Which Trees Are Link Graphs?

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The link of a vertex v of a graph G is the subgraph induced by all vertices adjacent to v. If all the links of G are isomorphic to L, then G has constant link and L is called a link graph. We investigate the trees of order $p \leqslant 9$ to see which are link graphs. Group theoretic methods are used to obtain constructions of graphs G with constant link L for certain trees L. Necessary conditions are derived for the existence of a graph having a given graph L as its constant link. These conditions show that many trees are not link graphs. An example is given to show that a connected graph with constant link need not be point symmetric.

LINK GRAPHS

We consider only finite graphs and follow the terminology of [4]. The *link of a vertex* v of a graph G, denoted by $\mathrm{link}(v, G)$, is the subgraph induced by the points adjacent to v, as in [1]. Zykov [6] posed what has become a well known open problem in graph theory which appears to be intractable,
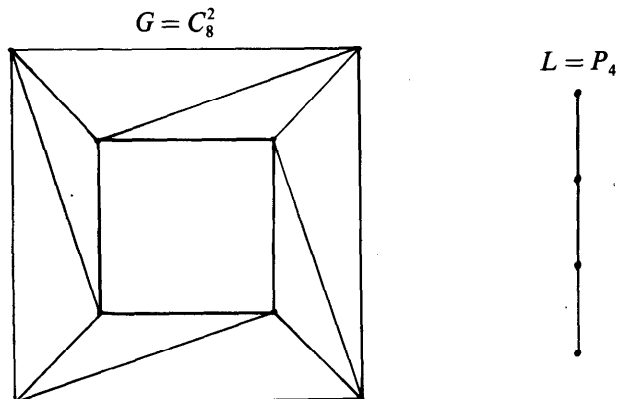
$$G = C_8^2$$

$$L = P_4$$

FIG. 1.   A graph G with constant link L.

namely, for what graphs H is there a graph G such that all links in G are isomorphic to H?

If a graph L is isomorphic to every link in G, then we call L the *link graph of G*, we write $L \to G$, and we say that G *has constant link L* (Fig. 1) and that L is a *link graph*. We sometimes denote link(v, G) by link v for brevity while the link of v in a proper subgraph H of G will always be denoted by link(v, H). The degree of v in a subgraph H of G is written deg(v, H).

Brown and Connelly [1] obtained solutions to Zykov's problem for certain classes of graphs, namely, linear forests, starlike trees, and cycles (Fig. 2). By definition, in a *linear forest* each component is a path.
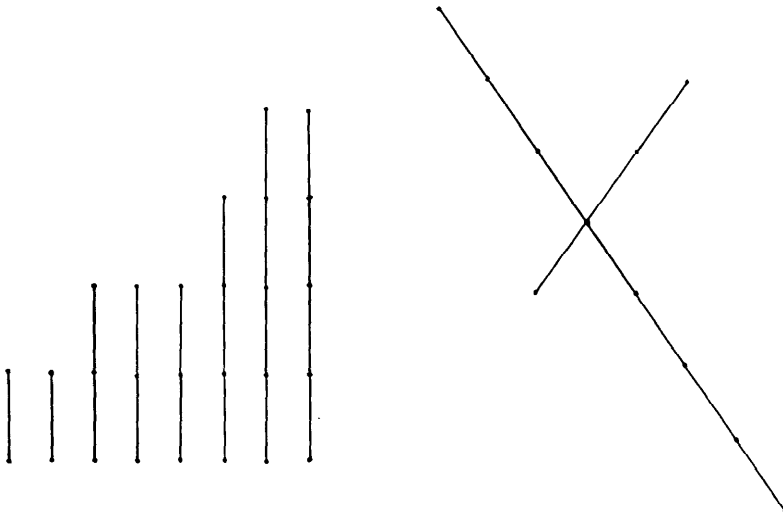
FIG. 2. A linear forest F and a starlike tree T, both of which are link graphs.

THEOREM A. *Let L be a linear forest, where n_i of the paths have length i. Then L is a link graph if and only if*

$$\dot{n}_2 \leqslant n_1 + \sum_{i=4}^{\infty} (i-3)n_i.$$

A tree T is *starlike* if it is homeomorphic to a star. The branches of T at its central point are its *arms*.

