Topology Vol. 16, pp. 417–421. Pergamon Press, 1977. Printed in Great Britain

A NOTE ON THE SPECTRA OF ALGEBRAIC K-THEORY

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(Received 21 February 1977)

IN [9], QUILLEN defined the K-groups of a ring A as the homotopy groups of a certain space

$$KA = BGl_\infty(A)^+ \times K_0(A).$$

Subsequently (cf. [10]) he characterized the space KA as follows: One takes the category $\mathscr{P}A$ of finitely generated projective modules over A together with all isomorphisms between them. Then one performs a sort of Grothendieck construction on the category $\mathscr{P}A$ to obtain a new category $\mathscr{H}\mathscr{P}A$. Quillen showed that

$$KA = B\mathscr{H}\mathscr{P}A$$

where $B\mathscr{H}\mathscr{P}A$ is the classifying space of the category $\mathscr{H}\mathscr{P}A$.

It has been shown in various places (cf. [1–4, 6, 11–13] that KA is an infinite loop space. This means that KA is the 0-th space of an Ω-spectrum, i.e. there is a sequence of spaces

$$\boldsymbol{E}A = \{E_nA \mid n \geqslant 0\}$$

and homotopy equivalences

$$\sigma_n: \quad E_nA \xrightarrow{\simeq} \Omega E_{n+1}A$$

such that $E_0A = KA$.

The most explicit construction of this sort is the Gersten–Wagoner approach [4 and 13]. Given a ring A we associate to it the ring CA of infinite matrices over A having only finitely many nonzero entries in each row and column. We then divide out by the ideal generated by matrices in CA having only finitely many nonzero entries. The resulting ring is denoted by SA. Gersten and Wagoner showed there is a fibration sequence

$$\cdots \to \Omega KA \to \Omega KCA \to \Omega KSA \xrightarrow{\eta} KA \to KCA \to KSA$$

and that KCA is homotopy trivial. Consequently η is a homotopy equivalence. Iterating this process they obtain an Ω-spectrum

$$\boldsymbol{GWA} = \{GW_nA = KS^nA \mid n \geqslant 0\}$$

with structure maps

$$\epsilon_n = \eta^{-1}: \quad KS^nA \xrightarrow{\simeq} \Omega KS^{n+1}A.$$

The competing constructions are not quite as explicit. However they have the advantage of applying to a wider context than just to the higher algebraic K-theory of rings. For instance, perhaps the most highly developed of these theories is that of May [6 and 7]. His theory associates to any symmetric monoidal category $\mathscr{A}$ an Ω-spectrum

$$\boldsymbol{M}\mathscr{A} = \{M_n\mathscr{A} \mid n \geqslant 0\}.$$

*Partially supported by NSF Grant No. MCS 76-07232.

If one applies his construction to the category $\mathscr{A} = \mathscr{P}A$ one gets an Ω-spectrum $\mathbf{M}A \doteq \mathbf{M}\mathscr{P}A$ whose 0-th space is KA.

A natural question which immediately comes to mind is whether the two spectra $\mathbf{M}A$ and $\mathbf{GW}A$ are equivalent (cf. [8, problem 9]). Unfortunately a moment's reflection shows that this cannot be the case. For the spectrum $\mathbf{M}A$ is (-1)-connected, i.e. each space M_nA is $(n-1)$-connected. On the other hand $\mathbf{GW}A$ is not (-1)-connected, since in general

$$\pi_0 GW_nA = \pi_0 KS^nA = K_0(S^nA) \neq 0.$$

Similar problems arise when comparing $\mathbf{GW}A$ with the spectra of [1–3, 11 and 12].

However we shall show that this is essentially the only problem which arises. Any Ω-spectrum $\mathbf{E}A$ which satisfies certain natural conditions will be equivalent to the spectrum $\mathbf{GW}A$ up to connectivity. In particular $\mathbf{M}A$ satisfies the conditions in question.

In what follows we will use the term *a higher algebraic K-theory* to denote a functor $\mathbf{E}$ which assigns to each ring A an Ω-spectrum $\mathbf{E}A = \{E_nA \mid n \geq 0\}$ such that $E_0A = KA$

UNIQUENESS THEOREM. *Let $\mathbf{E}$ be a higher algebraic K-theory which has an external tensor product. Then there is a natural map*

$$f = \{f_n\}\colon \quad \mathbf{E}A \to \mathbf{GW}A$$

of spectra such that f_0: $E_0A = KA \to KA$ is the identity map.

