JOURNAL OF NUMBER THEORY 9, 499–501 (1977)

Solution of a Problem of Skolem

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Communicated by D. J. Lewis

Received December 15, 1976

T. Skolem shows that there are at most six integer solutions to the Diophantine equation $x^5 + 2y^5 + 4z^5 - 10xy^3z + 10x^2yz^2 = 1$. The author shows here that there are precisely three integer solutions.

Skolem [1–3] shows that the equation Norm $(x + y\theta + z\theta^2) = x^5 + 2y^5 + 4z^5 - 10\,xy^3z + 10x^2yz^2 = 1$, where $\theta^5 = 2$, has at most six solutions in integers x, y, z of which he gives three, $(x, y, z) = (1, 0, 0)$, $(-1, 1, 0)$, $(1, -2, 1)$. We show here that there are no further solutions.

We have to find all integers m, n such that

$$\pm(x + y\theta + z\theta^2) = \epsilon_1{}^m\epsilon_2{}^n,$$

where ϵ_1, ϵ_2 are fundamental units of $\mathbf{Q}(\theta)$. By the calculations of Skolem [1], we may take $\epsilon_1 = -1 + \theta$, $\epsilon_2 = 1 + \theta + \theta^3$. We work p-adically, but it is expedient to be judicious in the choice of p; in fact we take $p = 251$, one of the first rational primes to split completely into first-degree prime factors in $\mathbf{Q}(\theta)$: Such primes include 5, 151, 241, 251,.... A small computer calculation shows that

$$\epsilon_1^{250} = 1 + 251\xi \qquad \text{with} \quad \xi \equiv 81\theta - 16\theta^2 + 76\theta^3 - 78\theta^4 \bmod 251,$$

$$\epsilon_2^{50} = 1 + 251\eta \qquad \text{with} \quad \eta \equiv 107\theta + 17\theta^2 - 14\theta^3 + 68\theta^4 \bmod 251,$$

and also that the only terms $\epsilon_1{}^r\epsilon_2{}^s$, $0 \leqslant r \leqslant 249$, $0 \leqslant s \leqslant 49$, having the coefficients of θ^3 and θ^4 both divisible by 251, are given by $(r, s) = (0, 0)$, $(1, 0)$, $(2, 0)$. Writing $m = 250M + r$, $n = 50N + s$, we immediately have

$$\pm(x + y\theta + z\theta^2) = \epsilon_1{}^r(1 + 251\xi)^M\,(1 + 251\eta)^N$$

with $r = 0$, 1, or 2.

ISSN 0022-314X

Write $(1 + 251\xi)^M (1 + 251\eta)^N = 1 + 251(M\xi + N\eta) + 251^2(\ ) + \cdots$
$= K_0 + K_1\theta + K_2\theta^2 + K_3\theta^3 + K_4\theta^4$

with

$$\begin{aligned}
K_0 &= 1 + 251(0 \cdot M + 0 \cdot N) + 251^2(\ ) + \cdots,\\
K_1 &= 251(81M + 107N) + 251^2(\ ) + \cdots,\\
K_2 &= 251(-16M + 17N) + 251^2(\ ) + \cdots,\\
K_3 &= 251(76M - 14N) + 251^2(\ ) + \cdots,\\
K_4 &= 251(-78M + 68N) + 251^2(\ ) + \cdots.
\end{aligned}$$

Equating coefficients of θ^3 and θ^4 to zero gives

(i) $K_3 = K_4 = 0$ when $r = 0$,

(ii) $K_2 - K_3 = K_3 - K_4 = 0$ when $r = 1$,

(iii) $K_1 - 2K_2 + K_3 = K_2 - 2K_3 + K_4 = 0$ when $r = 2$.

Point (i) implies

$$\begin{aligned}
0 &= (76M - 14N) + 251(\ ) + \cdots,\\
0 &= (-78M + 68N) + 251(\ ) + \cdots,
\end{aligned}$$

and since

$$\begin{vmatrix} 76 & -14 \\ -78 & 68 \end{vmatrix} \equiv 60 \bmod 251,$$

we have[1] [by 3, remark at end of proof of Theorem 11] that there is at most one solution, which is clearly $M = N = 0$.

Point (ii) implies

$$\begin{aligned}
0 &= (-92M + 31N) + 251(\ ) + \cdots,\\
0 &= (154M - 82N) + 251(\ ) + \cdots,
\end{aligned}$$

and

$$\begin{vmatrix} -92 & 31 \\ 154 & -82 \end{vmatrix} \equiv 9 \bmod 251,$$

so as above there is at most one solution, which is $M = N = 0$.

[1] For completeness, we state this remark in the form that we need:

Let $F_j(x, y) = \sum_{i=0}^{\infty} p^i f_{i,j}(x, y)$, $j = 1, 2$, where $f_{i,j}(x, y)$ are polynomials with integer coefficients, and p is prime. Suppose that $f_{0,1} = ax + by$, $f_{0,2} = cx + dy$ with

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \not\equiv 0 \bmod p.$$

Then $F_1(x, y) = F_2(x, y) = 0$ has at most one solution in integers x, y.

Point (iii) implies

$$0 = (189M + 59N) + 251(\ ) + \cdots,$$
$$0 = (-246M + 113N) + 251(\ ) + \cdots,$$

and

$$\begin{vmatrix} 189 & 59 \\ -246 & 113 \end{vmatrix} \equiv -22 \bmod 251,$$

so at most one solution, which is $M = N = 0$.

Accordingly, the only solutions are given by

$$(x + y\theta + z\theta^2) = (\theta - 1)^r, r = 0, 1, 2,$$

as required.

References

1. T. Skolem, Einige Sätze über Gewisse Reihenentwicklungen, Skrifter det Norske Videnskaps-Akademi, Oslo, 1933.
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3. T. Skolem, Ein Verfahren zur Behandlung gewisser exponentialer Gleichunger, 8de Skand. mat. Kongr. Forh. Stockholm, 1934.