THEOREM B. *Let L be starlike with n_i arms of length i. Then L is a link graph if and only if*

$$n_2 \leqslant n_1 + \sum_{i=4}^{\infty} (i-3)n_i,$$

$$n_1 \leqslant \sum_{i=2}^{\infty} n_i,$$

and

$$2n_1 \leqslant 2n_2 + 3n_3 + \sum_{i=4}^{\infty} (i-3)n_i.$$

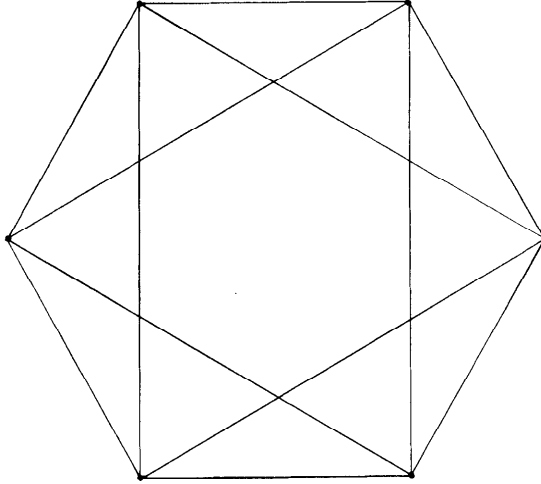THEOREM C. *All cycles are link graphs.*

FIG. 3.   The smallest graph whose link graph is C_4

Similar results are known for unions of stars and for unions of paths and cycles; see Hell [5].

1. NECESSARY CONDITIONS FOR LINK GRAPHS

The theorems proved in this section give conditions on the degrees of adjacent vertices of L which must be satisfied if L is to be a link graph.

We begin by observing that if v and w are adjacent vertices of a graph G, then $\mathrm{link}(v, \mathrm{link}(w, G)) = \mathrm{link}(w, \mathrm{link}(v, G))$. Indeed the two expressions define the same graph, namely, the subgraph induced by the set of points adjacent to both v and w, which is of course the intersection of their links.

For many graphs L, the following result allows us to prove, by mere inspection of L, that it is not a link graph. We define $G(n)$ to be the set of vertices of degree n in G.

THEOREM 1. *If L is a link graph with $L(n) \neq \varnothing$, and B is a set of positive integers such that*

$$n < \min_{v \in L(n)} |w \in \text{link}(v, L): \deg(w, L) \in B|$$

$$+ \max_{v \in L(n)} |w \in \text{link}(v, L): \deg(w, L) \in B|,$$

*then there is an edge of L joining two points whose degrees are in B.*

   Proof. Let $L \to G$. Also, let $u \in V(G)$ with v a point of degree n in $\text{link}(u) \cong L$ at which the maximum in the statement of the theorem occurs. Since $\deg(u, \text{link } v) = n$, our assumption tells us that

$$n < |w \in \text{link}(u, \text{link}(v, G)): \deg(w, \text{link } v) \in B|$$

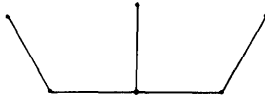$$+ |w \in \text{link}(v, \text{link}(u, G)): \deg(w, \text{link } u) \in B|. \qquad (1)$$

But $\text{link}(v, \text{link}(u, G)) = \text{link}(u, \text{link}(v, G))$, and this subgraph has just n points. Hence the two sets counted in (1) must have a common point w.
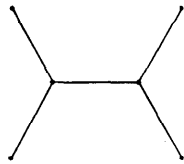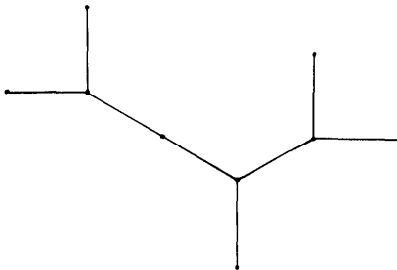

$n = 3$ $B = \{1\}$

$n = 2$ $B = \{1\}$

$n = 3$ $B = \{2\}$

$n = 3$ $B = \{1, 2\}$

FIG. 4. Four trees which are not link graphs by Corollary 1a.

Shifting our attention to link w, we see that

$$\deg(u, \text{link } w) = \deg(w, \text{link } u) \in B$$

and

$$\deg(v, \text{link } w) = \deg(w, \text{link } v) \in B.$$

Thus u and v are adjacent points of link w whose degrees in link w belong to the set B. ∎

COROLLARY 1a. *If L is a graph with $L(n) \neq \emptyset$ and B is a set of positive integers such that*

    (i) *For all $v \in L(n)$, $|w \in \text{link}(v, L): \deg(w, L) \in B| > n/2$,*

    (ii) *No two vertices with degrees in B are adjacent,*

then L is not a link graph. ∎

The corollary may be applied to the four trees in Fig. 4 to show that they cannot be link graphs.

Hell [5] independently found the special case of Corollary 1a for which B is a singleton.

For the two trees in Fig. 5, the full strength of the theorem is needed rather than just the special case in the corollary.

