Canonical Realization of General Time Systems*

BERNARD P. ZEIGLER

Department of Applied Mathematics,
The Weizmann Institute of Science,
Rehovot, Israel

and

Logic of Computers Group,
University of Michigan,
Ann Arbor, Michigan 48104

Communicated by John M. Richardson

ABSTRACT

The canonical realization of general time systems by dynamical systems is discussed employing the notation and definitions of Takahara and Mesarovic. Previous results are extended to the case of causal, but not necessarily stationary, systems and to stationary, strongly connected systems. Conditions are provided under which a time system is guaranteed to have a canonical representation from within a class of possible realizations.

INTRODUCTION

It has been shown [1] that by dropping the causality requirement for time systems one can establish that every time system has a dynamical system realization and likewise every stationary time system has a time-invariant dynamical system realization. The constructions employed in the proofs, however, make use of state spaces which, while serving for the general case, may be far from the smallest possible in each particular case. When causality is reintroduced, it remains unknown whether every time system has a causal dynamical system realization. Conditions are known, however, under which a stationary causal system can be guaranteed to have a canonical, that is to say, a unique minimal realization [2,3]. This paper improves the known results for causal stationary systems and extends them to the case of causal time-varying systems. We believe (but do not prove) that nothing meaningful can be said concerning canonical realization in the case of noncausal systems.

---

*This research was in part supported by NSF Grant No. DCR1-01997.

It should be noted that the problem of canonical realization centers around the uniqueness of assignment of response functions, since once such a family has been assigned, minimality is readily characterized along the lines of arguments familiar in sequential machine realization [see, e.g., 4, 5, 6].

For the purpose of uniformity, we shall employ the terminology and symbology presented in [1], to which paper we direct the reader for basic definitions and results.

1. UNIQUE REALIZATION OF CAUSAL SYSTEMS

We first reformulate the nonanticipatory function concept of [1] in more convenient terms.

PROPOSITION 1.1. *Let $\rho_t : C_t \times X_t \to Y_t$ be a response function. Then ρ_t is nonanticipatory, if and only if, there is a family of functions $\{f_{tt'} | t' \geqslant t\}$ such that*

(i) *for each $t' \geqslant t$,* $\quad f_{tt'} : C_t \times X_{tt'} \to Y_{tt'}$
(ii) *for each $t'' \geqslant t' \geqslant t$,* $\quad f_{tt'}(c_t, x_{tt'}) = f_{tt''}(c_t, x_{tt''}) | \bar{T}_{tt'}$
(iii) *for each $t' \geqslant t$,* $\quad f_{tt'}(c_t, x_{tt'}) = \rho_t(c_t, x_t) | \bar{T}_{tt'}$

where $\bar{X}_{tt'} = X | \bar{T}_{tt'}$ and $\bar{T}_{tt'} = T_{tt'} \cup \{t'\}$.

Proof. Obvious from the definition of nonanticipatory function (p. 111 of [1]). Q.E.D.

For causal systems, the relation between state and output is much more intimate than that holding in general. In fact, we have:

PROPOSITION 1.2. *Let $(\bar{\rho}, \bar{\varphi})$ be a nonanticipatory dynamical system and let $\{f_{tt'}\}$ be the family of functions defined by ρ according to Proposition 1.1. Then*

$$\rho_t(c_t, x_t) = f_{tt'}(c_t, x_{tt'}) \cdot \rho_{t'}(\varphi_{tt'}(c_t, x_{tt'}), x_{t'})$$

for each $t' \geqslant t$.

Proof. Obvious from (iii) of Proposition 1 and Definition 1.1 (ii) (α) of [1]. Q.E.D.

Given $(\bar{\rho}, \bar{\varphi})$, with every pair $(x_{tt'}, y_{tt'}) \in \bar{X}_{tt'} \times \bar{Y}_{tt'}$ we associate two sets:

$$\mathrm{INITIAL}(x_{tt'}, y_{tt'}) = \{c_t | c_t \in C_t, f_{tt'}(c_t, x_{tt'}) = y_{tt'}\}$$

$$\mathrm{FINAL}(x_{tt'}, y_{tt'}) = \{\varphi_{tt'}(c_t, x_{tt'}) | c_t \in \mathrm{INITIAL}(x_{tt'}, y_{tt'})\}.$$

INITIAL is the set of states which the system could have been in at time t, in order to produce $y_{tt'}$ given $x_{tt'}$ as the input. FINAL is the set of states it could be in at time t' under these conditions.

