A NOTE ON WAVE PROPAGATION IN BONE

In a recent work on wave propagation, Ambardar and Ferris [1] modelled long bone as a two-layered cylindrical structure, consisting of a thin inner layer of spongy material and a thick outer layer of more compact bone. Both types of bone were taken to be transversely isotropic and piezoelectric effects were included. The authors gave detailed information on frequency spectra, but no numerical data, or data on transients, were presented. Here an explicit algebraic, long wavelength approximation to the lowest mode of compressional wave propagation in an infinite two-layered cylinder, the layers being transversely isotropic, is given. Piezoelectric effects are not included, but imperfect layer bonding is allowed. Information on transients is also given.

The work is a direct extension of a previous paper by the author [2] in which a pressure shock problem for a cylinder made of isotropic layers was considered. In fact the solution procedures, and the form of the solution, are identical to those in reference [2]. Differences arise only through the structure of the parameters and only that detail will be presented here.

For the pressure shock problem specified in reference [2], the solutions now are

\[
\frac{c_1}{\sigma_0} \vec{\varepsilon}_{\sigma} = \frac{R_h}{U_1[1 + R_\mu(R_h - 1)]} \left[ 1 + \frac{\psi'}{3} \int_0^\phi \text{Ai}(-\eta) \, d\eta \right],
\]

\[
\frac{c_1}{\sigma_0} \vec{\varepsilon}_{zz} = \frac{R_h}{U_1[1 + R_\mu(R_h - 1)]} \left[ 1 + \frac{\psi'}{3} \int_0^\phi \text{Ai}(-\eta) \, d\eta \right],
\]

where Ai denotes the Airy function and

\[
\psi' = \left( \frac{2R_\mu \sqrt{U_1}}{3U_2} \right)^{1/3} \left( \frac{\sqrt{U_1 \tau - \xi'}}{\tau} \right), \quad \nu = \sqrt{\frac{c_1}{\rho_1}}, \quad \zeta = \frac{z}{b}, \quad \xi' = \frac{r}{b}.
\]

Further, following the sequence in reference [2], one has

\[
\rho_1 P_1 = c_1 U_1, \quad \rho_2 P_2 = c_1 b^2 U_2,
\]

\[
U_1 = 4\nu - \frac{b_1}{c_1} \xi + \frac{d_1}{c_1}, \quad \nu = \frac{\xi}{2} + \frac{R_\mu(1 - R^2_h)}{2\beta} - R_\mu(R^2_h - 1) \theta,
\]

\[
\left[ 2 \frac{(a_1 - e_1)}{c_1} - D_B 1 \right] \zeta = D_B 1 \left[ 1 + R^2_h \left( \frac{a_2 - e_2}{e_2} \right) \right] \frac{\delta}{\beta} + \frac{b_1 - c_1}{c_1} + \frac{D_B 1 R^2_h}{2} \left( \frac{b_2 - c_2}{e_2} \right).
\]

\[
R_\mu = c_2/c_1, \quad R_h = c/b,
\]

\[
\beta = \frac{(a_2 - e_2)}{c_2} (R^2_h - 1) \left[ 2 \frac{(a_1 - e_1)}{c_1} - D_B 1 \right] - \frac{D_B 1}{R_\mu} \left( \frac{a_1 - e_1}{c_1} \right) \left[ 1 + \frac{(a_2 - e_2)}{e_2} \right] R^2_h,
\]