To simplify the statement of the next theorem we introduce the following definitions. Let v and w be adjacent points in a graph G. We call the number of points adjacent to both v and w,

$$n = |\text{link}(v, \text{link}(w, G))| = |\text{link}(w, \text{link}(v, G))|,$$
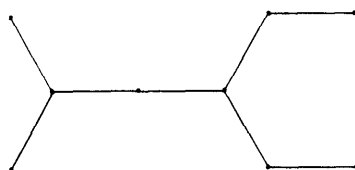
$$n = 3 \qquad B = \{2\}$$
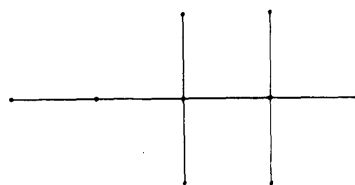
$$n = 4 \qquad B = \{1\}$$

FIG. 5. An application of Theorem 1.

the *relative degree* of v and w, and we denote this number by $\alpha(v, w)$. Furthermore, we say an edge (v, w) is "marked n" if $\alpha(v, w) = n$. For each $v \in V(G)$, let $e_i(v)$ be the number of edges in link v marked i, and let $\bar{e}_i$ be the average,

$$\bar{e}_i = \frac{1}{p(G)} \sum_v e_i(v).$$

Finally, let $p_i(L) = |L(i)|$ be the number of points in a graph L having degree i in L.

Note that $\alpha(v, w)$ is the degree of v in link w, and is also the degree of w in link v.

THEOREM 2.   *If $L \to G$, then for all i,*

$$2\bar{e}_i = i p_i(L).$$

Proof.   Fix an arbitrary i. We will express in two ways the number of ordered triples of points (u, v, w) forming a triangle where edge (u, v) is marked i. Call this number k.

For the first count, we focus on w. Each edge marked i in link w serves as the first edge of two such ordered triangles. This shows that

$$k = 2 \sum_{w \in V(G)} e_i(w) = 2p(G)\bar{e}_i.$$

For the second count, we focus on u. Any of the $p_i(L)$ edges from u to a point v of degree i in link u can serve as the first edge of a triangle. Then any one of the i edges in link u incident to v can serve as the second edge (v, w). The triangle is then completely determined. Thus the number of triangles is also

$$k = p(G) p_i(L) i.$$

Equating the two expressions for k, we get $i p_i(L) = 2\bar{e}_i$.   ∎

In order to apply Theorem 2 effectively, we need more information about the relative degrees of adjacent vertices. The following theorem supplies such information; in conjunction with Theorem 2, it will be used to show that certain graphs L satisfying the necessary conditions in Theorem 1 are nevertheless not link graphs.

THEOREM 3.   *Let $L \to G$ and $v \in V(G)$. For any $s \in$ link v there exists a $t \in L$ such that $\deg(t, L) = \deg(s, \text{link } v)$ and such that the numbers used to mark the lines of link v incident with s are the same (including multiplicity) as the numbers $\deg(u, L)$ for the points u adjacent to t in L.*

Proof.   Let $d = \alpha(s, v)$, and let δ be an isomorphism from link s to L. The t required by the proposition will be $\delta(v)$. Its degree in L is $\deg(\delta(v), L) = \deg(v, \text{link } s) = d$ as desired. The degrees of the vertices of L adjacent to t are the same as the degrees of the vertices of link s adjacent to v (because the isomorphism $\delta : L \to \text{link } s$ sends v to t). The latter degrees are, by definition, the marks on the edges of link v incident on s. ∎
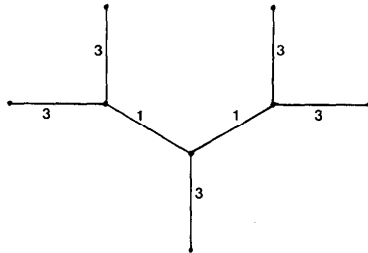
In a *cubic tree*, the degree of each point is 1 or 3.

FIG. 6.   A cubic tree with its edge markings.

COROLLARY 3.   *The cubic tree with eight points is not a link graph.*

Proof.   Let T (as in Fig. 6) be the unique cubic tree of order 8, and assume $T \to G$. Choose $v \in V(G)$ and consider $T = \text{link } v$. Every edge of T must be marked 1 or 3 since in the link of any point in G, all points have degree.1 or 3. In addition, the theorem implies that all edges of T incident to endpoints must be marked 3.

Again by the theorem, each vertex of degree 3 in T must be incident to at least one edge marked 1. Thus the edges of T must be marked as in Fig. 6. This being the case for all link v, $v \in G$, it follows that $\bar{e}_1 = 2$. By Theorem 2, $4 = 2\bar{e}_1 = 1 \cdot p_1(L) = 5$, a contradiction. ∎

