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A POLYNOMIAL THEORY OF RISK*

Teri A. BERGER
University of Michigan, Ann Arbor, U.S.A.

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There exists evidence that subjective expected utility (SEU) theory, the most general of the maximization theories, is an invalid model for decision making. Huang (1971) and Pollatsek (1971) have shown that subjects violate SEU theory if they prefer intermediate variance on a set of gambles when the level of expected value is held constant. There is substantial evidence in the studies cited below that subjects indeed do so. Consequently, the maximization theories do not provide a sufficiently general model for decision making behavior. It is, therefore, necessary to find a general model that will take into account more diverse types of decision making behavior.

There are several decision making models in the literature in which risk is an important variable (Royden et al. 1959; Coombs and Pruitt 1960; Pruitt 1962; Van der Meer 1963; Coombs and Meyer 1969; Coombs and Huang 1970b). In order to test these models, risk must be examined in more detail; otherwise, the interpretation of discrepant experimental results will not discriminate between incorrect assumptions about the nature of risk and incorrect assumptions about the decision process.

Coombs and Huang (1970a) propose a structure for perceived risk in their paper on the polynomial psychophysics of risk. They define three transformations on games of the form $g=(y,p,z)$, in which outcome y

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occurs with probability p, otherwise z occurs ($y \geqslant z$). They propose that these transformations induce corresponding transformations on perceived risk in which the joint effect has a particular polynomial form which can be tested using conjoint measurement methods (Krantz and Tversky 1971). The three transformations used in their study were:

$$a(g) = (y{+}a, 1/2, z - a)$$
$$b(g) = (y{+}b, 1/2, z + b)$$
$$c(g) = (y, 1/2, z)^c$$

where a and b are amounts of money and c is a nonnegative integer. The $a(g)$ transformation controls the range and variance while leaving the probabilities and expectation unchanged. The $b(g)$ transformation controls expected value while leaving the other variables unchanged. The $c(g)$ transformation is a multiple play transformation given by the convolution of the game g with itself c times.

The authors proposed that there exist real valued subjective functions α, β, and γ on the perceived risk of a game corresponding to the $a(g)$, $b(g)$, and $c(g)$ transformations respectively such that their joint effect can be characterized by the following distributive model:

$$R(g) = [\alpha(a) + \beta(b)]\ \gamma(c)$$

where $R(g)$ is the riskiness of the game g generated by the values of a, b, and c applied to the gamble g_0=(0,1/2,0). The experimental results from rank orderings of risk preferences supported this model over several other models.

Pollatsek and Tversky (1970) constructed an axiomatic theory for perceived risk which leads to the theorem that risk is a linear function of variance and expected value. Coombs and Bowen (1971) varied probability keeping variance and expectation constant and showed that risk varied concurrently, indicating that risk cannot be solely a function of variance and expectation but that probability plays a significant part.

The current paper tests a model for perceived risk that takes both range and probability into account keeping expected value constant. Two-outcome gambles were chosen so that the range of a game, a, and the probability of a game, p, could be varied while expected value remained constant. Expected value was not varied since the effect of expected value on the perceived risk of a game has been studied extensively (Coombs and Huang 1970a; Coombs and Bowen 1971; Pollatsek and Tversky 1970). These considerations led to the following derivation for the form of the gambles. Suppose we have a game $g=(y,p,z)$ and we set the expected value equal to zero, then:

(1) $yp + (1 - p)z = 0$, or $p(y - z) + z = 0$

or

(2) $z = -pa$, where a is the range ($a=y - z$)

Solving now for y yields:

(3) $y = a(1 - p)$

Consequently gambles where the expected value is zero and the range and probability are varied will have the form $g = (qa, p, - pa)$ where $0 < p < 1$, $q=1 - p$, and a is some positive amount of money. Gambles were constructed of the above form to investigate the effect of range and probability upon the perceived risk of a game. A positive amount b, 30 cents, was added to both outcomes fixing the expected value at 30 cents. A positive expected value was chosen so that there would not be a minus-one transformation for games with the same range and different probabilities. For example, the game (0.40, 2/3, − 0.80) becomes (0.80, 1/3, − 0.40) when the probability is varied and range remains constant with expected value equal to zero. However, when the expected value is set to 30 cents, the two games become (0.70, 2/3, − 0.50) and (0.10, 1/3, − 0.10) and it is not as easy to discern with a glance that the two ranges are equal.

