A NOTE ON WAVE PROPAGATION IN BONE

In a recent work on wave propagation, Ambardar and Ferris [1] modelled long bone as a two-layered cylindrical structure, consisting of a thin inner layer of spongy material and a thick outer layer of more compact bone. Both types of bone were taken to be transversely isotropic and piezoelectric effects were included. The authors gave detailed information on frequency spectra, but no numerical data, or data on transients, were presented. Here an explicit algebraic, long wavelength approximation to the lowest mode of compressional wave propagation in an infinite two-layered cylinder, the layers being transversely isotropic, is given. Piezoelectric effects are not included, but imperfect layer bonding is allowed. Information on transients is also given.

The work is a direct extension of a previous paper by the author [2] in which a pressure shock problem for a cylinder made of isotropic layers was considered. In fact the solution procedures, and the *form* of the solution, are identical to those in reference [2]. Differences arise only through the structure of the parameters and only that detail will be presented here.

For the pressure shock problem specified in reference [2], the solutions now are

$$\frac{c_{12}e_{rr}}{\sigma_0} = \frac{R_h[(\gamma R_h^2/\xi'^2) - \delta/\beta]}{U_1[1 + R_\rho(R_h - 1)]}\left[\frac{1}{3} + \int_0^{\psi'} \mathrm{Ai}\,(-\eta)\,\mathrm{d}\eta\right],$$

$$\frac{c_{11}e_{zz}}{\sigma_0} = \frac{c_{12}e_{zz}}{\sigma_0} = -\frac{R_h}{U_1[1 + R_\rho(R_h - 1)]}\left[\frac{1}{3} + \int_0^{\psi'} \mathrm{Ai}\,(-\eta)\,\mathrm{d}\eta\right],$$

where Ai denotes the Airy function and

$$\psi' = \left(\frac{2R_\rho\sqrt{U_1}}{3U_2}\right)^{1/3}\frac{(\sqrt{U_1}\tau - \zeta')}{\tau^{1/3}}, \qquad \tau = \sqrt{\frac{c_1}{\rho_1}}\frac{t}{b}, \qquad \zeta' = \frac{z}{b}, \qquad \xi' = \frac{r}{b}.$$

Further, following the sequence in reference [2], one has

$$\rho_1 P_1 = c_1 U_1, \qquad \rho_2 P_2 = c_1 b^2 U_2,$$

$$U_1 = 4\nu - \frac{2b_1}{c_1}\zeta + \frac{d_1}{c_1}, \qquad \nu = \frac{\zeta}{2} + \frac{R_\mu(1 - R_h^2)\delta}{2\beta} - R_\mu(R_h^2 - 1)\theta,$$

$$\left[2\left(\frac{a_1 - e_1}{c_1}\right) - D_{B1}\right]\zeta = D_{B1}\left[1 + R_h^2\left(\frac{a_2 - e_2}{e_2}\right)\right]\frac{\delta}{\beta} + \frac{b_1 - c_1}{c_1} + \frac{D_{B1}R_h^2}{2}\left(\frac{b_2 - c_2}{e_2}\right),$$

$$R_\mu = c_2/c_1, \qquad R_h = c/b,$$

$$\beta = \left(\frac{a_2 - e_2}{c_2}\right)(R_h^2 - 1)\left[2\left(\frac{a_1 - e_1}{c_1}\right) - D_{B1}\right] - \frac{D_{B1}}{R_\mu}\left(\frac{a_1 - e_1}{c_1}\right)\left[1 + \left(\frac{a_2 - e_2}{e_2}\right)R_h^2\right],$$

$$\delta = \frac{1}{2}\left[2\left(\frac{a_1 - e_1}{c_1}\right) - D_{B1}\right]\left[\left(\frac{b_2 - c_2}{c_2}\right)(1 - R_h^2) - \left(\frac{b_1 - c_1}{c_1}\right)\frac{1}{R_\mu}\right]$$
$$+ \frac{1}{R_\mu}\left(\frac{a_1 - e_1}{c_1}\right)\left[\frac{b_1 - c_1}{c_1} + \frac{D_{B1}R_h^2}{2}\left(\frac{b_2 - c_2}{e_2}\right)\right],$$