Note that Theorem 2 depends crucially on the fact that graphs are by definition finite. If infinite graphs are admitted, then the cubic tree of Fig. 6 can be shown to be a link graph.
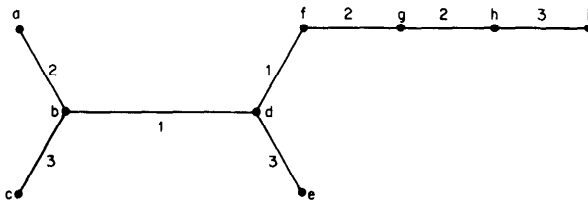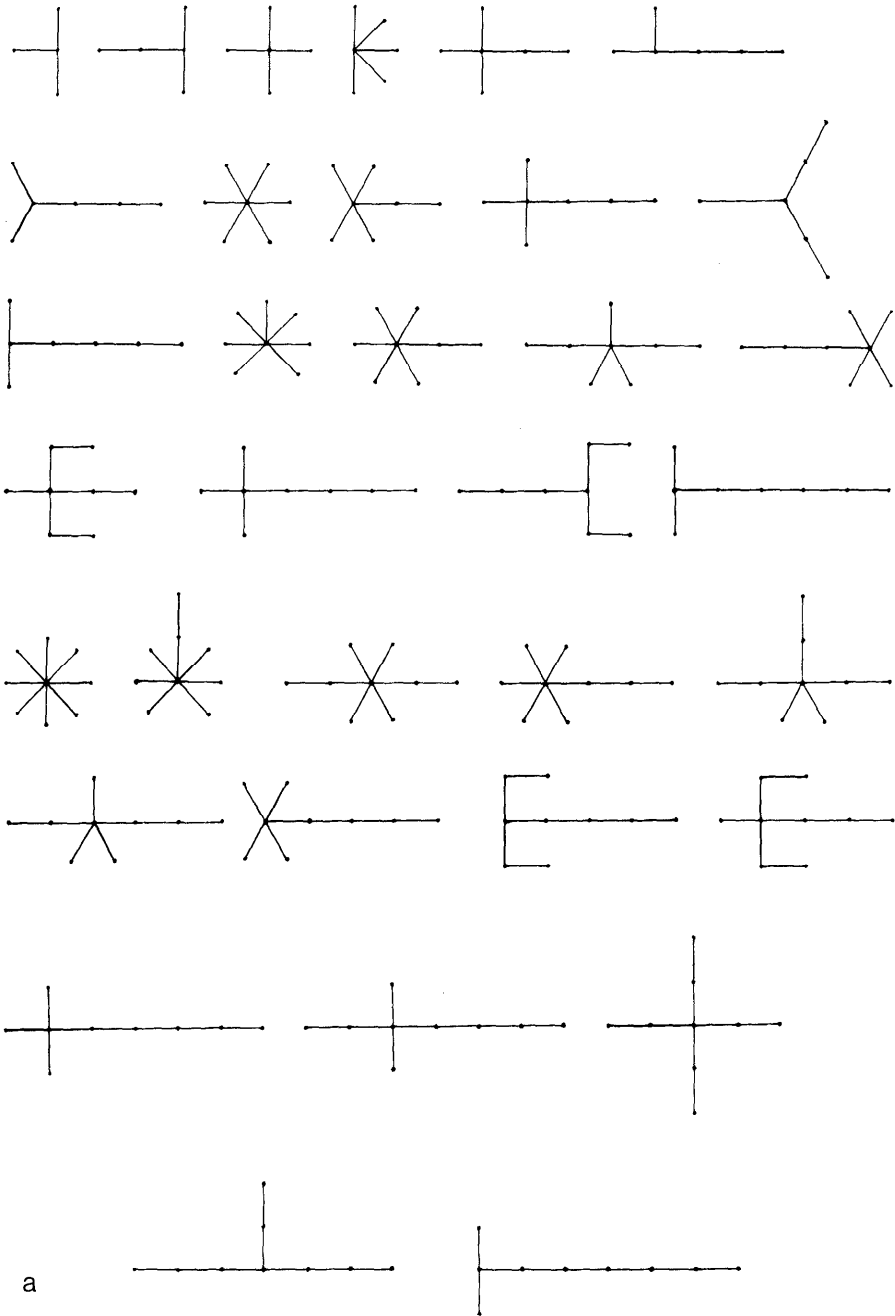

FIG. 7.   A tree which is not a link graph.