We say that $(x_{tt'}, y_{tt'})$ *identifies* $c_{t'}$, if FINAL $(x_{tt'}, y_{tt'}) = \{c_{t'}\}$. Having observed $y_{tt'}$ in response to $x_{tt'}$ we can uniquely identify the system state to be $c_{t'}$ at time t' in these circumstances.

It is convenient to consider the family

$$\bar{\rho} = \{\rho_t | \rho_t : C_t \times X_t \to Y_t\}$$

to be represented by the family

$$\{\rho_{c_t} | c_t \in C_t, t \in T\}$$

where $\rho_{c_t} : X_t \to Y_t$, defined by $\rho_{c_t} = \rho_t(c_t, \ )$, is the response function of state c_t.

States c_t and $\tilde{c}_t$ (not necessarily in the same dynamical system) are *equivalent* if $\rho_{c_t} = \rho_{\tilde{c}_t}$.

Given two representations $(\bar{\rho}, \bar{\varphi})$ and $(\bar{\rho}', \bar{\varphi}')$ of the same time system S we seek conditions under which states of the one representation are equivalent to states of the second representation.

Our fundamental result is the following:

THEOREM 1.1. *Let* $(\bar{\rho}, \bar{\varphi})$ *and* $(\bar{\rho}', \bar{\varphi}')$ *be nonanticipatory dynamical system realizations of a time system* S. *Let* c'_t *be identifiable in* $(\bar{\rho}', \bar{\varphi}')$. *Then* $\rho_{c'_t} \in \bar{\rho}$.

Proof. Let $(x_{tt'}, y_{tt'})$ identify $c'_{t'}$, i.e.,

$$\text{FINAL } (x_{tt'}, y_{tt'}) = \{c'_{t'}\} \text{ for } (\bar{\rho}', \bar{\varphi}').$$

Since FINAL$(x_{tt'}, y_{tt'})$ for $(\bar{\rho}_,', \bar{\varphi}')$ is not empty, INITIAL$(x_{tt'}, y_{tt'})$ is also not empty and $(x_{tt'}, y_{tt'}) \in S_{tt'}$. Reversing the argument for $(\bar{\rho}, \bar{\varphi})$ we see that INITIAL$(x_{tt'}, y_{tt'})$ and FINAL$(x_{tt'}, y_{tt'})$ are not empty for it, but of course, the latter need not be a singleton.

Let $c_{t'}$ be any element of FINAL$(x_{tt'}, y_{tt'})$ in $(\bar{\rho}, \bar{\varphi})$. We show now that $c_{t'}$ is equivalent to $c'_{t'}$.

Since $c_{t'} \in$ FINAL$(x_{tt'}, y_{tt'})$, there is a $c_t \in C_t$ such that $\varphi_{tt'}(c_t, x_{tt'}) = c_{t'}$ and $f_{c_t}(x_{tt'}) = y_{tt'}$. For any $x_t \in X_t$ we have $\rho_{c_t}(x_t) = y_{tt'} \rho_{c_{t'}}(x_{t'})$ by Proposition 1.2. Thus $(x_t, y_{tt'} \rho_{c_{t'}}(x_{t'}), \in S_{t'}$. Let c'_t produce this pair in $(\bar{\rho}', \bar{\varphi}')$, i.e.,

$$\rho'_{c'_t}(x_t) = y_{tt} \ \rho_{c_{t'}}(x_{t'}).$$

But by Proposition 1.2, again

$$\rho'_{c'_t}(x_t) = f'_{c'_t}(x_{tt'}) \rho'_{\varphi'_{tt'}(c'_t, x_{tt'})}(x_{t'}).$$

Equating the two expressions yields

(a) $y_{tt'} = f'_{c'_t}(x_{tt'})$

and

(b) $\rho_{c_{t'}}(x_{t'}) = \rho'_{\varphi'_{tt'}(c'_t, x_{tt'})}(x_{t'})$.