The model assumes that there exist real valued functions R, α, β, and γ, defined on g, a, p, and p respectively such that

$$R(g) = \alpha(a)\beta(p) + \gamma(p)$$

where α, β, and γ correspond to three psychophysical functions for the subjective effects of the a and p parameters on perceived risk, $R(g)$. This particular function that was chosen to characterize the risk of a game was mediated by the following considerations. Changing the range of a game will increase risk by increasing the amount to win and at the same time increase the amount to lose. Changing the probability of a game, however, will not only change the actual amounts of the outcomes, but it will also change the probability with which each of the outcomes occurs. For example, increasing the probability will not only decrease the amount to win and increase the amount to lose, it will also increase the probability of winning the positive outcome. In view of these considerations, the above model of risk was proposed, where $\alpha(a)\beta(p)$ represents the joint effect of the a and p parameters on risk that is mediated by the outcomes and $\gamma(p)$ corresponds to the subjective effect of odds alone on risk.

To test the theory, the difference in risk was compared between certain gambles chosen so that the $\gamma(p)$ term can be cancelled as follows: If one considers the two games, g_i with parameters a_i and p and g_j with parameters a_j and p, then according to this model, their difference in risk is given by:

$$\begin{aligned} D_{ij} = R(g_i) - R(g_j) &= [\alpha(a_i)\beta(p) + \gamma(p)] - [\alpha(a_j)\beta(p) + \gamma(p)] \\ &= \alpha(a_i)\beta(p) - \alpha(a_j)\beta(p) \\ &= \beta(p)\,[\alpha(a_i) - \alpha(a_j)] \end{aligned}$$

In characterizing differences in risk by the above model it has been assumed that the subjective process of evaluating differences in risk is a subtractive one. One sees that the difference in risk between two gambles with the same expected value has the structure of a simple distributive polynomial. This structure must satisfy certain properties which can be tested by conjoint measurement methods.

Method of analysis

There are four simple polynomial models in three variables as follows (Krantz and Tversky 1971):

$r_1 + r_2 + r_3$	additive model
$r_1 r_2 r_3$	multiplicative model
$r_1 (r_2 + r_3)$	distributive model
$r_1 r_2 + r_3$	dual distributive model

The necessary conditions for each of these models, which are summarized in table 1, can be tested by conjoint measurement methods;

consequently, conjoint measurement methods can be used to distinguish between these four models. Since the *p* and *a* parameters are assumed to induce subjective transformations which can be characterized by the distributive model when looking at differences in risk, attention is directed to those tests which are the necessary and distinguishing tests for that model.

A measure of each *S*'s consistency was constructed to provide a comparative error rate as a basis for testing significance of results where necessary and appropriate for the tests listed in table 1. These tests are described briefly in the results. For a more detailed explanation of the necessary conditions for the four polynomial models and the tests used in this experiment see Coombs and Huang (1970a) and Krantz and Tversky (1971).

The experiment

Method

Subjects

The *S*s were 20 paid male volunteers who were students at the University of Michigan.

Stimuli

The stimuli which are shown in table 2 are all pairs of two-outcome gambles with one level of expectation, b=0.30, two levels of p, and five levels of a. The number of levels chosen was the minimum number needed to test the properties of the distributive model (Krantz and Tversky 1971). A graphical representation of

Table 1
Necessary conditions for four polynomial models.

