ANNALS OF PHYSICS 128, 463–490 (1980)

Radiative Corrections to Parapositronium Decay*

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Received July 27, 1979

Radiative corrections to the two-gamma decay of parapositronium are examined. Special care is taken in the handling of the so-called binding diagram; in particular, the limiting procedure related to the infrared divergence is considered carefully. The general covariant gauges and the Fried–Yennie gauge are used in the computation to see that gauge invariance is accounted for. The order α correction of Harris and Brown is confirmed. In addition, from a sharp peak of the matrix element at low momentum and the low-momentum correction to the wave function, an $\alpha^2 \ln \alpha^{-1}$ correction is derived.

I. INTRODUCTION

The decay rate of parapositronium into two gammas was calculated to order α by Harris and Brown [1] on the basis of an earlier calculation of Compton scattering [2]. While the experimental measurement [3] made so far is not accurate enough to test the theoretical prediction, an ongoing precision measurement of the Michigan group is expected to test the result of Harris and Brown.

The development of the study of the orthopositronium decay into three gammas is more of a zigzag story. Although earlier experiments [4] agree with the order α calculation of Stroscio and co-workers [5], recent experiments of the Michigan group [6] indicate a significant discrepancy with the theoretical result as well as the earlier experiments. A subsequent calculation [7] give a result which was in disagreement with the previous calculation, and was consistent with the Michigan experiments [6] within 2 standard deviation. A more recent experiment [8] confirmed the result of the Michigan experiment.

In theoretical calculations of positronium decay, the treatment of the binding diagram has been the most subtle part. In earlier work [1], dropping the Sommerfeld factor was the key to eliminate double counting of the binding force and the radiative correction. A considerable refinement of the argument was attained in more recent work [7], but the limiting procedure related to the infrared divergence and the low-momentum expansion of the matrix element is less than clear cut.

In this article, we present a calculation of the order α correction to parapositronium decay, in order to clarify the subtlety of the binding diagram treatment. In order to see that gauge invariance is accounted for, we have used the general co-

* Work supported by the U.S. Department of Energy.

0003-4916/80/100463–28$05.00/0

variant gauge in the computation. Although the positronium decay matrix element should be infrared divergence free to order α, the usual procedure of on-mass-shell renormalization introduces artificial infrared divergences for individual diagrams. This makes some complication in the binding diagram in particular. We have used the Fried–Yennie gauge [9], too, since this gauge eliminates the infrared divergence from the individual diagrams.

The plan of the article is the following. Section II gives the order α correction to parapositronium decay and Section III derives an $\alpha^2 \ln \alpha^{-1}$ correction. A discussion is given in Section IV and the Appendix shows the details of the computation and relevant formulae [7].

II. Radiative Corrections of Order α

The amplitude for parapositronium decay into two gammas can be written

$$A = -i \frac{e^2}{(2k_0 2k_0')^{1/2}} \int \mathscr{U}_{\alpha\beta}(p; k, k')\, \psi_{\beta\alpha}(p)\, d^4p \delta^{(4)}(K - k - k') \tag{2.1}$$

with

$$\mathscr{U}(p; k, k') = (C\epsilon'\gamma)\, M(p; k, k')(\epsilon\gamma) + \begin{pmatrix} \epsilon \leftrightarrow \epsilon' \\ k \leftrightarrow k' \end{pmatrix}, \tag{2.2}$$

where (k_μ, ϵ_μ) and (k_μ', ϵ_μ') are the momentum and polarization vectors of the two photons, the momenta p_1 and p_2 of the electron and positron pair are related to $K = p_1 + p_2$, $p = (p_1 - p_2)/2$ which, in the c.m. system, gives

$$\frac{K^\circ}{2} = k = m - \frac{E}{2} = m - \frac{\gamma^2}{2m} + O(m\alpha^4),$$

with

$$\gamma = m\alpha/2.$$

The Bethe–Salpeter wave function for positronium is expressed as

$$\psi_{\alpha\beta} = \begin{pmatrix} \phi & -\phi(\boldsymbol{\sigma}\mathbf{p}/2m)^T \\ (\boldsymbol{\sigma}\mathbf{p}/2m)\, \phi & 0 \end{pmatrix} \tag{2.3}$$

to the order required [10]. The 4×4 matrix C is the charge conjugation matrix and is given by