FIG. 8. Non-link trees with fewer than 10 points. (a) Non-links by the Brown–Connelly theorems, (b) non-links by Theorem 1, (c) other non-links.
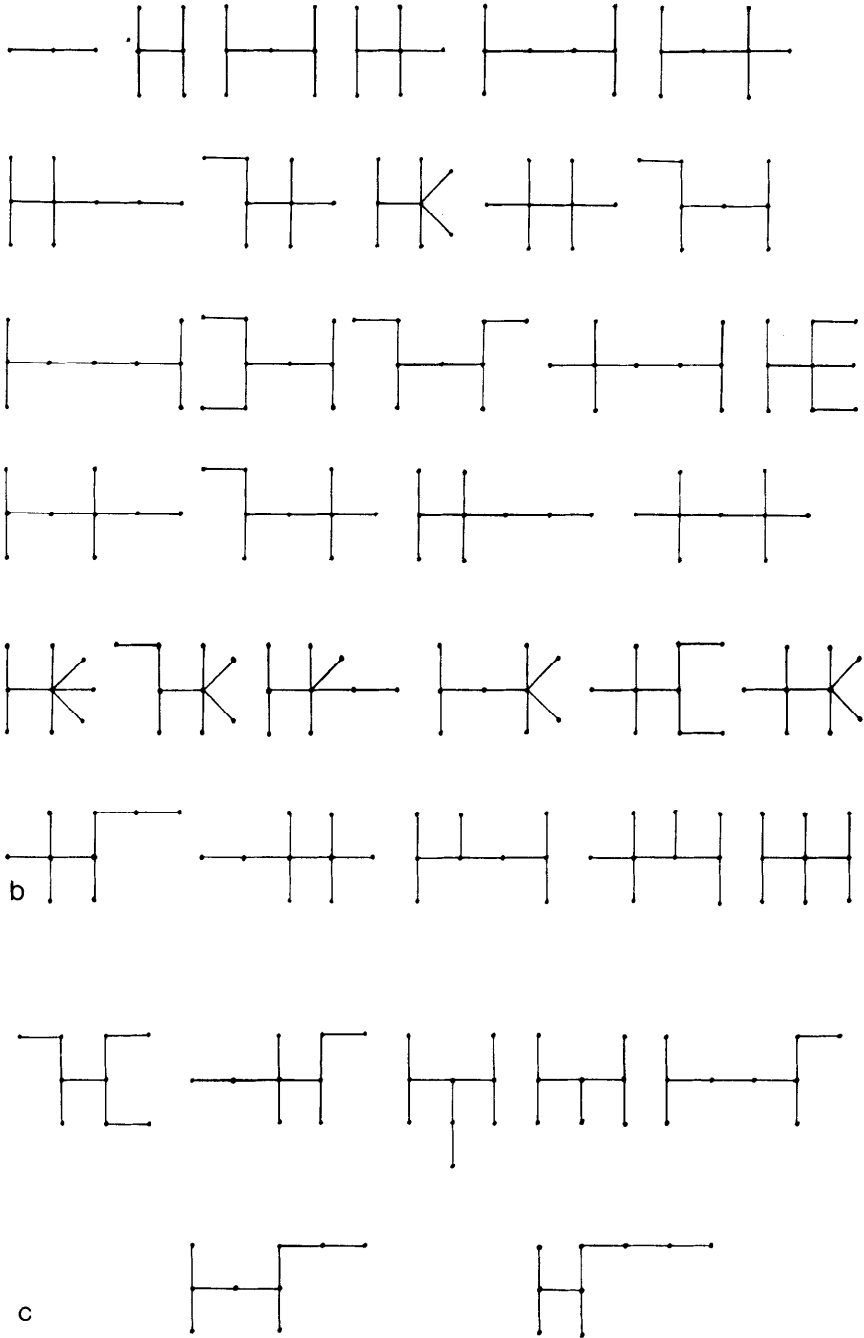
FIG. 8—*Continued.*

The next example of a negative finding is the most complicated one we have obtained yet. The proof that it is not a link graph uses Theorems 2 and 3 and an additional ad hoc argument.

THEOREM 4.   *The tree T of Fig. 7 is not a link graph.*

Sketch of proof.   Suppose $T \to G$. Let $v \in V(G)$ and, for ease of reference, identify link v with T. Theorem 3, applied to the vertices a, b and c of link v, shows that the edges incident with b must be marked as in Fig. 7 (possibly after interchanging a and c). Applied to the vertices of degree 2 in link v, the same theorem shows that at least two edges in the path from d to i are marked 2. In fact, exactly two of them are marked 2 and edge de is not marked 2, because Theorem 2 says $\bar{e}_2 = 3$. Therefore, de is marked 3 and the path di is marked 1223, 1212, 1232, 2123, or 2323, by Theorem 3. For each vertex x of G, let $\gamma(x)$ be the unique vertex of degree two in link x both of whose neighbors in link x also have degree two (so $\gamma(v) = g$). By counting in two ways the set of pairs $(x, \gamma(x))$, we find that the path di must be marked 1223 in every link and that every vertex of G is $\gamma(x)$ for exactly one x. But, knowing that the edges of every link are marked as in Figure 7, we easily see that $g = \gamma(f) = \gamma(v)$, a contradiction. ∎

The methods of this section, together with the Brown–Connelly theorems quoted earlier suffice to prove that none of the trees in Fig. 8 above are link graphs. The figure is divided into three parts. Trees in the first part are not link graphs by the Brown–Connelly Theorem B cited above. Trees in the second part are not link graphs by Theorem 1. Trees in the third part are not link graphs by Theorems 2 and 3 together with *ad hoc* arguments similar to (but generally easier than) the proof of Theorem 4.

The remaining trees with nine or fewer vertices are shown in Fig. 9. In the next section we shall see that all of these trees are link graphs.

## 2. CONSTRUCTION OF GRAPHS WITH PRESCRIBED LINK

In this section we present a group-theoretic method for determining a graph G with constant link L.