Additive	Multiplicative	Distributive	Dual distributive
Simple independence	Simple sign dependence	Simple sign dependence	Simple sign dependence
Joint independence of all parameters	Joint sign dependence of all parameters	Joint sign dependence of one pair of parameters	Joint sign dependence of one pair of parameters
Double cancellation	Double cancellation	Double cancellation	Double cancellation
		Distributive cancellation	Dual distributive cancellation

Table 2
The basic games.

	p_1=1/3	p_2=2/3
a_1=1.20	(1.10,1/3, – 0.10)	(0.70,2/3, – 0.50)
a_2=1.80	(1.50,1/3, – 0.30)	(0.90,2/3, – 0.90)
a_3=4.50	(3.30,1/3, – 1.20)	(1.80,2/3, – 2.70)
a_4=9.10	(6.90,1/3, – 3.00)	(3.60,2/3, – 6.30)
a_5=20.70	(14.10,1/3, – 6.60)	(7.20,2/3, – 13.50)

the design of the experiment is shown by the x's in the cells of fig. 1. Each cell represents the difference in risk between two games, one with the range equal to the column entry and one with the range equal to the row entry and both having the same probability of outcomes. Therefore, each stimulus can be characterized by a triple (a_i, a_j, p_k) where a_i and a_j are the respective values of the range of the two games and p_k is the probability of winning, which is the same for both games. Only the cells above the diagonal have been indicated as they will have the same ordering as those below the diagonal with a sign change.

Procedure

The *S* was presented with a deck of IBM cards with one pair of gambles on each card (see fig. 2). The cards were randomized within each deck. The order of the pair of games on each card was randomized. There were five different replications of each deck. The order of these five decks was also randomized.

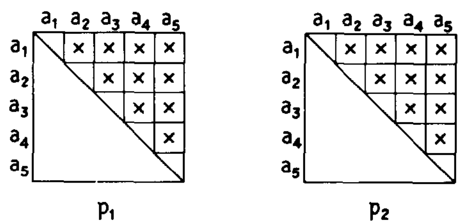

Fig. 1. The basic design.

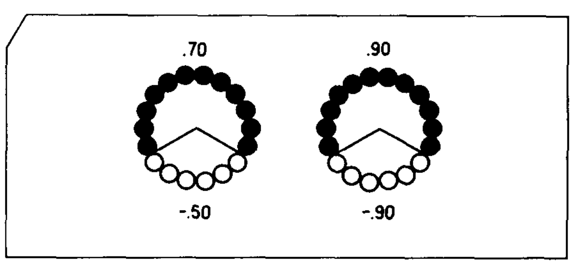

Fig. 2. One of the stimuli used in the experiment.

The *S*s attended six sessions on six consecutive days. The first session was a practice session. During the first part of the practice session, the *S*s became accustomed to the type of games they would be making judgements about. During the second part of this session, they were presented with a set of practice games which differed in expectation, range, and probability from the games used in the following sessions. They were asked to make the same type of judgements they would be making throughout the rest of the experiment.

On the following days the *S*s were presented with a deck of 114 IBM cards. They were asked to take off the top two cards from the deck, putting the top card on their left and the other card on their right. Then they were to decide whether the pair of games on the card on the left or on the card on the right had a greater difference in risk. Once they had made their decision, they were to place the card with the pair of games which had a greater difference in risk in a pile with the other card on top of it. They then took off the next two cards and proceeded in the same manner as before. They did this until they finished going through all 114 IBM cards. A total of 57 paired comparisons were made by each *S*. It took each *S* an average of 25 min to go through the deck of 114 IBM cards.

The data were analyzed between sessions to determine the *S*'s pay for the previous session. The *S*s were instructed that if they made careful judgements, then they would be paid $ 3.00 for the session. Since it was assumed that, if the *S*s made careful judgements, then they would satisfy one of the tests of independence needed for any of the simple polynomial models, satisfaction of this test was used as a measure of how carefully the *S*s were making their decisions. Although each *S*'s pay was determined by the violations made of this test of independence, 25 cents was deducted for each violation, they were never aware of how their pay was determined. At the end of the six sessions, each *S*'s responses for a given comparison were pooled across sessions to obtain the stochastically dominant ordering. This is the ordering that is used in reporting the results.

Results

Consistency

The level of consistency was calculated for each *S* using a folded binomial distribution (Coombs and Huang 1970a) which had a mean of 3.44 and a standard deviation of 0.15. All of the *S*s deviated significantly from chance (consistency at a chance level = 3.5) at the 0.01 level. The *S*s' consistency level is presented in table 3. They are ordered from most to least consistent.