$$C = -i\alpha_2 .$$

The zeroth-order amplitude is obtained by using the lowest-order matrix element

$$M(p; k, k') = 1/(i\gamma(p_1 - k) + m), \tag{2.4}$$

and computing at $p = 0$. We thus get

$$A = \frac{e^2}{4m^3} \int \mathrm{tr}\, \{(\sigma_2(\boldsymbol{\sigma\epsilon}')(\boldsymbol{\sigma}\mathbf{k})(\boldsymbol{\sigma\epsilon}) - \sigma_2(\boldsymbol{\sigma\epsilon})(\boldsymbol{\sigma}\mathbf{k})(\boldsymbol{\sigma\epsilon}'))\, \phi\}\, d^4p\,\delta^{(4)}(K - k - k')$$
$$= -\frac{(2\pi)^4\, 2^{1/2} e^2}{2m^3(\pi a^3)^{1/2}} (\mathbf{k} \cdot \boldsymbol{\epsilon} \times \boldsymbol{\epsilon}')\, \delta^{(4)}(K - k - k'), \tag{2.5}$$

where the singlet wave function is defined by

$$\int \mathrm{tr}(\sigma_2 \phi(p))\, dp_0 = 2\pi \phi_s(p) 2^{1/2} i \tag{2.6}$$

and the Schrödinger wave function for the singlet state

$$\phi_s^{(0)}(p) = \frac{1}{(\pi a^3)^{1/2}} \frac{8\pi\gamma}{(p^2 + \gamma^2)^2} \tag{2.7}$$

with

$$a = 2/m\alpha = 1/\gamma \tag{2.8}$$

is used. The decay rate computed from Eq. (2.5) gives

$$\Gamma^{(0)}_{^1S_0 \to 2\gamma} = m\alpha^5/2. \tag{2.9}$$

The diagrams contributing radiative corrections of order α are given in Fig. 1. They are (a) the self-energy correction, (b) the vertex correction, and (c) the binding diagram, in which the wavy line represents the photon propagator. From (c), the contri-

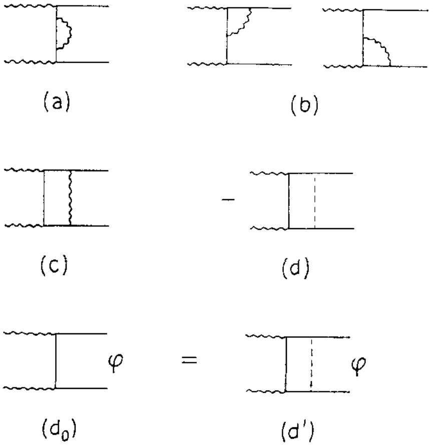

FIG. 1. Diagrams which contribute order α corrections to parapositronium decay. (Crossed photon diagrams are not shown.)

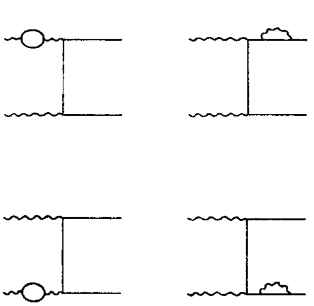

FIG. 2. Diagrams which are incorporated to eliminate the ultraviolet divergence and to give coupling constant renormalization. (Crossed photon diagrams and diagrams for bound-state wave function renormalization are not shown.)

bution of diagram (d) in which the dotted line stands for the Coulomb interaction γ_C, must be subtracted, since the Coulomb interaction is already included in the Bethe–Salpeter wave function. On the other hand, the Bethe–Salpeter equation enables us to write down the contribution from (the lowest order matrix element) $\times$ (wave function) as (d′). The reason for using the form (d′), instead of (d_0) is to avoid computing the correction to the wave function which would give an order α correction for the decay rate. In other words, using the diagram (d′), instead of (d_0) is equivalent to including the correction to the wave function in the computation. The total contribution to the decay amplitude is then symbolically expressed as

$$(a) + (b) + (c) - (d) + (d'). \tag{2.10}$$

To these, the diagrams of Fig. 2 should be added. But this addition is equivalent to absorbing the ultraviolet divergences into the Z renormalization constants and replacing the coupling constant by the renormalized one. (Wave function renormalization of the bound state is also necessary.)

