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An Example of a Rigid Partial Differential Equation*

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The motions of a partial differential equation $Lu = 0$ are those changes of the independent variables that take solutions to solutions. The study of the motions can shed light on the structure of the equation and on its space of solutions. An extreme case is when there are *no* motions except the identity transformation, in which case we say that the equation is *rigid*. We show here that this extreme case can actually occur. Indeed, it appears from our methods that this is probably the generic situation. Let us be more precise.

Let $Lu = 0$ be a partial differential equation satisfied by the function $u = u(x_1, ..., x_n)$ in a region $\Omega \subseteq \mathbb{R}^n$. We say that a C^∞ map φ: $\Omega \to \Omega$ is a C^∞-motion of the equation $Lu = 0$ if $L(u) = 0 \Rightarrow L(u \circ \varphi) = 0$. (It can be important to consider motions φ that are not C^∞, but we do not treat this theme here.) The problem of *analysis* is to find all the motions—the problem of *synthesis* is to reconstruct L, as much as possible, from its motions. (See [2, 4–7]). Such synthesis will be impossible if L is rigid, that is, has no motions except for the identity map. So it is natural to ask whether there exist any rigid partial differential equations. There is a growing interest in questions of this kind. For a blanket reference, we give [1], which the interested reader may take as a starting point for investigating the literature.

In this paper we provide a class of examples of linear partial differential equations with constant coefficients that are rigid on certain domains Ω. By contrast, Laplace's equation in two variables, for example, is far from rigid, allowing as motions any analytic (or conjugate analytic) map φ from Ω into Ω.

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Our method gives some insight into the general problem of analysis in the constant-coefficient case on an arbitrary region Ω. Perhaps, if pushed hard enough, it would enable the complete calculation of all the motions in this case. It shows that "in general," the Jacobian matrix $\mathscr{D}\varphi$ of any motion φ must lie in a certain group of invariants of the homogeneous polynomial given by the highest order terms in L(i.e., the principal *symbol* of L). For Laplace's equation, for example, this group is just the scalar multiples of the orthogonal transformations. Our main result is the following, in which we cannot yet handle the case $n > 2$ because of apparently technical difficulties.

THEOREM. *There is a constant-coefficient linear partial differential equation* $Lu = 0$ *which is rigid on every bounded region in* $\mathbb{R}^n$, *where* $n = 2$. *Further, if* $\Omega = \mathbb{R}^n$ *(again for* $n = 2$*), then the only motions of* L *are translations.*

Of course, each translation $\varphi(x) = x - \tau$, $\tau \in \mathbb{R}^n$, is a motion of any such equation, provided Ω is all of $\mathbb{R}^n$. Our proof proceeds by analyzing two semigroups of invariants.

DEFINITION. Let $Q(\xi) = Q(\xi_1, ..., \xi_n)$ be a homogeneous polynomial of degree m in n variables (real or complex). We let G_Q denote the class of all $n \times n$ matrices E such that for some scalar $\lambda \neq 0$, $Q(E\xi) = \lambda Q(\xi)$ for all vectors ξ. The class SG_Q is defined the same way, except that we allow $\lambda = 0$.

Note that G_Q contains all scalar multiples of the identity. Further, unless Q is a function of fewer than n variables, G_Q consists only of nonsingular transformations—hence the letter "S" in "SG_Q" for "singular," because SG_Q contains many singular transformations. For example if $Q(\xi_0) = 0$ and if E is any map into the line joining 0 to ξ_0, then $Q(E\xi) = 0 = 0 \cdot Q(\xi)$, so that $E \in SG_Q$.

DEFINITION. We say that Q is rigid when G_Q consists only of scalar multiples of the identity.

For an n-tuple $\alpha = (\alpha_1, ..., \alpha_n)$ of integers, we write $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$. Now let

$$P(\xi) = \sum_{|\alpha| \leqslant m} C_\alpha \xi^\alpha$$

be a polynomial of degree m, and let

$$Q(\xi) = \sum_{|\alpha| = m} C_\alpha \xi^\alpha$$

be the principal symbol of P. We will consider the linear constant-coefficient operator $L = P(D)$, where, as usual,

$$D = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$$

$$D^\alpha = \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial x_1{}^\alpha \cdots \partial x_n{}^\alpha}.$$

The main tool in our analysis of the motions is the following, which we will prove later.