\[
\delta = \frac{1}{2} \left[ 2 \left( \frac{a_1 - e_1}{c_1} - D_B 1 \right) \left( \frac{b_2 - c_2}{c_2} \right) (1 - R^2_h) \left( \frac{b_1 - c_1}{c_1} \right) \frac{1}{R_\mu} \right] + \frac{1}{R_\mu} \left( \frac{a_1 - e_1}{c_1} \right) \left( \frac{b_1 - c_1}{c_1} + \frac{D_B 1 R^2_h}{2} \left( \frac{b_2 - c_2}{e_2} \right) \right),
\]
\[ \theta = \{2R_p[\zeta + R_m(1 - R_h^2)(\delta/\beta)] - \eta\} \{4R_m[1 + R_p(R_h^2 - 1)]\}^{-1}, \]
\[ D_B1 = \alpha_1 b/c_1, \quad R_p = \rho_2/\rho_1, \]
\[ \eta = (d_2/c_1) - (d_1/c_1)R_p + (2b_2\delta/c_1\beta) + (2b_1R_p\delta/c_1), \]
\[ U_2 = U_{21} + U_{22}, \]
\[ U_{21} = \{(V_1 + V_2 + V_3 + V_4 + V_5 + V_6)/[1 - R_p(1 - R_h^2)]\} - 4R_mR_h^2\nu_1 + 2R_p b_2(\xi_2 + \xi_3 + \xi_4 + \xi_5)/c_2\xi_1, \]
\[ U_{22} = b_3R_p R_h^2\delta_1/c_2 + [(d_2/c_2)R_m - R_p[4\nu - (2b_1/c_1)\xi + d_1/c_1]]\Psi, \]
\[ V_1 = 8R_m(1 - R_h)\eta_1 + 2R_p R_h(1 - R_h^4)\delta_1 - 8R_p\xi_1, \]
\[ V_2 = (2R_p/\xi_1)(\xi_2 + \xi_3 + \xi_4 + \xi_5)[R_m(1 - R_h^2) - (b_2/c_2)] - 2R_p\Omega, \]
\[ V_3 = -2R_p R_h^2[R_m(1 - R_h^2 - 2R_h^2 \ln R_h - 2)]\nu_1, \]
\[ V_4 = \frac{2R_p(b_1 - c_1)}{c_1} \left[ \frac{c_1 R_m(a_2 - e_2)(1 - R_h^2)(\xi_2 + \xi_3 + \xi_4 + \xi_5)}{c_2(a_1 - e_1)\xi_1} \right], \]
\[ V_5 = -R_p[2R_p R_h^2 \ln R_h + (b_2/c_2)]R_h^2\delta_1, \]
\[ V_6 = 2R_p \left( \frac{b_1}{c_1} \zeta - 2\nu \right) \left[ \frac{d_2}{c_2} R_m - R_p \left( 4\nu - 2\frac{b_1}{c_1} \zeta + \frac{d_1}{c_1} \right) \right] \Psi, \]
\[ \eta_1 = \frac{\theta}{16} \left[ \frac{R_m}{R_p} \left( 4\nu - 2\frac{b_1}{c_1} \zeta + \frac{d_1}{c_1} \right) - \frac{d_2}{c_2} \right] - \frac{b_2}{4c_2} \beta_1, \]
\[ \beta_1 = \frac{c_2\delta}{8a_2\beta} \left[ R_p \left( 4\nu - 2\frac{b_1}{c_1} \zeta + \frac{d_1}{c_1} \right) - 1 \right] - \frac{\theta b_2}{4a_2}, \]
\[ \zeta_1 = (\nu/8)[2\nu - (b_1/c_1)\zeta] - (b_1/4c_1)\Omega, \]
\[ \Omega = -(b_1/4a_1)\nu + (\zeta c_1/8a_1)[1 - 4\nu + (2b_1/c_1)\xi - (d_1/c_1)], \]
\[ \xi_1 = \frac{c_1 R_m(1 - R_h^2)(a_2 - e_2)}{c_2(a_1 - e_1)} \left[ D_{B1} - 2\left( \frac{a_1 - e_1}{c_1} \right) \right] - D_{B1} \left[ 1 + \left( \frac{a_2 - e_2}{e_2} \right) R_h^2 \right], \]
\[ \xi_2 = \left[ 2\left( \frac{2a_1 - e_1}{c_1} \right) - D_{B1} \right] \Omega + \left( \frac{b_1 - c_1}{c_1} \right) \nu + \left( \frac{b_1 - c_1}{c_1} \right) \left[ \frac{d_2}{c_2} R_p(R_h - 1) \Psi \right], \]
\[ \phi = \frac{1}{2} \xi^2 - \frac{3}{2} \nu - \frac{3}{2} R_p(R_h^3 - 1) \theta + 2R_p R_h^2 [R_h(\ln R_h - 1) + 1] \xi \]
\[ + \frac{3}{2} R_p(R_h^3 - 1)(\delta^2/\beta^2) + R_p R_h^3(R_h - 1)\gamma^2 + 2R_p(R_h - 1)R_h^2\delta \gamma/\beta, \]
\[ \xi = 2\theta + \gamma + \delta/\beta, \]
\[ \gamma = [(a_2 - e_2)/e_2] (\delta/\beta) + (b_2 - c_2)/2e_2, \]
\[ \Psi = \left[ \nu + (\xi/D_{B2}) + \frac{\phi}{2} - \theta - \frac{2\nu}{D_{B2}} \right] \left[ 1 + R_p(R_h - 1) \right]^{-1}, \quad D_{B2} = \alpha_2 b/c_1, \]
\[ \xi_3 = \left[ \frac{2(a_1 - e_1)}{c_1} - D_{B1} \right] \nabla_1 + \nabla_2 + \nabla_3 + \nabla_4 + D_{B1} \left[ 1 + \left( \frac{2a_2 - e_2}{e_2} \right) R_h^4 \right] \beta_1, \]
\[ \nabla_1 = \frac{(b_1 - c_1)}{2(a_1 - e_1)} \left[ \frac{\phi}{2} - R_p(R_h - 1) \Psi \right] - \left( \frac{2a_1 - e_1}{a_1 - e_1} \right) \Omega - \frac{(b_1 - c_1)\nu}{2(a_1 - e_1)}, \]
\[ \nabla_2 = \left( \frac{2a_2 - e_2}{c_2} \right) \left( \frac{c_1}{a_1 - e_1} \right) \left( 1 - R_{h}^4 \right) R_{\mu} \beta_1 + \left( \frac{b_2 - e_2}{2c_2} \right) \left( \frac{c_1}{a_1 - e_1} \right) \left( 1 - R_{h}^4 \right) R_{\mu} \theta. \]