$$\theta = \{2R_\rho[\zeta + R_\mu(1 - R_h^2)(\delta/\beta)] - \eta\}\{4R_\mu[1 + R_\rho(R_h^2 - 1)]\}^{-1},$$

$$D_{B1} = \alpha_1 b/c_1, \qquad R_\rho = \rho_2/\rho_1,$$

$$\eta = (d_2/c_1) - (d_1/c_1)R_\rho + (2b_2\delta/c_1\beta) + (2b_1R_\rho\zeta/c_1),$$

$$U_2 = U_{21} + U_{22},$$

$$U_{21} = \{(V_1 + V_2 + V_3 + V_4 + V_5 + V_6)/[1 - R_\rho(1 - R_h^2)]\} - 4R_\mu R_h^2\nu_1$$
$$+ 2R_\mu b_2(\xi_2 + \xi_3 + \xi_4 + \xi_5)/c_2\xi_1,$$

$$U_{22} = b_2 R_\mu R_h^2\delta_1/c_2 + \{(d_2/c_2)R_\mu - R_\rho[4\nu - (2b_1/c_1)\zeta + d_1/c_1)]\}\Psi,$$

$$V_1 = 8R_\rho R_\mu(1 - R_h^4)\eta_1 + 2R_\rho R_\mu(1 - R_h^4)\beta_1 - 8R_\rho\zeta_1,$$

$$V_2 = (2R_\mu/\xi_1)(\xi_2 + \xi_3 + \xi_4 + \xi_5)[R_\rho(1 - R_h^2) - (b_2/c_2)] - 2R_\rho\Omega,$$

$$V_3 = -2R_\mu R_h^2[R_\rho(1 - R_h^2 - 2R_h^2\ln R_h) - 2)]\nu_1,$$

$$V_4 = \frac{2R_\rho(b_1 - c_1)}{c_1}\left[\frac{c_1 R_\mu(a_2 - e_2)(1 - R_h^2)(\xi_2 + \xi_3 + \xi_4 + \xi_5)}{c_2(a_1 - e_1)\xi_1} + \nabla_1 + \nabla_2 + \nabla_3 + \nabla_4\right],$$

$$V_5 = -R_\mu[2R_\rho R_h^2\ln R_h + (b_2/c_2)]R_h^2\delta_1,$$

$$V_6 = 2R_\rho\left(\frac{b_1}{c_1}\zeta - 2\nu\right)\left[\frac{\phi}{2} - R_\rho(R_h - 1)\Psi\right] - \left[\frac{d_2}{c_2}R_\mu - R_\rho\left(4\nu - 2\frac{b_1}{c_1}\zeta + \frac{d_1}{c_1}\right)\right]\Psi,$$

$$\eta_1 = \frac{\theta}{16}\left[\frac{R_\rho}{R_\mu}\left(4\nu - \frac{2b_1}{c_1}\zeta + \frac{d_1}{c_1}\right) - \frac{d_2}{c_2}\right] - \frac{b_2}{4c_2}\beta_1,$$

$$\beta_1 = \frac{c_2\delta}{8a_2\beta}\left[\frac{R_\rho}{R_\mu}\left(4\nu - \frac{2b_1}{c_1}\zeta + \frac{d_1}{c_1}\right) - 1\right] - \frac{\theta b_2}{4a_2},$$

$$\zeta_1 = (\nu/8)[2\nu - (b_1/c_1)\zeta] - (b_1/4c_1)\Omega,$$

$$\Omega = -(b_1/4a_1)\nu + (\zeta c_1/8a_1)[1 - 4\nu + (2b_1/c_1)\zeta - (d_1/c_1)],$$

$$\xi_1 = \frac{c_1 R_\mu(1 - R_h^2)(a_2 - e_2)}{c_2(a_1 - e_1)}\left[D_{B1} - 2\left(\frac{a_1 - e_1}{c_1}\right)\right] - D_{B1}\left[1 + \left(\frac{a_2 - e_2}{e_2}\right)R_h^2\right],$$

$$\xi_2 = \left[2\left(\frac{2a_1 - e_1}{c_1}\right) - D_{B1}\right]\Omega + \left(\frac{b_1 - c_1}{c_1}\right)\nu + \left(\frac{b_1 - c_1}{c_1}\right)\left[\frac{\phi}{2} - R_\rho(R_h - 1)\Psi\right],$$