PROPOSITION 1. *Let $\varphi: \Omega \to \Omega$ be a C^∞ motion of the equation $P(D)u = 0$. Then for every $x \in \Omega$, $(\mathscr{D}\varphi)(x) \in SG_Q$. Further,*

$$Q((\mathscr{D}\varphi)(x)\xi) = H(x)Q(\xi),$$

where H is a C^∞ function on Ω.

PROPOSITION 2. *When $n > 1$, there exists a rigid homogeneous polynomial Q in n variables.*

Proof. Replacing E by $\lambda^{-1/n}E$, we see that it is enough to study the condition $Q(E\xi) = Q(\xi)$. Each such condition gives an algebraic equation on the coefficients of Q and E. Hence, the condition that E must be the identity is *generic*. This means that either all homogeneous polynomials of degree m (for n fixed), with the exception of those whose coefficients lie in a proper algebraic variety in the space of coefficients, admit only $E = \alpha I$, where α is an nth root of unity, as a solution of $Q(E\xi) = Q(\xi)$ for all ξ, or else all homogeneous polynomials of this degree must admit a solution $E \neq \alpha I$. Thus, to prove that the generic Q of degree m is rigid, it suffices to exhibit one such Q.

What is actually true is that in two variables, the generic Q of degree $\geqslant 4$ is rigid, while those of degree 1, 2, 3 are not. This may be seen as follows. If $Q(x, y)$ has degree m, then $Q(x, y)$ is the product of m linear factors $l_i(x, y) = a_i x + b_i y$. Any linear transformation E leaving Q invariant must permute the lines $l_i(x, y) = 0$. Thus, thinking of E as a transformation of the projective space $\mathbb{P}^1$, we see that E must permute the points of $\mathbb{P}^1$ corresponding to the lines l_i. Now it is well known that any three points of $\mathbb{P}^1$ can be mapped to any other three points by a projective transformation, but for $m \geqslant 4$ this is no longer true, since the cross ratio of four points is preserved under a projective transformation. It follows that the generic polynomial $Q(x, y)$ in two variables, of degree $m \geqslant 4$, is rigid.

We now state, without proofs, some relevant facts that are known to algebraic geometers. In $n = 3$ variables, no homogeneous polynomial of degree $m \leqslant 3$ is rigid. However, for degree $m \geqslant 4$, the generic Q is rigid. An example is

$$Q(x, y, z) = (x^3 + 2y^3 + 3z^3)(x + y + z).$$

A similar example works in $n > 3$ variables. In this case ($n > 3$), the generic Q of degree $m \geqslant 3$ is rigid.

We now use Propositions 1 and 2 to prove the theorem, keeping $n = 2$ unspecified as long as possible.

Proof of the Theorem. Let Q be a rigid homogeneous polynomial of degree m. We will find P in the form $P = Q + R$, where $R \neq 0$ is a polynomial of degree $<m$. If φ is a C^∞-motion of $P(Du) = 0$, then by Proposition 1, we have $Q((\mathscr{D}\varphi)(x)\xi) = H(x)Q(\xi)$, where $H \in C^\infty(\Omega)$. Let

$$U = \{x \in \Omega \colon H(x) \neq 0\}.$$

We claim that U is open and closed in Ω. Since H is continuous, U is open. If $x \in U$, then $(\mathscr{D}\varphi)(x) = \lambda(x)I$. In particular, $\partial\varphi_i/\partial x_j = 0$ for $i \neq j$, and so $\varphi(x) = (\varphi_1(x_1),..., \varphi_n(x_n))$, locally on U. Let us fix i. Now $\partial\varphi_i/\partial x_i = \lambda(x)$, so that $\lambda(x)$ must be a function only of x_i. Since $n \geqslant 2$, this implies that λ is locally constant on U. Hence $\varphi_i(x) = cx_i + d_i$ for some constants c and d_i, locally on U. But the set of $x \in \Omega$ on which $\varphi_i(x) = cx_i + d_i$ is clearly closed, and $(\mathscr{D}\varphi)(x)$ is constant on this set. Thus, we must have either $U = \Omega$ or $U = \varnothing$. This leads to two cases.

If $U = \Omega$ (i.e., $\varphi_i(x) = cx_i + d_i$ for all $x \in \Omega$), then it is easy to complete the proof—we have to prove that $c = 1$. Suppose $c \neq 1$. Since $R \neq 0$, we can choose $\xi \in \mathbb{C}^n$ so that $P(\xi) = 0$, but $P(c\xi) \neq 0$. Then $u(x) = e^{\xi \cdot x}$ is a solution of $P(D)(u) = 0$. Hence also $(u \cdot \varphi)(x) = \text{const}\, e^{c\xi \cdot x}$ is a solution. But this implies $P(c\xi) = 0$, since $P(D_x)\, e^{\xi \cdot x} = P(\xi) \cdot e^{\xi \cdot x}$, a contradiction which proves that $c = 1$ so that φ is a translation. If Ω is bounded then φ must surely be the identity. This handles the first case.