One may think that the contributions of (d) and (d′) in Eq. (2.10) cancel each other exactly. This is not the case in a guage where an infrared divergence is artificially introduced by the on-mass-shell renormalization procedure. In fact, all diagrams (a)–(d) contain infrared divergences, which disappear in the sum. Obviously, diagram (d′) is infrared divergence free, so that (d) $\neq$ (d′). This is due to the way the limit is taken in the computation of (d) and (d′): In (d), the limit $\nu = m_\gamma^2/m_e^2 \neq 0$, $|\mathbf{p}| = 0$ is taken, while in (d′), the limit $\nu = 0$, $|p| \neq 0$ is considered, where m_γ is the photon mass. An exception occurs in the Fried–Yennie gauge, in which all the diagrams are infrared divergence free so that (d) $=$ (d′).

Summing the contributions of all the diagrams, the amplitude A is written as

$$A = -\frac{2^{1/2}(2\pi)^4 e^2}{2m^3} (\mathbf{k} \cdot \boldsymbol{\epsilon} \times \boldsymbol{\epsilon}') \int f(\mathbf{p})\, \phi_s(p) \frac{d^3p}{(2\pi)^3}, \tag{2.11}$$

TABLE I

Order α Corrections to Parapositronium Decay (See Fig. 1)

Photon propagator	Covariant gauge $\frac{-i}{k^2}\left(\delta_{\mu\nu} + \xi \frac{k_\mu k_\nu}{k^2}\right)$	Fried–Yennie gauge $\frac{-i}{k^2}\left(\delta_{\mu\nu} + 2 \frac{k_\mu k_\nu}{k^2}\right)$
(a)	$\frac{\alpha}{2\pi}\left[1 + \ln \nu + 4 \ln 2 + \xi\left(-1 - \frac{1}{2}\ln \nu + \ln 2\right)\right]$	$\frac{\alpha}{2\pi}(-3 + 6 \ln 2)$
(b)	$\frac{\alpha}{2\pi}\left[-4 + \frac{\pi^2}{4} - 2 \ln \nu - 4 \ln 2 + \xi(2 + \ln \nu - 2 \ln 2)\right]$	$\frac{\alpha}{2\pi}\left(4 + \frac{\pi^2}{4} - 8 \ln 2\right)$
(c)	$\frac{\alpha}{2\pi}\left[-2 + \ln \nu + \frac{2\pi}{\nu^{1/2}} + \xi\left(-1 - \frac{1}{2}\ln \nu + \ln 2\right)\right]$	$\frac{\alpha}{2\pi}(-6 + 2 \ln 2) + \frac{2\gamma}{p}\tan^{-1}\frac{p}{\gamma}$
(−d) + (d′)	$\frac{\alpha}{2\pi}\left(-\frac{2\pi}{\nu^{1/2}}\right) + \frac{2\gamma}{p}\tan^{-1}\frac{p}{\gamma}$	0
sum	$-\frac{\alpha}{2\pi}\left(5 - \frac{\pi^2}{4}\right) + \frac{2\gamma}{p}\tan^{-1}\frac{p}{\gamma}$	$-\frac{\alpha}{2\pi}\left(5 - \frac{\pi^2}{4}\right) + \frac{2\gamma}{p}\tan^{-1}\frac{p}{\gamma}$

where, to order α, $\phi_s(p)$ can be approximated by the nonrelativistic ground-state wave function

$$\phi_s(p) = \phi_s^{(0)}(p) = \left(\frac{\gamma^3}{\pi}\right)^{1/2} \frac{8\pi\gamma}{(p^2+\gamma^2)^2} \tag{2.12}$$

and $f(\mathbf{p})$ stands for the contribution from the diagrams (a), (b), (c), (d) and (d'). In Eq. (2.11), we used the transverse gauge for the external photons, since the gauge of the external photon can be chosen arbitrarily and independently from the gauge of the internal photon. This is a result of charge conservation.

The value of $f(\mathbf{p})$ for each diagram is given in Table I for the covariant gauge, where the photon propagator is given by

$$-\frac{i}{k^2}\left(\delta_{\mu\nu} + \xi\frac{k_\mu k_\nu}{k^2}\right), \tag{2.13}$$

and for the Fried–Yennie gauge, where the photon propagator is

$$-\frac{i}{k^2}\left(\delta_{\mu\nu} + 2\frac{k_\mu k_\nu}{k^2}\right). \tag{2.14}$$

In the table, we notice that the case of the Fried–Yennie gauge is not equal to the limit $\xi \to 2$ in the general covariant gauge. The details of this account are given elsewhere [11]. The computation which leads to the result of Table I is given in the Appendix.

