a Cauchy sequence (by 1_t)) and we may let

$$T_i(v_1) = \lim_t v_t \in m^i.$$

It is easy to check that

(a) T_i is K-linear,

(b) $T_i(v_1) \equiv v_1 \bmod m^{i+1}$ (by 2_t)), and

(c) $D(T_i(v_1)) = iT_i(v_1)$ (from 3_t)).

Moreover, one can easily check that if $v_1 \in m^{i+1}$, then $v_t \in m^{i+t}$ for all t, whence $T_i(v_1) = 0$, so that T_i kills m^{i+1} and so induces a K-linear map

$$h_i : R_i' = m^i/m^{i+1} \to m^i$$

such that

$$(m^i \to m^i/m^{i+1}) \circ h_i = \mathrm{id}_{R_i'}\,. \tag{#}$$

To establish our earlier claim, we must show that if $v \in m^i$, $v \equiv u$ modulo m^{i+1} and $D(v) = iv$, then $v = h_i(u)$, i.e., $v = T_i(v)$. But it is immediate from $(*)$ by induction on t that $v_t = v$ for all t in this case.

Now, if $u \in R_i'$, $u' \in R_j'$, then $y = h_i(u)\, h_j(u')$ has the properties

$$y \equiv uu' \text{ modulo } m^{i+j} \qquad \text{and} \qquad D(y) = (i + j)\, y.$$

Thus, $h_{i+j}(uu') = h_i(u)\, h_j(u')$. It follows that the h_i together yield a K-homomorphism h of rings

$$R' \xrightarrow{h} R.$$

It is easy to check that h induces an isomorphism $\hat{R}' \simeq R$. Q.E.D.

Remark 2. We simply want to make explicit the observation that if there is a Cohen–Macaulay counterexample to the Zariski–Lipman conjecture, there is also a Gorenstein counterexample. In fact, when R is Cohen–Macaulay normal of finite type over K (say char $K = 0$), and $\dim R = d$, then $(\Omega^d_{R/K})^{**}$ (where $*$ is $\mathrm{Hom}_R(\ , R)$) is a canonical module, and this is canonically isomorphic with $(\wedge^d(\Omega^1_{R/K}))^{**} \cong (\wedge^d((\Omega^1_{R/K})^*))^* \cong (\wedge^d \mathrm{Der}_K(R, R))^*$. For all P such that $(\mathrm{Der}_K(R, R))_P$ is free, we have that

$$((\Omega^d_{R/K})^{**})_P \cong R_P ,$$

so that R_P is Gorenstein.

Remark 3. We record the following observation (see [1]) of Becker and Rego. If R is, say, an analytic local ring, and $\mathrm{Der}_{\mathbb{C}}(R, R)$ is free, then the ring of higher order derivations is free as an R-module and generated by the 1-derivations $\mathrm{Der}_{\mathbb{C}}(R, R)$. Hence, Nakai's conjecture (generation of the ring of higher derivations by the 1-derivations $\Rightarrow$ regular) implies the Zariski–Lipman conjecture. (The Becker–Rego result is proved thus: let $D_1 ,..., D_d$ be a free basis for $\mathrm{Der}_{\mathbb{C}}(R, R)$. Let $\mathscr{D}_n$ be the set of higher derivations of order $\leqslant n$. Let F be the free module on the basis of all d-tuples $(i_1 ,..., i_d)$ of nonnegative integers with $\sum_r i_r \leqslant n$, and map $F \to \mathscr{D}_n$ by $(i_1 ,..., i_d) \mapsto D_1^{i_1} \cdots D_d^{i_d}$. One checks easily that this map is an isomorphism off the singular locus. Since R is normal, the singular locus has codimension 2, and F, $\mathscr{D}_n$ are reflexive, it follows that $F \to \mathscr{D}_n$ is an isomorphism for all n. Q.E.D.

Remark 4. Probably, the next case of the conjecture one should attack is that of a two-dimensional complete intersection. For simplicity, let us assume that R is a reduced complete intersection which is a complete local ring of dimension 2 and embedding dimension n. We may assume $n \geqslant 4$, since the result is known for hypersurfaces. Moreover, we later assume that R is normal (has an isolated singularity). We also assume, for simplicity, that the residue class field is $\mathbb{C} \subset R$.

We want to give criteria for $\mathrm{Der}_{\mathbb{C}}(R, R)$ to be free. We have in mind the possibility of giving a counterexample to the Zariski–Lipman conjecture (and, hence, also, to the Nakai conjecture).

We fix some notation.