In the second case, we have

$$Q((\mathscr{D}\varphi)(x)\xi) = 0,$$

for all $x \in \Omega$ and all $\xi \in \mathbb{C}^n$. Here is where technical troubles lead us to assume, from now on, that $n = 2$, so that Q is a product of linear functions. Let $v_1 ,..., v_k$ be unit vectors in $\mathbb{C}^2$ along the lines on which Q vanishes. Let $u_1 ,..., u_k$ be their orthogonal unit vectors. We suppose now that R satisfies $R(0) \neq 0$, and that $Q(\xi) + R(\xi) = P(\xi)$ does not reduce to a constant on any of the lines $\{\xi = tu_j\colon t \in \mathbb{C}\}$. For example, the affine function

$$R(\xi) = 1 + \beta\xi_1 + \gamma\xi_2$$

will always have this property if β and γ are suitably chosen constants. Now if $(\mathscr{D}\varphi)(x) \equiv 0$ on Ω, then φ is a constant map. Thus, if ξ is any zero of $P(\xi) = 0$, then $u(x) = e^{\xi \cdot x}$ is a solution of $P(D)u = 0$, and so $u \circ \varphi$ is constant. But $P(0) \neq 0$, so the constant function is not a solution of $P(D)u = 0$. This rules out $\varphi = \text{const}$. Thus $(\mathscr{D}\varphi)(x)$ must have rank 1 on an open set V in Ω. From the equation $Q((\mathscr{D}\varphi)(x)(\xi)) \equiv 0$, we see that the range of $\mathscr{D}\varphi(x)$ must be contained in the union of the lines that gives the zero set of Q. So, for some v_j, in terms of the orthonormal basis v_j, u_j, we have

$$(\mathscr{D}\varphi)(x) = \begin{pmatrix} a(x) & b(x) \\ 0 & 0 \end{pmatrix}.$$

This implies that $\varphi(x) = \lambda(x)\, v_j + cu_j$, on some open set V, where $\lambda(x)$ is a C^∞ function and c is a constant.

Now the map $t \mapsto P(tu_j)$ does not reduce to a constant, so we can find $\xi = \delta u_j$ so that $P(\xi) = 0$. Hence, $e^{\delta u_j \cdot \varphi(x)}$ is a constant on V, since u_j is orthogonal to v_j. But this is a contradiction because $P(0) \neq 0$ and thus $P(D_x)u = 0$ has no solution $u = \text{const}$ on any open set.

This proves the Theorem in the stated case $n = 2$. It would be good to find an argument that handles our second case ($\mathscr{D}\varphi$ is singular) for $n > 2$.

It remains to prove Proposition 1, a "division lemma." For this, we introduce what is essentially the symbol of P after the change of coordinates under φ,

$$F(x, \xi) = e^{-\xi \cdot \varphi(x)} P(D_x)\, e^{\xi \cdot \varphi(x)}.$$

Note that $F\colon \Omega \times \mathbb{C}^n \to \mathbb{C}$ is then a C^∞ map on $\Omega \times \mathbb{C}^n$, and a polynomial in ξ of degree $\leqslant m = \deg P$.

LEMMA 1. *If $P(\xi) = 0$, and if φ is a C^∞-motion of $P(D)u = 0$, then $F(x, \xi) = 0$ for all $x \in \Omega$, even when multiplicities are taken into account.*

Proof. This means that for all $x \in \Omega$

$$\frac{\partial^j}{\partial \xi_i^{\,j}} P(\xi) = 0, \quad 0 \leqslant j < k \Rightarrow \frac{\partial^j}{\partial \xi_i^{\,j}} F(x, \xi) = 0, \quad 0 \leqslant j < k. \tag{1}$$

It is clear from the definition of F that $F = 0$ if ξ is chosen so that $P(\xi) = 0$. To handle higher multiplicities, we use the Leibnitz formula

$$\frac{\partial^j}{\partial \xi_i^{\,j}} (e^{\xi \cdot \varphi(x)} F(x, \xi)) = \sum_{l=0}^{j} \binom{j}{l} \frac{\partial^{j-l}}{\partial \xi_i^{\,j-l}} [e^{\xi \cdot \varphi(x)}] \frac{\partial^l}{\partial \xi_i^{\,l}} [F(x, \xi)]. \tag{2}$$