The final result for $f(\mathbf{p})$ can be read off from Table I,

$$f(\mathbf{p}) = -\frac{\alpha}{2\pi}\left(5-\frac{\pi^2}{4}\right) + \frac{2\gamma}{p}\tan^{-1}\frac{p}{\gamma} \tag{2.15}$$

and hence

$$\int f(\mathbf{p})\, \phi_s^{(0)}(p)\frac{d^3p}{(2\pi)^3} = \left(\frac{\gamma^3}{\pi}\right)^{1/2}\left(1-\frac{\alpha}{2\pi}\left(5-\frac{\pi^2}{4}\right)\right). \tag{2.16}$$

This leads to the decay rate

$$\Gamma_{{}^1S_0\to 2\gamma} = \frac{m\alpha^5}{2}\left(1-\frac{\alpha}{\pi}\left(5-\frac{\pi^2}{4}\right)\right) \tag{2.17}$$

which is the result of Harris and Brown [1].

The second term of Eq. (2.15) is responsible for producing the zeroth-order amplitude, the first term of the r.h.s. of Eq. (2.16). In fact, the relevant integral

$$\int \frac{2\gamma}{p}\tan^{-1}\frac{p}{\gamma}\frac{8\pi\gamma}{(p^2+\gamma^2)^2}\frac{d^3p}{(2\pi)^3} = \frac{8}{\pi}\int_0^\infty \frac{x\tan^{-1}x}{(x^2+1)^2}\,dx = 1 \tag{2.18}$$

proves this assertion. If, however, one takes the limit $\gamma \to 0$ in Eq. (2.15), one obtains

$$\ln_{\gamma\to 0} f(\mathbf{p}) = -\frac{\alpha}{2\pi}\left(5 - \frac{\pi^2}{4}\right) + \frac{\pi\gamma}{p}. \tag{2.19}$$

The last term of Eq. (2.19),

$$\frac{\pi\gamma}{p} = \frac{\alpha}{2\pi}\frac{m\pi^2}{p} = \frac{\pi\alpha}{2v}, \tag{2.20}$$

is precisely half of the Sommerfeld factor for the Compton scattering cross section. The integration

$$\int \frac{\pi\gamma}{p}\frac{8\pi\gamma}{(p^2+\gamma^2)^2}\frac{d^3p}{(2\pi)^3} = 4\int_0^\infty \frac{x}{(x^2+1)^2}\,dx = 2 \tag{2.21}$$

indicates that the limit (2.19) should not be taken. As a matter of fact, Eq. (2.15) is, though sharply peaked, finite at $p = 0$ as long as $\gamma \neq 0$.

It is the sharp peak at $p = 0$ of the function $(2\gamma/P)\tan^{-1}(p/\gamma)$ as well as that of the bound-state wave function which gives the zeroth-order amplitude by integration, despite the fact that it originates from a diagram of order α. (See especially the case of the Fried–Yennie gauge.) This is similar to the mechanism of reproducing the nonrelativistic wave function by iteration in the Bethe–Salpeter equation. It may also suggest that a correction to the wave function should lead to a quantity larger than the order α^2 correction. That this is indeed the case will be shown in the next section.

Before closing this section, let us comment on the calculation of the binding diagram in Refs. [5, 7]. They obtained a singular term $\pi\gamma/2p$ which is half that of Eq. (2.20), so that the correct zeroth-order term is obtained by integration. However, that involves a separation of an infrared divergent term and $1/p$ singular term starting with an integral which is infrared divergent but p finite, or $1/p$ singular (in the limit $\gamma \to 0$) but infrared convergent. As is shown clearly in Table 1, in particular for the Fried–Yennie gauge, our computation gives Eq. (2.15) without any ambiguity. This indicates that the procedure in Refs. [5, 7] may be sensitive. Certainly, it does not give any result different from ours to order α, but it definitely leads to a different prediction in the order $\alpha^2 \ln \alpha^{-1}$ calculation, as we see in the next section.