Although it was assumed that the more consistent *S*s would have fewer errors, as was found after collecting the data, no error theory was incorporated in the model.

Independence

Two types of independence were tested, simple independence and joint independence. Simple independence for each variable was tested by comparing the orderings induced on that variable with the remaining variables held constant at

Table 3
Consistency.

Subject number	Consistency level
24	5.00
1	5.00
14	4.98
17	4.98
6	4.98
8	4.98
21	4.96
10	4.96
15	4.95
22	4.95
7	4.94
9	4.84
18	4.82
3	4.78
16	4.77
2	4.66
11	4.61
23	4.58
4	4.22
5	4.03

each level in turn. These orderings must all be the same. Joint independence was tested by comparing the orderings induced on the joint effect of two variables with the third variable held constant at each level in turn. These orderings must all be the same. The results on the two tests of simple independence and the two tests of joint independence are presented in table 4.

The last test of joint independence, A × P;A, consisted only of six new tests in addition to the previous 16 tests of simple independence. Therefore, if a *S* made three or more violations of this test of joint independence and made no violations of the tests of simple independence, this was considered to be a serious violation. As can be seen from table 4, 16 of the 20 *S*s substantially violated this test of joint independence. As all of these tests of independence are necessary conditions for the additive model, that model can be ruled out for characterizing the difference in risk between two games.

Sign dependence

This test, a more general form of independence, requiring that independence be satisfied except for the sign of the variable held constant, was carried out on all possible combinations. The results on the tests of sign dependence are presented in table 5. As may be seen from the table 15 of the 20 *S*s substantially violated the test of sign dependence for A × P;A. Since all of the tests of sign dependence must be satisfied if the multiplicative model is a viable one, this model may be rejected.

Table 4
Results on four tests of independence.

		Simple independence		Joint independence	
		P; A×A	A; A×P	A×A; P	A×P; A
No. of violations	0	16	18	17	1
	1	3	0	0	3
	2	0	1	1	0
	3 or more	1	1	2	16
No. of possible violations		8	8	13	22

Table 5
Results on four tests of sign dependence.

		Simple sign dependence		Joint sign dependence	
		P; A×A	A; A×P	A×A; P	A×P; A
No. of violations	0	16	18	17	1
	1	3	1	0	4
	2	0	1	1	0
	3 or more	1	0	2	15
No. of possible violations		8	8	13	22

Double cancellation

All twenty *S*s satisfied the four tests of double cancellation.

Distributive cancellation

Distributive cancellation was the final property tested. This property is a necessary condition for all of the models except for the dual distributive one. However, if distributive cancellation is satisfied, the dual distributive model may still be a viable model. All 20 *S*s satisfied all nine tests of distributive cancellation.

Discussion

Using conjoint measurement methods the multiplicative and additive models have been shown not to be viable models for characterizing differences in risk. These conclusions are supported by a priori considerations also as follows.

If one makes the assumption that the difference in risk between two identical games is zero, and one assumes that differences in risk can be characterized by an additive model, then the following is true for the game g_i with parameters a_i and p.

$$\begin{aligned} D_{ii} &= \beta(p) + \alpha(a_i) - \alpha(a_i) \\ &= \beta(p) \\ D_{ii} &= 0 \Rightarrow \beta(p) = 0 \end{aligned}$$

The above derivation shows that if the model for characterizing differences in risk is an additive one, then the $\beta(p)$ term is an irrelevant variable. Since probability was shown to be a relevant variable when looking at the difference in risk between two games, this provides support for the conclusion that differences in risk cannot be characterized by an additive model. A similar analysis reveals that the multiplicative model cannot characterize differences in risk.

All of the tests for the distributive and dual distributive model were satisfied. Since distributive cancellation was satisfied for all 20 subjects, this leads one to believe that the distributive model is the correct model for characterizing the difference in risk between two games. However, these two models must be examined in more detail.