Recall that a permutation group H is said to be *sharply transitive* on a set S if H is transitive on S and each permutation in H is uniquely determined by its action on any single element of S.

Let H be a group and Z a generating subset closed under inverses and not containing the identity. The *Cayley graph* $[H; Z]$ is defined to be that graph whose vertex set is H, with u and v adjacent when $u^{-1}v \in Z$. (This is the underlying graph of the symmetric digraph defined in [4, Chap. 14].) Clearly, H is a sharply transitive group of automorphisms of $[H; Z]$ by left transla-
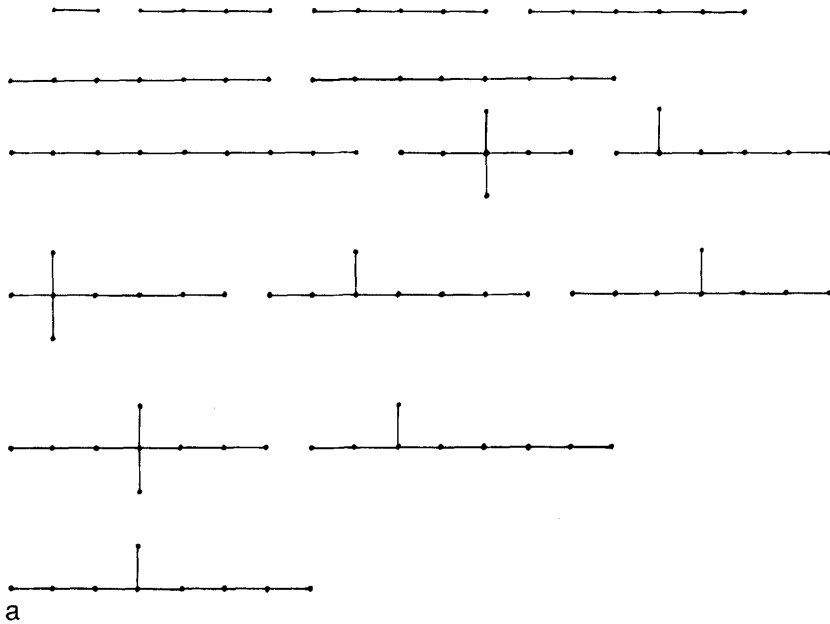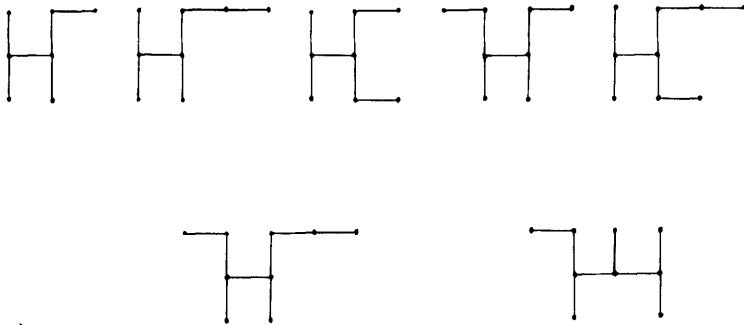
FIG. 9.   Link trees with fewer than 10 points. (a) Links by the Brown–Connelly theorems, (b) other links.

tion. Hence, $[H; Z]$ is a graph of constant link. As Z generates H, it follows that $[H; Z]$ is connected.

Conversely, if H is a sharply transitive group of automorphisms of a connected graph G, then there is a generating subset Z of H, as above, such that $[H; Z]$ is isomorphic to G. Indeed, for any vertex v of G, one can take $Z = \{h \in H: h(v) \in \text{link}(v, G)\}$.

The idea of our constructive method will be to find a group H and subset Z with the property that $[H; Z]$ will have a given L as link graph. In practice, we begin with L and assume that $L \to G$ for some G on which a group H acts in a sharply transitive way. The methods of the preceding section often provide enough information about H (and Z) to suggest a reasonable H to try.

As an example, we will use this method to find a graph G having P_4 as link graph. Such graphs are already known [1], and one is shown in Fig. 1. However we use this relatively simple example to show the method un-obscured by the complications that arise when it is applied to larger links.

Suppose then that $P_4 \to G$ and that the group H acts sharply transitively on G. By Theorem 3, we see easily that, for any fixed $v \in V(G)$, the marking of the edges of $\langle v, \text{link } v \rangle$ must be as shown in Fig. 10.
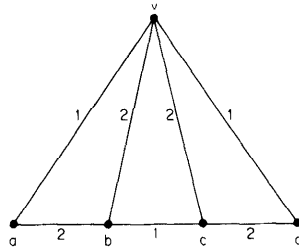

FIG. 10.   Edge marking within a graph having P_4 as constant link.

