ASYMPTOTIC EVAPORATION OF SPHERICAL DROPS

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ABSTRACT
    The equations governing spherically symmetric vaporization of liquid
    drops in an infinite hot environment are solved for conditions typi-
    cal of drop shrinkage in the asymptotic stage which is controlled by
    the transport of heat and mass. The Jakob number for drop evapora-
    tion is derived. The vaporization constant is determined whose mag-
    nitude is confined within the upper and lower limits. The lower limit
    coincides with the previously reported quasi-steady approximation, while
    the upper limit is a constant multiple of the lower limiting value.
    The effects of radial convection resulting from unequal phase den-
    sities, high system pressure, and vapor diffusion in the environment
    are established and the regions of applicability of the quasi-
    steady approximation are determined.

Introduction

    Liquid drops are commonly observed in nature and in human activities,
particularly in many industrial and biological systems and processes. The
diminution of liquid droplets by evaporation is important in various fields
of technology, for example combustion, manufacturing, and spray cooling.
Fuchs [1] has conducted a comprehensive survey of the literature on the kin-
etics of droplet evaporation at quasi-steady state. Recent works on the
quasi-steady phase of droplet evaporation are numerous [for example, 2
through 7].

    Spalding [2] has introduced a transfer number as the driving force for
property transport between the drop surface and the surrounding gas medium.
This dimensionless quantity has been found to be useful in droplet evapora-

tion as well as combustion. Godsave [4] has proposed two distinct mechanisms
for determining the rates of decrease in size of the drops during the processes
of evaporation and combustion: the evaporation rate is determined by mass-
diffusion processes when the temperature of the surrounding gas is substan-
tially the same as that of the individual drops (the temperature concerned
being low in relation to the boiling point of the drops). On the other hand,
when the difference in temperatures between the drops and the surrounding
gas is considerable, the shrinkage rate is determined by the rate of heat
transfer to the drop. The quasi-steady model [3-6] predicts that the shrinkage
of liquid droplets evaporating in a hot environment obeys the "square law":
a linear variation in the square of drop diameter with time. The predicted
evaporation times agree well, in the asymptotic stage (at large times), with
the results of laboratory-scale tests on single drops [3-7]. However, the
quasi-steady model fails under zero gravity condition [8] or high gas pres-
sure [9]. References 8, 9 and 10 have proposed the transient diffusion model
taking into account the influence of the transient processes in the gas.
While both references 8 and 9 have ignored the convective effects on drop
evaporation, the complicated formulation in reference 10 prohibits the possi-
bility of finding the exact solution.

The present work solves the equations governing the asymptotic stage of
spherically symmetric drop evaporation limited by heat and mass transfer. The
solutions are exact and provide adequate descriptions of drop evaporation
during all but the earliest stages in which fluid inertia, viscosity and sur-
face tension may be important. Since the duration of the asymptotic phase
dominates the droplet lifetime, the errors introduced by employing the solu-
tions to predict the evaporation times would be insignificant.

Analysis

Consider a spherical drop of initial radius R_0 being placed in an infinite
hot environment at temperature T_∞ . Let the origin of spherical coordinates
be fixed at the drop center, which is at rest. r measures the radial distance.
$T(t,r)$ is the temperature of the surrounding gas near the drop, where t de-
notes the time.

The equation of continuity for spherical symmetry and an incompressible
fluid can be integrated to give

$$ur^2 = f(t)$$

in which u represents the radial velocity of the gas. The quantity ur^2 can be evaluated in terms of its value at the drop radius R(t) as .

$$ur^2 = -E\dot{R}R^2 \qquad (1)$$

wherein E is defined as the density ratio of the liquid and gas phases minus unity, i.e. $E = \rho_\ell/\rho_g - 1$, and $\dot{R}$ is the velocity of the drop surface resulting from evaporation.