III. A Correction of Order $\alpha^2 \ln \alpha^{-1}$

A correction to the bound-state wave function of the B.S. equation is obtained by iteration of the nonrelativistic wave function as the zeroth-order approximation. For the Coulomb kernel

$$-\frac{ie^2}{|\mathbf{k}|^2}\gamma_4^{(a)}\gamma_4^{(b)}, \tag{3.1}$$

Karplus and Klein derived the solution

$$\phi(\mathbf{p}) = \frac{m}{(p^2 + m^2)^{1/2}} \left[\left(1 + \frac{\boldsymbol{\alpha}^{(a)}\mathbf{p}}{2m}\right)\left(1 - \frac{\boldsymbol{\alpha}^{(b)}\mathbf{p}}{2m}\right) + \frac{\mathbf{p}^2}{4m^2}\right] \phi^{(0)}(\mathbf{p}). \tag{3.2}$$

For the large–large component $\phi_{11}(\mathbf{p})$, we have

$$\phi_{11}(\mathbf{p}) = \phi_{\mathrm{NR}}^{(0)}(\mathbf{p})\left(1 - \frac{p^2}{4m^2} + O\left(\left(\frac{p}{m}\right)^4\right)\right). \tag{3.3}$$

The author proved that the solution (3.2) or (3.3) is, in fact, the exact solution of the B.S. equation to the order $p^2/4m^2$, provided a term of order α^2 is neglected. This is proved by showing that its form is invariant under further iteration [12].

The integral

$$\int_0^{\Lambda} \frac{2\gamma}{p} \tan^{-1}\frac{p}{\gamma} \frac{8\pi\gamma}{(p^2+\gamma^2)^2} \frac{p^2}{4m^2} \frac{d^3p}{(2\pi)^3} = -\frac{\gamma^2}{m^2}\ln\gamma + O(\gamma^2) \tag{3.4}$$

or

$$\begin{aligned} &\int_0^{\infty} \frac{2\gamma}{p} \tan^{-1}\frac{p}{\gamma} \frac{8\pi\gamma}{(p^2+\gamma^2)^2} \frac{p^2}{4m^2 + Ap^2} \frac{d^3p}{(2\pi)^3} \\ &= -\frac{\gamma^2}{m^2}\left[\frac{1}{(1 - A\gamma^2/4m^2)^2} \ln\frac{2\gamma}{2m/A^{1/2} + \gamma} - \frac{1}{4}\frac{1}{1 - A\gamma^2/m^2}\right] \\ &= -\frac{\gamma^2}{m^2}\ln\gamma + O(\gamma^2) \end{aligned} \tag{3.5}$$

shows that the last term of Eq. (2.15) and the corrected wave function (3.5) give a correction to the decay amplitude.

$$\delta A = (-(\alpha^2/4)\ln\alpha^{-1} + O(\alpha^2))\, A^{(0)}. \tag{3.6}$$

Note that the integral in Eq. (3.4) or (3.5) can be replaced by

$$\begin{aligned} \int_0^{\Lambda} \frac{\pi\gamma}{p} \frac{8\pi\gamma}{(p^2+\gamma^2)^2} \frac{p^2}{4m^2} \frac{d^3p}{(2\pi)^3} &= \frac{\gamma^2}{2m^2}\ln\frac{\Lambda^2+\gamma^2}{\gamma^2} + O(\gamma^4) \\ &= -\frac{\gamma^2}{m^2}\ln\gamma + O(\gamma^4). \end{aligned} \tag{3.7}$$

Hence, if we use the expression of Refs. [5, 7], $\pi\gamma/2p$ for the singular part of Eq. (2.15), then we would have obtained one half of Eq. (3.6) for the $\alpha^2 \ln \alpha^{-1}$ correction.

We have obtained an $\alpha^2 \ln \alpha^{-1}$ term from the binding diagram singularity and the $p^2/4m^2$ correction to the bound-state wave function. However, similar terms can be obtained also from p^2 corrections to the matrix element and the small–small component of the positronium wave function. The details of that calculation will be reported elsewhere.

IV. Discussion

We have presented a careful treatment of the radiative correction to parapositronium decay, in particular of the binding diagram. Because of the sensitivity of the limiting procedure involving the infrared divergence, we kept the infrared parameter finite for the binding diagram. The p singular term which corresponds to the Sommerfeld factor appears from the subtraction of the Coulomb kernel for the general covariant gauge. A logically simpler argument was given in the Fried–Yennie gauge where there is no infrared divergence.

Additional $\alpha^2 \ln \alpha^{-1}$ corrections to the positronium decay rate and hfs were obtained in Ref. [14]. Those authors use a formalism different from ours, and we think it would be useful to derive such corrections by an independent method. In a forthcoming paper, we will compare our result for the $\alpha^2 \ln \alpha^{-1}$ corrections and those of earlier works.