On the other hand, the left-hand side of (2) equals

$$\frac{\partial^j}{\partial \xi_i^{\,j}} P(D_x)[e^{\xi \cdot \varphi(x)}] = P(D_x)\left(\frac{\partial^j}{\partial \xi_i^{\,j}} e^{\xi \cdot \varphi(x)}\right) = P(D_x)(u_j(\varphi(x))), \tag{3}$$

where

$$u_j(x) = \frac{\partial^j}{\partial \xi_i^{\,j}} e^{\xi \cdot x} = x_i^{\,j} e^{\xi \cdot x}.$$

Next, recall the general Leibnitz formula (see [3, Eq. (1.4.12), p. 10]) that

$$P(D_x)[a(x)\, e^{\xi\cdot x}] = \sum_{\gamma} (D_x{}^{\gamma} a)\, \frac{P^{(\gamma)}(D_x)}{\gamma!}\, [e^{\xi\cdot x}],$$

where the sum is over all multi-indices $\gamma = (\gamma_1, ..., \gamma_n)$, and $P^{(\gamma)}(\xi) = D_\xi{}^{\gamma} P(\xi)$. Applying this formula with $a(x) = x_i{}^j$, we see that the sum is only over those γ of the form $\gamma = (0, ..., 0, \gamma_i, 0, ..., 0)$, with $0 \leqslant \gamma_i \leqslant j$. Thus,

$$\begin{aligned} P(D_x)[u_j(x)] &= \sum_{l=0}^{j} \frac{1}{l!} \frac{\partial^l}{\partial x_i{}^l} [x_i{}^j] \cdot P^{(0,\dots,0,l,0,\dots,0)}(D_x)[e^{\xi\cdot x}] \\ &= \sum_{l=0}^{j} \frac{1}{l!} \frac{\partial^l}{\partial x_i{}^l} [x_i{}^j] \cdot \frac{\partial^l P(\xi)}{\partial \xi_i{}^l}\, e^{\xi\cdot x}. \end{aligned} \tag{4}$$

(The index vectors $(0, ..., 0, l, 0, ..., 0)$, etc., have their non-zero entry in the ith place.) The assertion (1) follows from (2), (3), and (4). For if the condition on the left-hand side of (1) holds, then by (4), we have $P(D_x)[u_j(x)] = 0$ for $0 \leqslant j < k$. Then because φ is a motion, it follows from (3) that the left-hand side of (2) vanishes for all $x \in \Omega$ when $0 \leqslant j < k$. Applying this successively for $j = 0, 1, 2, ..., k - 1$, we then obtain

$$0 = F(x, \xi) = \frac{\partial F(x, \xi)}{\partial \xi_i} = \cdots = \frac{\partial^j}{\partial \xi_i{}^j} F(x, \xi)$$

for $0 \leqslant j < k$, which proves (1).

As a corollary of Lemma 1, we have the following fact.

LEMMA 2. *If φ is a C^∞-motion of $P(D_x)u = 0$, then*

$$F(x, \xi) = H(x) P(\xi),$$

where H is a C^∞ function on G.

Proof. The lemma is a standard "division lemma," although we do not know an exact reference to this particular result. If F were analytic in x, then it would follow directly from, say, Theorem 9J, p. 29 of [8], and Lemma 1. A sketch of this kind of argument follows. First, use the Euclidean algorithm to divide $F(x, \xi)$ by $P(\xi)$, $F(x, \xi) = H(x, \xi)P(\xi) + r(x, \xi)$, where r is a polynomial in ξ of smaller degree than P. However, because F and P vanish to the same order, it then follows that $r(x, \xi)$ must vanish to the same order as P, but this is impossible since it has smaller degree. Thus $r(x, \xi) \equiv 0$ so $F(x, \xi) = H(x, \xi)P(\xi)$. The fact that $H(x, \xi) = H(x)$ follows by comparing degrees.

For the sake of completeness, we include a more detailed version of the argument just outlined.

Let $m = \deg P$. We shall prove that F has the form

$$F(x, \xi) = H(x, \xi)P(\xi), \tag{\#}$$

where H is a C^∞ function on $\Omega \times \mathbb{C}^n$ and P is a polynomial in ξ for each $x \in \Omega$. Then because $\xi \mapsto F(x, \xi)$ is a polynomial of degree $\leqslant m$, it follows that the degree of $\xi \mapsto H(x, \xi)$ must be zero, and thus $H(x, \xi) = H(x)$.