The energy equation for spherical symmetry reads

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial r} = \frac{a}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r}\right) \qquad (2)$$

a is the thermal diffusivity of the gas phase. The initial and boundary conditions are

$$T(0,r) = T_\infty \qquad (3)$$

$$T(t,\infty) = T_\infty, \qquad T(t,R) = T_s \qquad (4)$$

where T_s signifies the saturation temperature of the liquid. The heat balance equation for the drop is

$$k \frac{\partial T(t,R)}{\partial r} = \rho_\ell \dot{R}L \qquad (5)$$

with k and L denoting the gas thermal conductivity and the latent heat of vaporization, respectively.

Now, a similarity transformation according to

$$T^*(N) = \frac{T_\infty - T(t,r)}{T_\infty} ; \quad N = \frac{r}{2(at)^{1/2}} ; \quad R = -2B(at)^{1/2} \qquad (6)$$

is performed on Eqs. (2) through (5). It yields

$$\ddot{T}^* = 2(-N - EB^3N^{-2} - N^{-1})\dot{T}^* \qquad (7)$$

subject to

$$T^*(\infty) = 0; \quad T^*(-B) = T_0^*; \quad \dot{T}^*(-B) = 2BM \qquad (8)$$

Here, the superscript · indicates the derivatives with respect to N; B, vaporization constant; $T_0^* = (T_s - T_\infty)/T_\infty$; $M = \rho_\ell L/(\rho_g C_p T_\infty)$; and C_p, the constant-pressure specific heat of the gas phase. Integrating Eq. (7) twice, the solution is found as

$$T^* = -2MB^3 \exp(B^2 - 2EB^2) \int_N^\infty x^{-2} \exp(-x^2 - 2EB^3 x^{-1}) dx \qquad (9)$$

The constant B is determined from the second expression of Eq. (8) to be

$$\frac{T_o^*}{M} = J_a = \phi(E,B) = -2B^3\exp(B^2-2EB^2) \int_{-B}^{\infty} x^{-2}\exp(-x^2-2EB^3x^{-1})dx \qquad (10)$$

Ja is the Jakob number for drop evaporation defined as $\rho_g C_p \Delta T/(\rho_\ell L)$, where ΔT is the degree of superheating $(T_\infty - T_s)$.

It is obvious that the radius-time history given in Eq. (6), $R = -2B(at)^{1/2}$ does not satisfy the initial condition $R(0) = R_o$. This anomaly arises from the complete neglect of inertia, surface tension and viscous terms, namely the equation of motion in the formulation. As a remedy, one writes the radius time history as

$$R = R_o - 2B(at)^{1/2} \qquad (11)$$

which is mathematically inapplicable to the very beginning of the shrinkage. The droplet lifetime can then be expressed as

$$t_{life} = \frac{R_o^2}{4B^2 a} \qquad (12)$$

When the ambient temperature T_∞ is substantially the same as the drop temperature, the evaporation rate is strongly influenced by mass diffusion of the vapor from the drop surface into the surrounding medium [4]. Let C be the mass concentration of the vapor in the surrounding gas medium; C_o, the initial value and the value at large r; and C_s, at saturated state. Then, the mass balance equation for the vapor in the mixture in which chemical reaction effects are absent reads

$$\frac{\partial C}{\partial t} - E\dot{R}\frac{R^2}{r^2}\frac{\partial T}{\partial r} = \frac{D}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial C}{\partial r}) \qquad (13)$$

wherein D is the mass diffusivity of the vapor. The initial and boundary conditions are

$$C(0,r) = C_o \qquad (14)$$

$$C(t,\infty) = C_o, \quad \rho_\ell\dot{R} = D\frac{\partial C(t,R)}{\partial r} \qquad (15)$$

The assumption of thermodynamic equilibrium at the droplet surface allows the use of the Clausius-Clapeyron equation. It yields

$$\frac{C(t,R)}{C_s} = \exp\{\frac{ML}{RT_s}[1 - \frac{T_s}{T(t,R)}]\} \qquad (16)$$

in which M is the molecular weight of the vapor and R denotes the gas constant. Equation (16) can be approximated as

$$\frac{C(t,R)-C_s}{C_s} = \frac{ML}{\bar{R}T_s^2} [T(t,R)-T_s] \tag{17}$$

wherein $C_s ML/(\bar{R}T_s^2) = (\partial C/\partial T)_{p,s}$. The thermal and concentration fields are coupled through Eq. (17).