There is an additional condition which can be used to distinguish the distributive model from the dual distributive model; the distributive model requires that there exist a level of P which induces a degenerate ordering on A × A and a level of A × A which induces a degenerate ordering on P. It is argued that A × A induces a degenerate ordering on P when the two ranges are equal, that is, those cells which form the diagonal of the data matrix. If the two ranges are equal, then the two games are the same game and there is no difference in risk between them. If the level of P is 1, then P will induce a degenerate ordering on A × A; the reason being that there will be no risk involved. This provides support for the distributive model over the dual distributive model.

A further examination of the dual distributive model provides support for the rejection of this model as a viable one for characterizing differences in risk. If one assumes that the difference in risk between two identical games is zero and that differences in risk can be characterized by a dual distributive model, then the following is true for the game g_i with parameters a_i and p:

$$\begin{aligned} D_{ii} &= \beta(p)\alpha(a_i) - \alpha(a_i) \\ &= \alpha(a_i)\,[\beta(p) - 1] \\ D_{ii} &= 0 \Rightarrow \text{either } \alpha(a_i) = 0 \text{ or } \beta(p) = 1 \end{aligned}$$

The above derivation shows that if the above dual distributive is proposed then either $\beta(p)$ or $\alpha(a)$ is a constant function. However, if either of these two functions are constant then this implies that either probability or range is not a relevant variable when looking at differences in risk. This has been shown to be an invalid conclusion. A similar analysis follows for other possible dual distributive models. The above results provide substantial support for the rejection of a dual distributive model for characterizing differences in risk.

The above evidence supports the claim that of the four simple polynomials, the distributive one is the only viable model for characterizing differences in risk. Through the use of conjoint measurement methods it was found that all the necessary conditions for the distributive model were satisfied. Furthermore, the distributive model is the only model which satisfies the assumption that the difference in risk between two identical games is zero. Consequently, if differences in risk are characterized by a distributive model and the cognitive process of taking differences in risk is a subtractive one, then the only viable model for characterizing the risk of a single game is a multiplicative or a dual distributive one.

An alternative model that could be proposed for determining the risk of a game is the additive one, as follows:

$$R'(g) = \alpha(a) + \beta(p) + \gamma(p)$$

for all $a \in$ A and for all $p \in$ P where α, β, and γ are real valued functions defined on A, P, and P respectively. Since the model proposed in this paper is a multiplicative one for the functions $\alpha(a)$ and $\beta(p)$, it could be argued that one cannot distinguish between the model proposed earlier and the one proposed above. However, the following analysis shows that the above model is not a viable one for the two games, g_i with parameters a_i and p and g_j with parameters a_j and p.

$$D_{ij} = R(g_i) - R(g_j) = [\alpha(a_i) + \beta(p) + \gamma(p)] - [\alpha(a_j) + \beta(p) + \gamma(p)]$$
$$= \alpha(a_i) - \alpha(a_j)$$

This implies that the difference in risk between two games with the same probability can be completely characterized by the difference in range between two games and that probability is not a relevant variable. However, the previous analysis has shown that probability is indeed a relevant variable when one evaluates the difference in risk between two games. Consequently, the above model can be ruled out as a viable model for a theory of risk.

In earlier studies (Coombs and Huang 1970a, b; Coombs and Meyer 1969), it has been assumed that increasing the range of a game increases the risk of a game if expected value is held constant. This assumption was supported in this study. It does not seem likely that such a simple monotonic relationship holds when the probability of a game is increased. If risk is characterized by a dual distributive model, where changing the probability of a game produces two different effects on the risk of a game, then it seems unlikely that any simple relationship between changes in risk and changes in probability will be found. Furthermore, if risk is characterized by a dual distributive model, it is not obvious how one would test the effect of changes in probability on the risk of a game.

The additive, multiplicative, and dual distributive models have been shown to be invalid models for characterizing differences in risk. The additive model has also been shown to be an invalid model for characterizing the risk of a game. There has been shown to be substantial support for the hypothesis that the difference in risk between two games has the structure of a distributive model and that the risk of a game has the structure of a multiplicative or dual distributive model. Therefore, there appears to be substantial support for the proposed empirical relational system for perceived risk.

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