But from (a), $c'_t \in \mathrm{INITIAL}(x_{tt'}, y_{tt'})$ and since $(x_{tt'}, y_{tt'})$ identifies $c'_{t'}$, we have $\{c'_{t'}\} = \mathrm{FINAL}(x_{tt'}, y_{tt'}) = \{\varphi'_{tt'c'_t}, x_{tt'})\}$. Thus from (b), $\rho_{c_{t'}}(x_{t'}) = \rho'_{c'_{t'}}(x_{t'})$. Since this holds for arbitrary x we have $\rho_{c_{t'}} = \rho_{c'_{t'}}$ as promised. Q.E.D.

A dynamical system $(\bar{\rho}', \bar{\varphi}')$ is *identifiable* if each of its states is identifiable, i.e., for each $c'_{t'}$ there is a pair $(x_{tt'}, y_{tt'})$ which identifies it.

Our main theorem follows as a corollary of the preceding one:

THEOREM 1.2. *Let $(\bar{\rho}, \bar{\varphi})$ and $(\bar{\rho}', \bar{\varphi}')$ be nonanticipatory dynamical system realizations of a time system S. If $(\bar{\rho}', \bar{\varphi}')$ is identifiable, then $\bar{\rho}' \subseteq \bar{\rho}$ (each state of the primed representation has an equivalent state in the unprimed system).*

Theorem 1.2 asserts that if a nonanticipatory time system has an identifiable realization, then the set of response functions employed by the realization is a minimal one.

2. HOMOMORPHIC IMAGE AND CANONICAL REALIZATION

Let $(\bar{\rho}, \bar{\varphi})$ and $(\bar{\rho}', \bar{\varphi}')$ be dynamical systems over the same base X, Y, T. We say that $(\bar{\rho}', \bar{\varphi}')$ is a *homomorphic image of a subsystem of* $(\bar{\rho}, \bar{\varphi})$ if there is a family of mappings $\{h_t | t \in T\}$ such that for each $t \in T$

(i) $h_t : \bar{C}_t \xrightarrow{\text{onto}} C'_t$ where $\bar{C}_t \subseteq C_t$

(ii) $\rho_{c_t} = \rho'_{h(c_t)}$ for each $c_t \in \bar{C}_t$

(iii) $h_{t'}(\varphi_{tt'}(c_t, x_{tt'})) = \varphi'_{tt'}(h_t(c_t), x_{tt'}))$ for each $c_t \in \bar{C}_t, x_{tt'} \in X_{tt'}$, $t' \geqslant t$.

If $\bar{C}_t = C_t$ for each $t \in T$, we say that $(\bar{\rho}, \varphi')$ is a *homomorphic image* of $(\bar{\rho}, \bar{\varphi})$.

A time system S is said to have a *canonical dynamical system* representation $(\bar{\rho}', \bar{\varphi}')$ over Σ if for every dynamical representation $(\bar{\rho}, \bar{\varphi})$ of S, such that $(\bar{\rho}, \bar{\varphi}) \in \Sigma$, it is the case that $(\bar{\rho}', \bar{\varphi}')$ is a homomorphic image of a subsystem of $(\bar{\rho}, \bar{\varphi})$.

In this case S is said to be *canonically representable in the class* Σ.

THEOREM 2.1. *Let $(\bar{\rho}, \bar{\varphi})$ and $(\bar{\rho}', \bar{\varphi}')$ be dynamical systems over the same base such that $\bar{\rho}' \subseteq \bar{\rho}$. If $(\bar{\rho}', \bar{\varphi}')$ is reduced then $(\bar{\rho}', \bar{\varphi}')$ is a homomorphic image of a subsystem of $(\bar{\rho}, \bar{\varphi})$. [If $\bar{\rho}' = \bar{\rho}$, then $(\bar{\rho}', \bar{\varphi}')$ is a homomorphic image of $(\bar{\rho}, \bar{\varphi})$.]*

Proof. Define the relation $h_t \subseteq C_t \times C'_t$ for each $t \in T$, where $(c_t, c'_t) \in h_t \Leftrightarrow \rho_{c_t} = \rho_{c'_t}$. By standard arguments (e.g., Theorem 2.3 of [1]) it is readily shown that

$$(c_t, c'_t) \in h_t \rightarrow (\varphi_{tt'}(c_t, x_{tt'}), \varphi'_{tt'}(c'_t, x_{tt'})) \in h_t,$$

for all $t' \geqslant t$, and $x_{tt'} \in X_{tt'}$.