APPENDIX A: Self Energy

(i) *Renormalization*

The self energy is given by [11]

$$\Sigma(p) = \frac{-ie^2}{(2\pi)^4} \int \gamma_\mu \frac{1}{i\gamma(p-k)+m-i\epsilon} \gamma_\nu \left(\frac{\delta_{\mu\nu}}{k^2+\lambda^2-i\epsilon} + \frac{\xi k_\mu k_\nu}{(k^2+\lambda^2-i\epsilon)^2} \right) \times \frac{d^{2\omega}k}{(2\pi\mu)^{2(\omega-2)}}$$

$$= A_\xi + B_\xi(i\gamma p + m) + C_\xi(i\gamma p + m)^2, \tag{A.1}$$

where

$$A_\xi = \frac{\alpha m}{4\pi}(3D+4),$$

$$B_\xi = \frac{\alpha}{4\pi}[D + 4 + 2\ln\nu + \xi(D - \ln\nu)],$$

$$C_\xi = \frac{\alpha}{4\pi m}\Big[\frac{1}{1-\rho} - \frac{2-3\rho}{(1-\rho)^2}\ln\rho + \frac{(-i\gamma p+m)}{\rho m}\left\{\frac{-4+3\rho}{1-\rho} + \frac{4-4\rho-\rho^2}{(1-\rho)^2}\ln\rho - 2\ln\nu\right\} + \xi\Big[\frac{1}{1-\rho} + \left(\frac{1}{1-\rho}+\frac{\rho}{(1-\rho)^2}\right)\ln\rho + \frac{(-i\gamma p+m)}{\rho m}\left\{-\frac{\rho}{1-\rho} - \frac{2-2\rho+\rho^2}{(1-\rho)^2}\ln\rho + \ln\nu\right\}\Big]\Big]. \tag{A.2}$$

Here

$$D = \frac{1}{2-\omega} + \psi(1) + \ln \frac{4\pi\mu^2}{m^2} \tag{A.3}$$

represents the ultraviolet divergence in the dimensional regularization method [13], $\psi(1) = -\gamma = -0.57721...$, μ is a quantity of the dimension of mass, λ is the photon mass and

$$\nu = \lambda^2/m^2. \tag{A.4}$$

Whenever permissible, the limits $\omega \to 2$ and $\nu \to 0$ are taken.

In the Fried–Yennie gauge ($\xi = 2$), the infrared-divergent terms disappear. In Ref. [11] we noted the facts

$$B_{\mathrm{FY}} = B_{\xi=2} - \alpha/2\pi,$$

$$C_{\mathrm{FY}} = C_{\xi=2} + \frac{\alpha}{2\pi} \frac{1}{i\gamma p + m}$$

because of the on-mass-shell expansion of the renormalization and the infrared divergence thus introduced. In fact, the self energy in the Fried–Yennie gauge is given by

$$\Sigma_{\mathrm{FY}}(p) = A_{\mathrm{FY}} + B_{\mathrm{FY}}(i\gamma p + m) + C_{\mathrm{FY}}(i\gamma p + m)^2, \tag{A.5}$$

where

$$\begin{aligned} A_{\mathrm{FY}} &= \frac{\alpha m}{4\pi}(3D + 4), \\ B_{\mathrm{FY}} &= \frac{3\alpha}{4\pi} D, \\ C_{\mathrm{FY}} &= \frac{3\alpha i\gamma p}{4\pi m^2}\left(\frac{1}{1-\rho} + \frac{\rho}{(1-\rho)^2}\ln\rho\right). \end{aligned} \tag{A.6}$$

(ii) *Contribution to the Decay Amplitude*

The contribution to the decay amplitude for parapositronium due to the self-energy diagram, Fig. 1a, can be expressed in terms of M defined in Eq. (2.1) and (2.2),

$$\begin{aligned} M(p; k, k') &= \frac{1}{i\gamma(p_1 - k) + m}(-i\Sigma_f(p_1 - k)) \frac{-i}{i\gamma(p_1 - k) + m} \\ &= -C_\xi(p_1 - k) \qquad \text{for the general covariant gauge} \\ &= -C_{\mathrm{FY}}(p_1 - k) \qquad \text{for the Fried–Yennie gauge,} \end{aligned} \tag{A.7}$$

where Σ_f stands for the third term of Eq. (A.1) or (A.5). Since $\mathbf{p}_1 = \mathbf{K}/2 + \mathbf{p} = \mathbf{p}$ and we need to evaluate the amplitude for $\mathbf{p} = 0$, the parameter ρ in $C_\xi(p_1 - k)$ or $C_{\mathrm{FY}}(p_1 - k)$ is given by

$$\rho = \frac{\mathbf{k}^2 + m^2}{m^2} \cong 2, \qquad \left(|\,\mathbf{k}\,| = m - \frac{\gamma^2}{2m} \right). \tag{A.8}$$