Let $S = \mathbb{C}[[x_1 ,..., x_n]]$, let $m = (x_1 ,..., x_n)S$, let $f_1 ,..., f_{n-2}$ be an S-regular sequence in m^2, let $I = (f_1 ,..., f_{n-2})S$, let $^-$ denote reduction modulo I, and let $R = \bar{S} = S/I$. If R is to yield a counterexample to the Zariski–Lipman conjecture, it must be a normal domain. Hence, assume also that I is prime and that R is normal. Since R is a complete intersection and, so, Cohen–Macaulay, this is equivalent to assuming that R has an isolated singularity at m, i.e., that the $(n-2)$-size minors of

$$J = ((\partial f_i / \partial x_j)^-)$$

generate an ideal Q in R primary to $\bar{m}$. Thus, we know $\text{depth}_Q R = \text{depth}_m R = 2$.

We have an exact sequence:

$$0 \longrightarrow \text{Der}_{\mathbf{C}}(R, R) \longrightarrow R \xrightarrow{J} R^{n-2}$$

where, as indicated, the map $R^n \to R^{n-2}$ has matrix J. We have a map

$$J^t : (R^{n-2})^* \to (R^n)^*,$$

and hence

$$\begin{array}{ccc} \Lambda^{n-2}(R^{n-2})^* & \longrightarrow & \Lambda^{n-2}(R^n)^* \\ \uparrow \cong & & \downarrow \cong \\ R & & \Lambda^2(R^n) \end{array} \qquad (\#)$$

This determines an element of $\Lambda^2(R^n)$, unique up to multiplication by units of R. (The isomorphisms $R \cong \Lambda^{n-2}(R^{n-2})^*$ and $\Lambda^{n-2}(R^n)^* \cong \Lambda^2(R^n)$ are not canonical: the second is determined by a choice of generator for $\Lambda^n(R^n) \cong R$.)

CRITERION. $\text{Der}_{\mathbf{C}}(R, R)$ *is free if and only if the element of* $\Lambda^2(R^n)$ *determined in this way is decomposable, i.e., has the form* $\lambda \wedge \mu$, *where* $\lambda, \mu \in \Lambda^1(R^n)$.

Proof. $\text{Der}_{\mathbf{C}}(R, R)$ has rank 2. Hence, it is free if and only if there is a $2 \times n$ matrix M over R such that

$$0 \longrightarrow R^2 \xrightarrow{M} R^n \xrightarrow{J} R^{n-2}$$

is exact. The results of [2] assert that this sequence is exact if and only if $MJ = 0$ and depth $I_2(M) \geq 2$. The conditions on λ, μ imply that we may take $M = \left[\begin{smallmatrix}\lambda \\ \mu\end{smallmatrix}\right]$. The coordinates of $\lambda \wedge \mu$ in the usual basis for $\Lambda^2 R^n$ are the 2×2 minors of M (up to sign), and hence these are the same (up to sign) as the $(n-2) \times (n-2)$ minors of J, i.e., $I_2(M) = Q$ has depth 2.

On the other hand, given the existence of $M = \left[\begin{smallmatrix}\lambda \\ \mu\end{smallmatrix}\right]$, the results of [3] yield at once the indicated element of $\Lambda^2(R^n)$ (constructed in (#)) is a multiple of $\lambda \wedge \mu$; since depth $Q = 2$, the multiplier must be a unit, which can be absorbed into μ. Q.E.D.

Remark 5. We retain all the notation and hypotheses of the fourth paragraph of Remark 4, but we now want to specialize the preceding remark to the case $n = 4$. Let

$$N = (\partial f_i / \partial x_j),$$

so that $\bar{N} = J$, and let Δ_{pq} be the determinant of the 2 by 2 submatrix of N formed from the pth and qth rows if $p < q$ ($\Delta_{pq} = -\Delta_{pq}$). Using the bases already chosen to make identifications, we see that Im $\Lambda^2(J^t)$ is generated by

$\alpha = \sum_{i<j} \bar{\Delta}_{ij} e_i^* \wedge e_j^*$. Of course, α is a priori decomposable in $\wedge^2(R^4)^*$. But the corresponding element β in $\wedge^2 R^4$ under the identification induced by $\wedge^2 R^4 \otimes \wedge^2 R^4 \to \wedge^4 R^4 \cong R$ (where $e_1 \wedge e_2 \wedge e_3 \wedge e_4 \mapsto 1$) is $\sum_{i<j} (-1)^{i+j+1} \bar{\Delta}_{p,q} e_i \wedge e_j$ where for each $i < j$, p, q are chosen so that $p < q$ and $\{i, j, p, q\} = \{1, 2, 3, 4\}$. Change bases: let $f_1 = e_2$, $f_2 = -e_1$, $f_3 = -e_4$, $f_4 = e_3$. Then

$$\beta = \sum_{i<j} \bar{\Delta}_{p,q} f_i \wedge f_j,$$

where

$$\begin{aligned} p, q &= 3, 4 && \text{if } i, j = 1, 2, \\ p, q &= 1, 2 && \text{if } i, j = 3, 4, \\ p, q &= i, j && \text{in all other cases.} \end{aligned}$$