$$\phi = \tfrac{1}{3}\zeta^2 - \tfrac{2}{3}\nu - \tfrac{2}{3}R_\rho(R_h^3 - 1)\theta + 2R_\rho R_h^2[R_h(\ln R_h - 1) + 1]\xi$$
$$+ \tfrac{1}{3}R_\rho(R_h^3 - 1)(\delta^2/\beta^2) + R_\rho R_h^3(R_h - 1)\gamma^2 + 2R_\rho(R_h - 1)R_h^2\delta\gamma/\beta,$$

$$\xi = 2\theta + \gamma + \delta/\beta,$$

$$\gamma = [(a_2 - e_2)/e_2](\delta/\beta) + (b_2 - c_2)/2e_2,$$

$$\Psi = \left[\nu + (\zeta/D_{B2}) + \frac{\phi}{2} - \theta - \frac{2\nu}{D_{B2}}\right][1 + R_\rho(R_h - 1)]^{-1}, \qquad D_{B2} = \alpha_2 b/c_1,$$

$$\xi_3 = \left[\frac{2(a_1 - e_1)}{c_1} - D_{B1}\right](\nabla_1 + \nabla_2 + \nabla_3 + \nabla_4) + D_{B1}\left[1 + \left(\frac{2a_2 - e_2}{e_2}\right)R_h^4\right]\beta_1,$$

$$\nabla_1 = -\frac{(b_1 - c_1)}{2(a_1 - e_1)}\left[\frac{\phi}{2} - R_\rho(R_h - 1)\Psi\right] - \left(\frac{2a_1 - e_1}{a_1 - e_1}\right)\Omega - \frac{(b_1 - c_1)\nu}{2(a_1 - e_1)},$$

$$\nabla_2 = \left(\frac{2a_2 - e_2}{c_2}\right)\left(\frac{c_1}{a_1 - e_1}\right)(1 - R_h^4)R_\mu \beta_1 + \left(\frac{b_2 - c_2}{2c_2}\right)\left(\frac{c_1}{a_1 - e_1}\right)(1 - R_h^4)R_\mu \theta,$$

$$\nabla_3 = R_\mu\left(\frac{c_1}{a_1 - e_1}\right)\left[\frac{a_2}{2c_2}(1 - R_h^2) - \left(\frac{a_2 - e_2}{c_2}\right)R_h^2 \ln R_h\right]\delta_1 R_h^2,$$

$$\delta_1 = \frac{c_2}{2a_2}\left[\frac{R_\rho}{R_\mu}\left(4\nu - \frac{2b_1}{c_1}\zeta + \frac{d_1}{c_1}\right) - 1\right]\gamma + \frac{b_2}{2a_2}\xi,$$

$$\nabla_4 = R_\mu\left(\frac{b_2 - c_2}{2c_2}\right)\left(\frac{c_1}{a_1 - e_1}\right)[(1 - R_h^2)\Psi + \xi R_h^4 \ln R_h],$$

$$\xi_4 = \frac{(b_2 - c_2)D_{B1}R_h^4\theta}{2e_2} + \frac{D_{B1}R_h^4}{2}\left[\frac{a_2}{e_2} + \frac{2(a_2 - e_2)}{e_2}\ln R_h\right]\delta_1,$$

$$\xi_5 = [(b_2 - c_2)/2e_2]R_h^2 D_{B1}[\Psi - R_h^2\xi \ln R_h],$$

$$\nu_1 = \tfrac{1}{4}\xi\{(R_\rho/R_\mu)[4\nu - (2b_1/c_1)\zeta + d_1/c_1] - (d_2/c_2)\} + b_2\delta_1/2c_2,$$

$$a_j = {}_jc_{11}, \quad b_j = {}_jc_{13}, \quad c_j = {}_jc_{44}, \quad d_j = {}_jc_{33}, \quad 2e_j = {}_jc_{11} - {}_jc_{12}, \quad j = 1, 2,$$

the $_jc$ being the elastic constants of layer j. Explicitly, the low frequency, large wavelength approximation that was used in arriving at the above is $\omega = k\sqrt{P_1 - k^2 P_2}$, where k denotes wavenumber.

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REFERENCES

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