It follows that f induces an isomorphism on homotopy groups in nonnegative degrees. Upon killing the homotopy groups of $\mathbf{E}A$ and $\mathbf{GW}A$ in negative degrees, f becomes an equivalence. Thus the (-1)-connected covers of $\mathbf{E}A$ and $\mathbf{GW}A$ are equivalent spectra.

A higher algebraic K-theory is said to have an external tensor product if it recognizes the following construction: Let A and B be two rings. If P is a finitely generated projective A-module and Q is a finitely generated projective B-module, the $P \otimes_Z Q$ is a finitely generated projective $A \otimes_Z B$-module. This defines a functor

$$\otimes\colon \quad \mathscr{P}A \times \mathscr{P}B \to \mathscr{P}(A \otimes_Z B)$$

called the external tensor product. It is easily checked that this induces a functor

$$\otimes\colon \quad \mathscr{H}\mathscr{P}A \times \mathscr{H}\mathscr{P}B \to \mathscr{H}\mathscr{P}(A \otimes_Z B).$$

Passing to classifying spaces we get a map

$$KA \times KB \to K(A \otimes_Z B)$$

which factors through the smash product as a map

$$\mu\colon \quad KA \wedge KB \to K(A \otimes_Z B).$$

Definition. A higher algebraic K-theory $\mathbf{E}$ is said to have an external tensor product if there is a natural pairing of spectra (in the sense of G. W. Whitehead [14])

$$\boldsymbol{\lambda}\colon \quad (\mathbf{E}A, \mathbf{E}B) \to \mathbf{E}(A \otimes_Z B)$$

which extends the map

$$\mu\colon \quad KA \wedge KB \to K(A \otimes_Z B)$$

of 0th spaces

In [5] Loday found a more explicit formulation of the structure maps in $\mathbf{GW}A$.

LODAY'S LEMMA. *The structure map*

$$\epsilon_N:\quad GW_nA = KS^nA \xrightarrow{\simeq} \Omega GW_{n+1}A = \Omega KS^{n+1}A$$

is the adjoint of the composite map

$$S^1 \wedge KS^nA \xrightarrow{\alpha\wedge 1} KS\mathbf{Z} \wedge KS^nA \xrightarrow{\mu} K(S\mathbf{Z} \otimes_{\mathbf{Z}} S^nA) = KS^{n+1}A$$

where $\alpha \in \pi_1\ KS\mathbf{Z} \cong \pi_0 K\mathbf{Z} = K_0(\mathbf{Z}) \cong \mathbf{Z}$ is the standard generator.

Using this result he was able to show that the higher algebraic K-theory $\boldsymbol{GW}$ has an external tensor product. More recently May has shown that his theory $\boldsymbol{M}$ has an external tensor product (cf. latest version of [7]). Thus by the uniqueness theorem the theory $\boldsymbol{M}$ is equivalent to $\boldsymbol{GW}$ up to connectivity. It is probable that the various other theories [1, 2, 3, 11 and 12] can be shown to have an external tensor product and are therefore also equivalent to $\boldsymbol{GW}$ up to connectivity.

Before we prove the uniqueness theorem, we prove another result from which the uniqueness theorem immediately follows.

Definition. An ***Ω-bispectrum*** is an Ω-spectrum of Ω-spectra. More precisely an Ω-bispectrum $\boldsymbol{X}_*$ is a sequence of Ω-spectra

$$\boldsymbol{X}_* = \{\boldsymbol{X}_n | n \geq 0\} = \{X_{n,m} | n, m \geq 0\}$$

together with maps of spectra

$$\zeta_n:\quad \boldsymbol{X}_n \to \Omega \boldsymbol{X}_{n+1}$$

such that the induced maps of 0-th spaces

$$\zeta_{n,0}:\quad X_{n,0} \to \Omega X_{n+1,0}$$

are homotopy equivalences.

There are two spectra we can naturally associate with a bispectrum $\boldsymbol{X}_*$: the 0-th spectrum $\boldsymbol{X}_0 = \{X_{0,n} | n \geq 0\}$ (the going -up spectrum), and the spectrum

$$\boldsymbol{X}^0 = \{X_{n,0} | n \geq 0\}$$

(the going-across spectrum) with structure maps

$$\zeta_{n,o}:\quad X_{n,0} \to \Omega X_{n+1,0}.$$

The following result relates these two spectra.