With the aid of Eq. (1), Eqs. (2) through (5) and (13) through (15) are solved using the method of similarity transformation: utilizing Eq. (6) and

$$C^*(N_c) = \frac{C(t,r)-C_o}{C_o}, \quad N_c = \frac{r}{2(Dt)^{1/2}} = NL_e, \quad Q = \frac{\rho_\ell}{C_o} \tag{18}$$

one gets Eq. (9) and

$$C^*(Le^{1/2}N) = 2Q(BLe^{1/2})^3 \exp[B^2Le(1-2E)] \int_{Le^{1/2}N}^{\infty}$$

$$x^{-2}\exp(-x^2-2EB^3Le^{3/2}x^{-1})dx \tag{19}$$

wherein Le denotes the Lewis number. The constant B for the combined heat and mass transfer mechanism is evaluated by means of Eq. (17), i.e.

$$T_o^* = M\phi(E,B) + Q[\phi(E,BLe^{1/2}) - C_o^*]/m \tag{20}$$

in which

$$m = -(\partial C/\partial T)_{p,s}(T_\infty/C_o); \quad C_o^* = (C_s-C_o)/C_o$$

Results and Discussion

The Jakob number is the product of the gas-liquid density ratio and the dimensionless superheat $C_p\Delta T/L$. The function ϕ in Eq. (10) has been numerically integrated with the aid of a digital computing machine for values of E spanning the range of potential application. The results are presented in Fig. 1. It is important to note that

(i). For a given value of E, there exists an upper limit in the evaporation constant B which can be expressed as $(30 Ja)^{1/2}$. However, one must bear in mind that the value of E varies as Ja changes.

(ii). The result obtained from the quasi-steady model [for example 5] $B = (Ja/2)^{1/2}$ represents a lower limiting value in the range of B.

(iii). The upper limit and the quasi-steady approximation are different by the factor of $(60)^{1/2}$ or 7.746 for a given value of Ja.

(iv). The quasi-steady model gives an accurate approximation only when the dimensionless superheat $C_p\Delta T/L$ is sufficiently small and ρ_g is much less than ρ_ℓ.

(v). If there is no change in density accompanying phase transition, i.e., $\rho_g/\rho_\ell = 1$ and $E = 0$, there is no convective motion in the system and the solution simplifies to

$$T^* = 2MB^3 \exp B^2 \pi^{1/2} [\text{erfc } N - N^{-1} \exp(-N^2)/(\pi)^{1/2}]$$

$$\phi(0,E) = 2(\pi)^{1/2} B^3 \exp B^2 \text{erfc } B - 2B^2$$

The deviation of the zero E curve from the quasi-steady approximation is attributed to the contribution of the transient term. On the other hand, the difference between the non-zero and zero E curves is a measure of the effect of convective motion in the system.

Spalding [9] has pointed out that the validity and utility of the quasi-steady formulas are doubtful at the high pressures prevailing in current engines. As the system pressure approaches the critical pressure of the droplet, the liquid-gas density ratio approaches unity and the latent heat of vaporization decreases to zero. As a result, the value of E approaches zero, while Ja tends to infinity. Figure 1 shows that the deviation between the zero E curve and the quasi-steady approximation grows with an increase in Ja.