Since $(\bar{\rho}', \bar{\varphi}')$ is reduced, it is easy to show that h_t is a function, and since $\bar{\rho}' \subseteq \bar{\rho}$, it is onto. Thus the requirements for homomorphism are all satisfied. Q.E.D.

THEOREM 2.2. *Let S be a time system which has an identifiable nonanticipatory dynamical system representation. Then S is canonically representable in the class of all nonanticipatory dynamical systems.*

Proof. Let $(\bar{\rho}', \bar{\varphi}')$ be an identifiable nonanticipatory representation of S. It is easy to show that the reduced version $(\hat{\rho}', \hat{\varphi}')$ of $(\bar{\rho}', \bar{\varphi}')$ (Theorem 3.3 of [1]) is also identifiable. Let $(\bar{\rho}, \bar{\varphi})$ be any nonanticipatory realization of S. By Theorem 1.2, $\hat{\rho} \subseteq \bar{\rho}$ and by Theorem 2.1, $(\hat{\rho}', \hat{\varphi}')$ is a homomorphic image of a subsystem of $(\bar{\rho}, \bar{\varphi})$. Thus S is canonically representable in the class of all nonanticipatory dynamical systems. Q.E.D.

3. IDENTIFIABILITY AND COUNTABLE STATE SETS

We now study the relationship between the properties of "being reduced" and "being identifiable."

PROPOSITION 3.1. *Let $(\bar{\rho}, \bar{\varphi})$ be a reduced nonanticipatory dynamical system. If there is a pair $(x_{tt'}, y_{tt'}) \in S_{tt'}$ for which* FINAL$(x_{tt'}, y_{tt'})$ *has finite cardinality, then there is at least one identifiable state.*

Proof. Let c_t be an arbitrary state in INITIAL$(x_{tt'}, y_{tt'})$ and list the states in FINAL$(x_{tt'}, y_{tt'})$ beginning with $\varphi(c_t, x_{tt'})$, i.e., FINAL$(x_{tt'}, y_{tt'}) = c_0, c_1, \ldots, c_n$ where $c_0 = \varphi(c_t, x_{tt'})$. Construct a pair $(\bar{x}, \bar{y}) = (x_{tt'} \cdot x_1 \cdot x_2 \cdots x_n, y_{tt'} \cdot y_1 \cdot y_2 \cdots y_n)$, where each (x_{i+1}, y_{i+1}) is chosen such that

$$y_{i+1} = f(\varphi(c_0, x_1 \cdot x_2 \cdots x_i), x_{i+1})$$

and

$$y_{i+1} \neq f(\varphi(c_{i+1}, x_1 \cdot x_2 \cdots x_i), x_{i+1})$$

[we interpret $\varphi(c_0, x_1, \cdot x_2 \cdots x_0) = c_0$ and similarly for $\varphi(c_1, x_1 x_2 \cdots x_0) = c_1$].

Note that such a choice (x_{i+1}, y_{i+1}) always exists because $\bar{\rho}$ is reduced.

We claim that $(\bar{x}, \bar{y})$ identifies $\varphi(c_t, \bar{x})$.

Certainly, by repeated applications of Proposition 1.2, we find that $\varphi(c_t,\bar{x}) \in \text{FINAL}(\bar{x},\bar{y})$. But also, if some $\varphi(\bar{c}_t,\bar{x}) \in \text{FINAL}(\bar{x},\bar{y})$, then $f(\bar{c}_t, x_{tt'} \cdot x_1 x_2 \cdots x_n) = y_{tt'} \cdot y_1 \cdot y_2 \cdots y_n$. From this it follows from Proposition 1.2 that $\bar{c}_t \in \text{INITIAL}(x_{tt'}, y_{tt'})$ and hence that $\varphi(\bar{c}_t, x_{tt'}) = c_j$ for some $j \in [0,n]$. Either $j=0$ as desired or else let $j=i+1 \in [1,n]$. By Proposition 1.2, again,

$$f(\varphi(c_{i+1}, x_1 \cdot x_2 \cdots x_i), x_{i+1}) = y_{i+1},$$

a contradiction. Q.E.D.