\[ \nabla_3 = R_{\mu} \left( \frac{c_1}{a_1 - e_1} \right) \left[ \frac{a_2}{2c_2} \left( 1 - R_{h}^2 \right) - \left( \frac{a_2 - e_2}{c_2} \right) R_{h}^2 \ln R_{h} \right] \delta_1 R_{h}^2. \]

\[ \delta_1 = \frac{c_2}{2a_2} R_{\mu} \left( 4 \nu - \frac{2b_1}{c_1} \zeta + \frac{d_1}{c_1} \right) \chi + \frac{b_2}{2a_2} \xi, \]

\[ \nabla_4 = R_{\mu} \left( \frac{b_2 - c_2}{2c_2} \right) \left( \frac{c_1}{a_1 - e_1} \right) \left[ \left( 1 - R_{h}^2 \right) \Psi + \xi R_{h}^4 \ln R_{h} \right]. \]

\[ \xi_4 = \frac{(b_2 - c_2)D_{B1}R_{h}^4}{2e_2} + \frac{D_{B1}R_{h}^4}{2} \left[ \frac{a_2}{e_2} + \frac{2(a_2 - e_2)}{e_2} \ln R_{h} \right] \delta_1, \]

\[ \xi_5 = \left( \frac{(b_2 - c_2)}{2e_2} \right) R_{h}^2 D_{h1} \left[ \Psi - R_{h}^2 \xi \ln R_{h} \right]. \]

\[ \nu_1 = \frac{1}{2} \xi \left( \frac{R_{h}}{R_{\mu}} \right) \left[ 4 \nu - \left( 2b_1/c_1 \right) \zeta + \left( d_1/c_1 \right) - \left( d_2/c_2 \right) \right] + b_2 \delta_1/2c_2, \]

where the \( c \) are the elastic constants of layer \( j \). Explicitly, the low frequency, large wavelength approximation that was used in arriving at the above is \( \omega = k \sqrt{P_1 - k^2 P_2} \), where \( k \) denotes wavenumber.

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