Let A, B, C, D be the unique automorphisms in H sending v to a, b, c, d, respectively. Using repeatedly the fact that these automorphisms preserve ad-jacency and relative degrees, we compute some relations between them. There are two cases, depending on whether $B^{-1}(v)$ is b or c. If it is b, then $B^2 = 1$, $BC = A$, $C^2 = 1$, and $CB = D$. (All of these equations are obtained by observing that both sides send v to the same vertex and then invoking the assumed sharp transitivity of H.) It is, therefore, natural to take for H a group generated by elements B, C with $B^2 = C^2 = 1$ and to take for Z the set $\{A, B, C, D\} = \{BC, B, C, CB\}$. Further relations must be imposed on the generators B and C to make the group H and, therefore, the graph $[H; Z]$ finite.

To choose the relations appropriately, note that $[H; Z]$ is in any case a graph with constant link $L = \text{link } 1 = Z$ with the adjacency relation $u^{-1}v \in Z$. In order that this link be P_4 as desired, we must arrange the defin-ing relations of H so that the four elements BC, B, C, CB of Z are distinct and so that no unwanted adjacencies occur; for example $(BC)^{-1}(CB)$ must not equal any member of Z lest the two endpoints of the path be adjacent. Straightforward calculation shows that what is required of H, beyond the

equations $B^2 = C^2 = 1$, is that $BC \neq CB$ and $(CB)^3 \neq 1$. These conditions are satisfied by the dihedral groups of order $\geqslant 8$. For example, we can take the group D_4 of symmetries of a square, where B is the reflection in a diagonal and C is the reflection in a line parallel to an edge. The resulting graph $[D_4, Z]$ is shown in Fig. 1.

Had we chosen $B^{-1}(v) = c$ rather than b, we would have obtained the relations $A = B^2$, $C = B^{-1}$, $D = B^{-2}$, so we could take H cyclic, generated by B, and $Z = \{B^2, B, B^{-1}, B^{-2}\}$. To prevent unwanted adjacencies in the link, we must take the order p of B to be at least 7. The resulting graph is C_p^2. If p is even, this is the same as what we got from the dihedral groups. If $p = 7$, we obtain the smallest possible graph with constant link P_4, namely, C_7^2.

The procedure for constructing a graph with constant link L is in general analogous to what we have done for P_4. One extracts from the results in Section 1 as much information as possible about the edge markings in the proposed link. If this does not lead to a contradiction, it imposes fairly strong constraints on the generators of a hypothetical group H and subset Z for which $[H; Z]$ has the desired link. Finally, one imposes additional relations on the generators to make H finite, but without producing unwanted identifications or adjacencies between elements of Z.

In many cases, the existence of these "finitizing" relations is assured by the results in [3], which imply that, for any free product of finitely many cyclic groups,

$$P = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z} * \mathbb{Z}_{n_1} * \cdots * \mathbb{Z}_{n_k},$$

and any finite subset $R \subseteq P$ such that $1 \notin R$, there exists a homomorphism f of P onto a finite group such that $1 \notin f(R)$. The theorem is applicable in the example above and in all examples to be given below; the provisional group generated by Z subject to the relations that give the desired adjacencies will have the required structure of a free product. In the example this provisional group was $\mathbb{Z}_2 * \mathbb{Z}_2$ in the first case and $\mathbb{Z}$ in the second.

Figure 9 shows all the trees, with nine or fewer vertices, that are not prevented from being link graphs by the results in Section 1 and the Brown–Connelly theorems. The paths and stars in the first part of the figure are link graphs by Theorems A and B. For all but the last of the others, the group-theoretic method outlined above provided a Cayley graph with the desired link. Table I lists appropriate H and Z for each of these trees and exhibits link$(1, [H; Z])$. The symbol "$< \infty$" in each H indicates additional relations to make H finite, as discussed above; in each case, the result quoted from [3] provides such relations. For example, the finitizing relations in the first line of Table I can be taken to be $B^8 = 1$ and $AB = B^3 A$. The resulting graph $[H, Z]$, which we believe to be the smallest graph with this link is shown in Fig. 11.