COROLLARY. *Every finite-state reduced causal dynamical system has at least one identifiable state.*

The effect of constraining the state space to a countable set is now examined.

Since the definition of time system given by [1] allows infinitely long segments for $T=[0,\infty)$ we can extend a fundamental result for finite-state automata [7] to general systems with countable state sets.

LEMMA. *Let $(\bar{\rho},\bar{\varphi})$ and $(\bar{\rho}',\bar{\varphi}')$ be nonanticipatory systems having no equivalent states in common. Suppose that $(\bar{\rho},\bar{\varphi})$ has a countable state set C_t for some $t \in T$. Then if S and S' are the time systems generated by $(\bar{\rho},\bar{\varphi})$ and $(\bar{\rho}',\bar{\varphi}')$, respectively, we have that there is a pair $(x,y) \in S'$ which is not in S.*

Proof. Let $c_t' \in C_t'$ be an arbitrary state of $(\bar{\rho}',\bar{\varphi}')$. Enumerate the states $c_1, c_2, \ldots, c_i, \ldots$ of C_t. Construct

$$(x,y) = (x_1, x_2 \cdots x_i \cdots, y_1 \cdot y_2 \cdots y_i \cdots)$$

by letting (x_i, y_i) be a pair such that

$$y_i = f(\varphi'(c_t', x_1 \cdot x_2 \cdots x_{i-1}), x_i)$$

and

$$y_i \neq f(\varphi(c_i, x_1 \cdot x_2 \cdots x_{i-1}), x_i).$$

Such a pair (x_i, y_i) exists, since by assumption,

$$\varphi'(c_t', x_1 \cdot x_2 \cdots x_{i-1})$$

and

$$\varphi(c_i, x_1 \cdot x_2 \cdots x_{i-1})$$

are not equivalent. As in Proposition 3.1, it is easy to see that $(x,y)\in S'$ but not in S.

PROPOSITION 3.2. *Let $(\bar\rho,\bar\varphi)$ and $(\bar\rho',\bar\varphi')$ be nonanticipatory systems such that $S'\subseteq S$. If $(\bar\rho,\bar\varphi)$ has a countable state set C_t for some $t\in T$, then there is at least one state $c'_{t'}\in C'_{t'}$ for some $t'\geqslant t$ such that $\rho_{c'_{t'}}\in\bar\rho$.*

Proof. Contrapositive of the previous lemma. Q.E.D.

4. CONDITIONS FOR CANONICAL REPRESENTABILITY

We now employ the results of Section 3 to refine the conditions for canonical representability given by Theorem 2.2.

That all the preceding theorems can be readily specialized to the case of time-invariant systems is apparent from the results of [2]. In addition, the concept of "strongly connected system" is meaningful in this case.

A time-invariant dynamical system is *strongly connected* if for every pair $c,c'\in C$, there exists x^t such that $\varphi_{0t}(c,x^t)=c'$.

THEOREM 4.1. *Let S be a stationary system having a nonanticipatory strongly connected realization and a nonanticipatory realization such that for some pair $(x^t,y^t)\in S^t$, $\mathrm{FINAL}(x^t,y^t)$ has finite cardinality. Then S is canonically representable in the class of all nonanticipatory time-invariant dynamical systems.*

Proof. Let $(\bar\rho,\bar\varphi)_{\mathrm{str}}$ and $(\bar\rho,\bar\varphi)_{\mathrm{fin}}$ be the (possibly identical) realizations given in the hypothesis. Since $(\bar\rho,\bar\varphi)_{\mathrm{fin}}$ satisfies Proposition 3.1, it has an identifiable state c. By Theorem 1.1 $\rho_c=\rho_{\bar c}$ for some state $\bar c$ of $(\bar\rho,\bar\varphi)_{\mathrm{str}}$.