The decomposability of this element β obtained by switching 2 "complementary" Plücker coordinates in the decomposable α is not easy to decide, with one notable exception: if $\bar{\Delta}_{12} = \bar{\Delta}_{34}$, then, evidently, the decomposability of α implies the decomposability of β. Hence:

COROLLARY. *With the notation and hypotheses of Remark* 4, *fourth paragraph, with* $n = 4$, *if* $\bar{\Delta}_{12} = \bar{\Delta}_{34}$, *then* $\mathrm{Der}_{\mathbb{C}}(R, R)$ *is free*; *i.e., if*

$$\partial(f_1, f_2)/(\partial(x_1, x_2)) \equiv \partial(f_1, f_2)/(\partial(x_3, x_4)) \text{ modulo } (f_1, f_2),$$

then $\mathrm{Der}_{\mathbb{C}}(R, R)$ *is free.*

Thus, if f_1, f_2 are an S-sequence in m^2, the Zariski–Lipman conjecture implies that if $S/(f_1, f_2)$ has an isolated singularity at the origin then

$$\partial(f_1, f_2)/(\partial(x_1, x_2)) \neq \partial(f_1, f_2)/(\partial(x_3, x_4)).$$

I do not know whether even this is true.

Finally, we give one criterion for the freeness of $\mathrm{Der}_{\mathbb{C}}(R, R)$ intermediate between the corollary above and the decomposability of β.

PROPOSITION. *With the notation and hypotheses of Remark* 4, *fourth paragraph, with* $n = 4$, *if* $r_{ij} \in R$, $1 \leqslant i < j \leqslant 4$, *give a relation* $\sum_{ij} r_{ij} \bar{\Delta}_{ij} = 0$ *which is "nondegenerate" in the sense that* $r = r_{12} r_{34} - r_{13} r_{24} + r_{14} r_{23} \not\equiv 0$ *modulo* $\bar{m}$, *then* $\mathrm{Der}_{\mathbb{C}}(R, R)$ *is R-free.*

Proof. Let C_i be the column

$$\begin{bmatrix} \overline{\partial f_1/\partial x_i} \\ \overline{\partial f_2/\partial x_i} \end{bmatrix}.$$

Let E_{ij}, $i < j$, be the 2×4 matrix whose ith column is C_j, whose jth column is $-C_i$, and whose other columns are 0. Let $E = \sum_{ij} r_{ij} E_{ij}$. We show that the sequence

$$0 \longrightarrow R^2 \xrightarrow{E} R^4 \xrightarrow{J} R^2$$

is exact. By [2], it suffices to show that $EJ = 0$ and that $I_2(E)$ (the ideal generated by the 2×2 minors of E) is equal to $I_2(J) = Q$.

Let $U = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Then $E_{ij}J = \bar{\Delta}_{ij} U$, whence

$$EJ = \sum r_{ij} E_{ij} J = \left(\sum r_{ij} \bar{\Delta}_{ij}\right) U = 0U = 0.$$

It remains to show that $I_2(E) = I_2(J)$. Let D_{ij} be the 2×2 minor of E formed from the ith and jth columns, $i < j$.

Define $r_{ii} = 0$ and $r_{ji} = -r_{ij}$, so that $A = (r_{ij})$ is skew-symmetric. Then the ith column E_i of E is

$$\sum_s r_{is} C_s ,$$

whence

$$D_{ij} = \sum_{s,t} r_{is} r_{jt} \bar{\Delta}_{st}$$

or

$$D_{ij} = \sum_{s<t} (r_{is} r_{jt} - r_{it} r_{js}) \bar{\Delta}_{st} .$$

We can view this as a system of six linear equations in six unknowns. The matrix is $\Lambda^2 A$, whence the determinant is $\det(\Lambda^2 A) = (\det A)^r$ and $\det A = (r_{12}r_{34} - r_{13}r_{24} + r_{14}r_{23})^2$, i.e., $\det(\Lambda^2 A) = r^6$. Since $r \notin \bar{m}$, we can solve for the $\bar{\Delta}_{st}$ in terms of the D_{ij}, and, of course, conversely. Q.E.D.

The earlier corollary is the special case $r_{12} = -r_{34} = 1$, $r_{ij} = 0$ otherwise.

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