Upon comparison of Eqs. (10) and (20), it becomes evident that the effect of mass transfer on drop shrinkage is represented by the second term on RHS of equation (20), which can be separated into two factors

$$\phi(E, BLe^{1/2}) - (C_s/C_o - 1)$$

and

$$\frac{Q}{m} = \frac{-\rho_\ell \tilde{R} T_s^2}{T_\infty L C_s M}$$

As expected, the lower the mass diffusivity of the vapor or the higher the vapor concentration in the ambient, the greater is the temperature difference $T_\infty - T_s$ required to obtain a given drop shrinkage constant B. When the temperature of the ambient atmosphere is substantially the same as that of the individual drops, (meaning that the temperature concerned being low in relation to the boiling point of the drop T_s, i.e. $T_o^* = 0$), Eq. (20) reduces to

$$\frac{\phi(E, BLe^{1/2}) - C_o^*}{\phi(E, B)} = \frac{L^2 C_s M}{\rho C_p \tilde{R} T_s^2} \tag{21}$$

Under these conditions, the evaporation rate is determined by mass diffusion processes. This process of drop evaporation is well known. On the other hand, when the difference in temperature between the drop and the environment is considerable, i.e. $T_\infty \gg T_S$, Q/m approaches zero and the second term in Eq. (20) vanishes. The rate of decrease in size of the drops is then determined by heat transfer processes. Only when the ratio of T_∞/T_S is moderate, then both heat and mass transfer are of comparable importance in evaporation processes.

Conclusions

The exact solutions of the heat and mass transfer equations governing spherically symmetric drop evaporation in an infinite hot environment are found to be precisely the expression for the square law of droplet shrinkage: the position of the phase boundary is proportional to the square root of time. The drop-radius time history is amended to the form of Eq. (11) in order to satisfy the initial condition on drop size. The result is inapplicable at the very beginning of the drop shinkage, however. The evaporation constant B is presented in graphical form for convenience in practical uses. The effects of radial convection resulting from unequal phase densities, system pressure, diffusion of the evaporated mass are established. The earlier approximate formulas based on the quasi-steady model is in considerable error for large superheats and large vapor densities.

Acknowledgment

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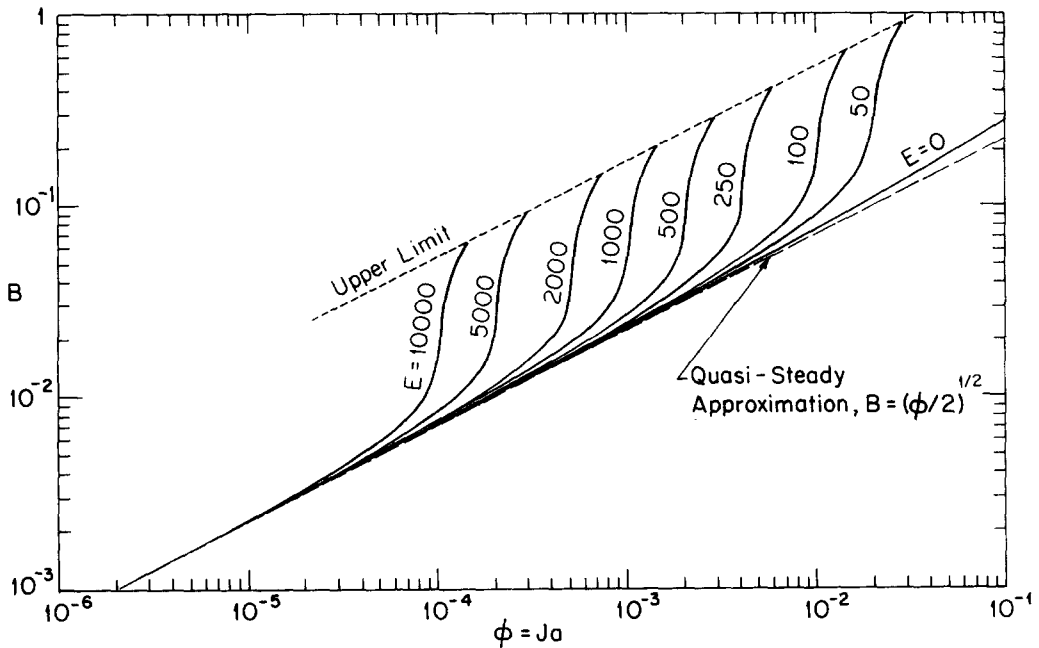
FIG. 1
The Vaporization Constant B as a Function
of the Jakob Number Ja