UP OR ACROSS LEMMA. *If $\boldsymbol{X}_*$ is an Ω-bispectrum, there is a natural map of spectra*

$$f:\quad \boldsymbol{X}_0 \to \boldsymbol{X}^0$$

such that the map of 0-th spaces

$$f_0:\quad X_{0,0} \to X_{0,0}$$

is the identity.

It is an immediate consequence that f induces an equivalence of spectra between the (-1)-connected covers of $\boldsymbol{X}_0$ and $\boldsymbol{X}^0$.

Proof of the up or across lemma. Let us denote by

$$\zeta_n:\quad \mathbf{X}_0 \to \Omega^n \mathbf{X}_n$$

the composite map of spectra

$$\mathbf{X}_0 \xrightarrow{\zeta_0} \Omega \mathbf{X}_1 \xrightarrow{\Omega\zeta_1} \Omega^2 \mathbf{X}_2 \to \cdots \to \Omega^n \mathbf{X}_n.$$

Here ξ_0: $\mathbf{X}_0 \to \mathbf{X}_0$ is understood to be the identity map. On the n-th space level the map ξ_n and the structural equivalence of the spectrum $\mathbf{X}_n$ yield a map

$$f_n:\quad X_{0,n} \to \Omega^n X_{n,n} \simeq X_{n,0}.$$

A straightforward diagram chase demonstrates the commutativity of the diagrams (up to sign)

$$\begin{array}{ccc} X_{0,n} & \xrightarrow{f_n} & X_{n,0}, \qquad n \geq 0 \\ \Big\downarrow{\scriptstyle \sigma_{0,n}} & & \Big\downarrow{\scriptstyle \xi_{n,0}} \\ \Omega X_{0,n+1} & \xrightarrow{\Omega f_{n+1}} & \Omega X_{n+1,0} \end{array}$$

Thus $f = \{\pm f_n\}$: $\mathbf{X}_0 \to \mathbf{X}^0$ is a map of spectra as required.

Proof of the uniqueness theorem. We use the up or across lemma and Loday's lemma.

Let A be any ring and let α: $S' \to KS\mathbf{Z} = E_0 S\mathbf{Z}$ be the map of Loday's lemma. The pairing $\boldsymbol{\lambda}$ restricts to give the composite map of spectra

$$S^1 \wedge \mathbf{E}S^n A \xrightarrow{\alpha \wedge 1} E_0 S\mathbf{Z} \wedge \mathbf{E}S^n A \xrightarrow{\boldsymbol{\lambda}} \mathbf{E}(S\mathbf{Z} \otimes_{\mathbf{Z}} S^n A) = \mathbf{E}S^{n+1} A$$

(where the smash product of a space and a spectrum is defined as in G. W. Whitehead[14]). We then have the adjoint map

$$\zeta_n:\quad \mathbf{E}S^n A \to \Omega \mathbf{E}S^{n+1} A.$$

Since on the level of 0-th spaces $\boldsymbol{\lambda}$ reduces to the map μ, the 0-th map

$$\zeta_{n,o}:\quad E_0 S^n A \to \Omega E_0 S^{n+1} A$$

of ζ_n coincides with the map

$$\epsilon_n:\quad KS^n A \to \Omega KS^{n+1} A$$

of Loday's lemma (and is thus an equivalence).

Now define an Ω-bispectrum

$$\mathbf{X}_* = \{\mathbf{X}_n = \mathbf{E}S^n A \mid n \geq 0\}$$

having as structure maps the maps

$$\zeta_n:\quad \mathbf{E}S^n A \to \mathbf{E}S^{n+1} A.$$

Clearly $\mathbf{X}_0 = \mathbf{E}A$, while by Loday's lemma $\mathbf{X}^0 = \mathbf{GW}A$. The up or across lemma then gives us the required map of spectra

$$f:\quad \mathbf{E}A \to \mathbf{GW}A.$$

I would like to take this opportunity to acknowledge useful conversations with Max Karoubi and Peter May which contributed to this paper. I would especially like to thank the latter for numerous suggestions which immeasurably improved the organization and exposition of this paper.

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