TABLE I

Some Trees That Are Link Graphs of Cayley Diagrams

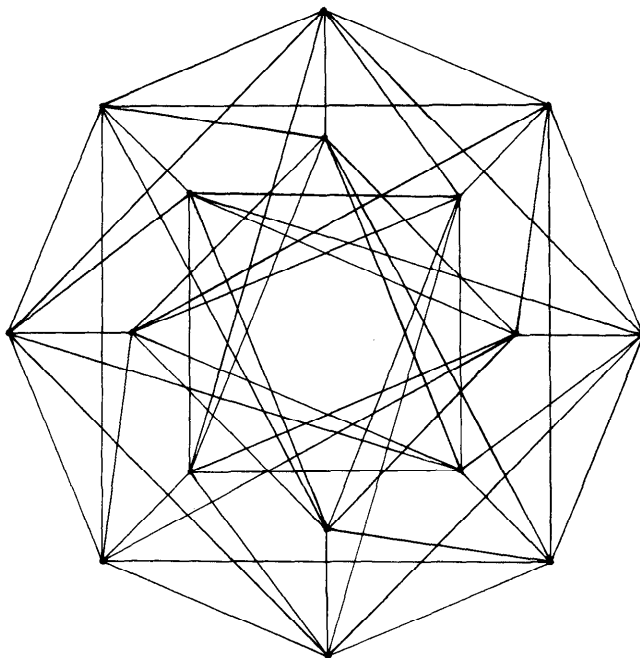| H	Z	link$(1, [H;Z])$
$\langle A, B : A^2 = 1, <\infty \rangle$	$\{A, B^{-2}, B^{-1}, B, B^2, BA, AB^{-1}\}$	$BA - A - AB^{-1}$, $\ \mid B^{-1}$, $B - B^2$, B^{-2}
$\langle A, B : A^3 = 1, <\infty \rangle$	$\{A, A^2, B^{-2}, B^{-1}, B, B^2, BA, A^2B^{-1}\}$	$BA - A - A^2 - A^2B^{-1}$, $\ \mid B^{-1}$, $B - B^2$, B^{-2}
$\langle A, B, C : A^2 = B^2 = C^3 = 1, <\infty \rangle$	$\{A, B, C, C^2, CA, AC^2, CB, BC^2\}$	$CA - A - AC^2$, $C - C^2$, $CB - B - BC^2$
$\langle A, B, C : A^2 = B^2 = C^3 = 1, <\infty \rangle$	$\{A, B, C, C^2, CA, AC^2, BC, C^2B\}$	$BC - B - A - AC^2$, $C - C^2$, $CA\quad C^2B$
$\langle A, B, C : A^2 = B^3 = C^3 = 1, <\infty \rangle$	$\{A, B, B^2, C, C^2, CB, B^2C^2, CA, AC^2\}$	$CB - B - B^2 - B^2C^2$, $C - C^2$, $CA - A - AC^2$
$\langle A, B, C : A^2 = B^3 = C^3 = 1, <\infty \rangle$	$\{A, B, B^2, C, C^2, CB, B^2C^2, AC, C^2A\}$	$AC - A - B - B^2 - B^2C^2$, $C - C^2$

FIG. 11.   A graph having as constant link the first tree in Table I.

This graph can be efficiently described, following Frucht [2], as $8(1, 2) \xrightarrow{0,1,5} 8(2, 3)$. In this notation, $8(1, 2)$ refers to a graph with eight vertices v_i $(0 \leqslant i \leqslant 7)$ with v_i adjacent to v_j when $|i - j|$ is congruent to 1 or 2 modulo 8; this is the outer part of Fig. 11. Similarly, $8(2, 3)$ describes the inner graph of 8 vertices w_i $(0 \leqslant i \leqslant 7)$. The 0, 1, 5 indicates that v_i is adjacent to w_j when $j - i$ is congruent to 0, 1, or 5 modulo 8.

Using Frucht's notation, Fig. 12 exhibits a graph of order 48 whose link is the last tree in Fig. 9.

$$12(4) \xrightarrow{0,3,4,9} 12(3)$$
$$\uparrow{\scriptstyle 0,4,9} \qquad\qquad \downarrow{\scriptstyle 0,4,9}$$
$$12(3) \xleftarrow{0,3,4,9} 12(4)$$

FIG. 12.   The proof that the last tree in Fig. 9 is a link graph.

3. Unsolved Problems

PROBLEM 1.   We expect that as $p \to \infty$, the probability that a tree T of order p is a link graph should approach 0 or 1. We conjecture that the answer is 0.

PROBLEM 2.   What is $\min p(G)$ such that there exists a connected graph G with constant link, but G is not point symmetric?

We know that the last tree of Fig. 9 is not a link graph in any point-symmetric G. Furthermore the graph of order 16 in Fig. 13 has constant link $\bar{K}_3$ but is not point symmetric. Hence the answer to Problem 2 is at most 16.
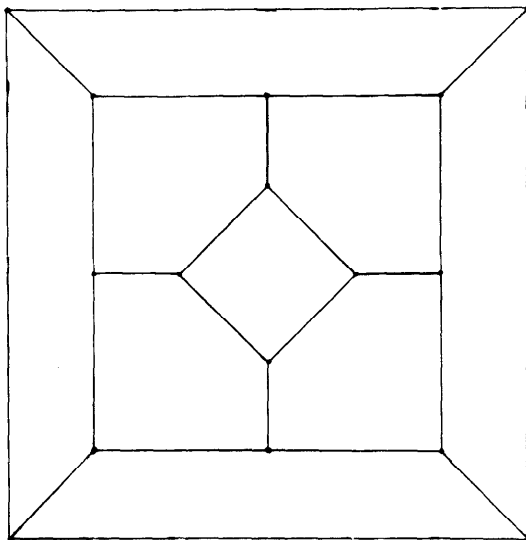

FIG. 13.   A graph of order 16 with constant link $\bar{K}_3$ that is not point symmetric.

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