It is easy to show that there follows $\rho_{\varphi_{\mathrm{fin}}(c,x^t)}=\rho_{\varphi_{\mathrm{str}}(\bar c,x^t)}$ for all $x^t\in X^t$ and $t\in T$. But since $(\bar\rho,\bar\varphi)_{\mathrm{str}}$ is strongly connected, $\varphi_{\mathrm{str}}(\bar c,x^t)$ runs through all of its state space as x^t is varied over X^t and t is varied over T. Thus $\bar\rho_{\mathrm{str}}\subseteq\bar\rho_{\mathrm{fin}}$. Obviously, $\bar\rho_{\widehat{\mathrm{str}}}\subseteq\bar\rho_{\mathrm{fin}}$, where $(\bar\rho,\bar\varphi)_{\widehat{\mathrm{str}}}$ is the reduced version of $(\bar\rho,\bar\varphi)_{\mathrm{str}}$. Thus from Theorem 2.1, $(\bar\rho,\bar\varphi)_{\widehat{\mathrm{str}}}$ is a homomorphic image of a subsystem of $(\bar\rho,\bar\varphi)_{\mathrm{fin}}$. As in Theorem 2.2, it is easy to show that c is identifiable for $(\bar\rho,\bar\varphi)_{\widehat{\mathrm{fin}}}$ implies that $h(c)$ is identifiable for $(\bar\rho,\bar\varphi)_{\widehat{\mathrm{str}}}$, where h is the homomorphic mapping just shown to exist. We conclude that $(\bar\rho,\bar\varphi)_{\widehat{\mathrm{str}}}$ is identifiable, since it is easy to show that all states which are accessible from an identifiable state are themselves identifiable and moreover $(\bar\rho,\bar\varphi)_{\widehat{\mathrm{str}}}$ is strongly connected.

Applying Theorem 2.2 completes the proof. Q.E.D.

Our final result applies to stationary systems with countable state sets.

THEOREM 4.2. *Let S be a stationary system having a strongly connected nonanticipatory realization. Then S is canonically representable over the class of nonanticipatory time-invariant systems with countable state sets.*

Proof. Let $(\bar{\rho},\bar{\varphi})_{str}$ be the strongly connected realization and let $(\bar{\rho},\bar{\varphi})_{count}$ be any realization with a countable state set. Then by Proposition 3.2, there is a state $c \in C_{str}$ such that $\rho_c \in \bar{\rho}_{count}$. As in Theorem 4.1, since $(\bar{\rho},\bar{\varphi})_{str}$ is strongly connected, there follows $\bar{\rho}_{str} \subseteq \bar{\rho}_{count}$. By Theorem 2.1, $(\bar{\rho},\bar{\varphi})_{str}$ is a homomorphic image of a subsystem of $(\bar{\rho},\bar{\varphi})_{count}$, and the theorem is proved. QE.D.

DISCUSSION

We have previously shown [3] that identifiability is implied by the finite memory and definite memory properties [6]. It is also a minimal condition for inferring uniqueness of response function assignment in the sense that there are representations $(\bar{\rho},\bar{\varphi})$ and $(\bar{\rho}',\bar{\varphi}')$ of the same system such that neither is identifiable, and neither response family is contained in the other [3]. However, the question of whether there is a causal time system with no identifiable realizations and with no canonical representation remains open.

REFERENCES

1. Y. Takahara and M. Mesarovic, Realizability and general dynamical systems, *Info. Sci.* **8**, 109–121 (1975).
2. B. P. Zeigler and S. Aggarwal, Canonical realization of I-O relations, *Proceedings of the Eighth Annual Princeton Conference on Information Sciences and Systems*, Princeton University Press, Princeton, N. J. 1974, p. 161.
3. B. P. Zeigler, *Theory of Modelling and Simulation*, John Wiley, New York, 1976, Chapter 15.
4. M. A. Arbib, *Theories of Abstract Automata*, Prentice-Hall, Englewood Cliffs, N. J., 1969, Chapter 4.
5. J. N. Gray and M. Harrison, The theory of sequential relations, *Inf. Control* **9**, 435–468 (1966).
6. A. Gill, *Introduction to the Theory of Finite State Machines*, McGraw-Hill, New York, 1973.
7. S. Ginsburg, *An Introduction to Mathematical Machine Theory*, Addison-Wesley, Reading, Mass., 1966.

Received September, 1976