To prove (#), we make a choice of coordinates $\xi = (\xi_1, ..., \xi_n)$ on $\mathbb{C}^n$ such that

$$P(\xi) = c\xi_1{}^m + \sum_{j=0}^{m-1} c_j(\xi_2, ..., \xi_n)\, \xi_1{}^j, \qquad c \neq 0, \tag{*}$$

where $c_0, ..., c_{m-1}$ are polynomials in $\xi_2, ..., \xi_n$. Apply the Euclidean division algorithm to the polynomial $\xi_1 \mapsto F(x, \xi)$ with divisor $\xi_1 \mapsto P(\xi)$. We obtain

$$F(x, \xi) = H(x, \xi)P(\xi) + r(x, \xi), \tag{$\wedge$}$$

where H, r are polynomials in ξ_1, r having degree $<m$. The coefficients of these polynomials are polynomials in the coefficients of $\xi_1 \mapsto H(x, \xi)$ and $\xi_1 \mapsto P(\xi)$. In particular, H is a C^∞ function on $\Omega \times \mathbb{C}^n$ and a polynomial in ξ. We will prove that r vanishes identically by proving that for almost all choices of $(\xi_2, ..., \xi_n)$, the polynomial of degree $<m$, given by $\xi_1 \mapsto r(x, \xi)$ has at least m zeros, counting multiplicity.

Consider the factorization of $P(\xi)$ as a product of prime powers,

$$P(\xi) = \prod_{i=1}^{l} P_i(\xi)^{m_i}, \tag{α}$$

where $m = \deg P = \sum m_i d_i$, $d_i = \deg P_i$, and the P_i are distinct irreducible polynomials. Each P_i must also have the form (*) with m replaced by m_i. For $\xi' = (\xi_2, ..., \xi_n)$ fixed, let $\alpha_{ij} = \alpha_{ij}(\xi')$, for $1 \leqslant j \leqslant d_i$, denote the roots of $\xi_1 \mapsto P_i(\xi) = P_i(\xi_1, \xi')$. For almost all choices of ξ', the roots $\{\alpha_{ij}\colon 1 \leqslant j \leqslant d_i, 1 \leqslant i \leqslant l\}$ are distinct. (See [8, Theorem 10E, p. 347]). From the representation (α), it is clear also that for such ξ', with $\xi = \xi_{ij} = (\alpha_{ij}(\xi'), \xi')$, the polynomial

$$\xi_1 \mapsto P(\xi_1, \xi') = c \prod_{i=1}^{l} \left[\prod_{j=1}^{d_i} (\xi_1 - \alpha_{i,j}(\xi')) \right]^{m_i}$$

must satisfy

$$\frac{\partial^k}{\partial \xi_1{}^k} P(\xi) = 0 \qquad \text{for} \quad 0 \leqslant k < m_i .$$

Because of Lemma 1, the polynomial $\xi_1 \mapsto F(x, \xi)$ then also satisfies

$$\frac{\partial^k}{\partial \xi_1} F(x, \xi) = 0 \qquad \text{for} \quad 0 \leqslant k < m_i, \xi = \xi_{ij},$$

so that by ($\wedge$), $\xi_1 \mapsto r(x, \xi)$ has at least $\sum_{i=1}^{l} m_i d_i = m$ zeros (counting multiplicities.) Hence $r \equiv 0$, and the Lemma is proved.

Proof of Proposition 1. (Now easy.) By the above Lemma, $F(x, \xi) = H(x)P(\xi)$, and equating the terms of degree m yields the Proposition.

We conclude the paper with some open problems.

PROBLEM 1. What if we allow motions φ that are continuous, but not C^∞? For this we would count as solutions to $Lu = 0$ any function u that is the uniform limit on compact sets of C^∞ solutions u_n. Can L be rigid in this stronger sense? One way to prove this would be to prove that every motion is the limit of C^∞-motions. This does not appear easy.

PROBLEM 2. Classify, up to isomorphism, the semigroups $S(L, \Omega)$ of motions that arise for all choices of L and Ω (see [4] for the case $n = 1$, where there are exactly nine isomorphic types possible).

PROBLEM 3. Is every $S(L, \Omega)$ isomorphic to an $S(L', \Omega')$, where L' has constant coefficients?

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