For

$$C_s(p) = a_s + i\gamma p b_s\,, \qquad s = \xi \text{ or FY}, \tag{A.9}$$

we obtain

$$A_a = \frac{-ie^2}{(2k_0 2k_0')^{1/2}} \int \mathrm{Tr}(C\epsilon'\gamma(-a_s - i\gamma(p_1 - k)\, b_s)(\epsilon\gamma)\, \psi(p))\, d^4p\delta^{(4)}(K - k - k')$$
$$+ \begin{pmatrix} k \leftrightarrow k' \\ \epsilon \leftrightarrow \epsilon' \end{pmatrix}. \tag{A.10}$$

Using

$$\mathrm{Tr}(C\epsilon'\gamma[-a_s - i\gamma(p_1 - k)\, b_s]\, \epsilon\gamma\psi)$$
$$= \mathrm{Tr}\begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i\boldsymbol{\sigma\epsilon}' \\ i\boldsymbol{\sigma\epsilon}' & 0 \end{pmatrix}\left(-a_s - b_s\begin{pmatrix} 0 & -\boldsymbol{\sigma}\mathbf{k} \\ \boldsymbol{\sigma}\mathbf{k} & 0 \end{pmatrix}\right)\begin{pmatrix} 0 & -i\boldsymbol{\sigma\epsilon} \\ i\boldsymbol{\sigma\epsilon} & 0 \end{pmatrix}\begin{pmatrix} \psi_{11} & 0 \\ 0 & 0 \end{pmatrix}$$
$$= ib_s\, \mathrm{Tr}(\sigma_2(\boldsymbol{\sigma\epsilon}')(\boldsymbol{\sigma}\mathbf{k})(\boldsymbol{\sigma\epsilon})\, \psi_{11}) \tag{A.11}$$

and

$$(\boldsymbol{\sigma\epsilon}')(\boldsymbol{\sigma}\mathbf{k})(\boldsymbol{\sigma\epsilon}) - (\boldsymbol{\sigma\epsilon})(\boldsymbol{\sigma}\mathbf{k})(\boldsymbol{\sigma\epsilon}') = 2i\mathbf{k} \cdot (\boldsymbol{\epsilon} x \boldsymbol{\epsilon}'). \tag{A.12}$$

We get

$$A_a = -\frac{(2\pi)^4\, 2^{1/2} e^2}{2m^3(\pi a^3)^{1/2}}\, \mathbf{k} \cdot (\boldsymbol{\epsilon} \times \boldsymbol{\epsilon}')\, \delta^{(4)}(K - k - k') \cdot (2b_s m^2), \tag{A.13}$$

where $2b_s m^2$ can be computed from the coefficients of $i\gamma p$ in C_ξ and C_{FY} (Eqs. (A.2) and (A.6)) for $\rho = 2$,

$$2b_\xi m^2 = \frac{\alpha}{2\pi}\left\{1 + 4 \ln 2 + \ln \nu + \xi\left(-1 + \ln 2 - \frac{1}{2}\ln \nu\right)\right\} \tag{A.14}$$

and

$$2b_{\mathrm{FY}} m^2 = \frac{\alpha}{2\pi}(-3 + 6 \ln 2). \tag{A.15}$$

These are the entries (a) in Table I.

APPENDIX B: Vertex Correction

(i) *Renormalization*

The vector correction is expressed as

$$\Gamma_\mu(p', p) = ie \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}(\mu^2)^{\omega-2}} \left(\gamma_\kappa \frac{1}{i\gamma(p'-q)+m-i\epsilon}\gamma_\mu \frac{1}{i\gamma(p-q)+m-i\epsilon}\gamma_\nu\right)$$

$$\times \left(\frac{\delta_{\kappa\nu}}{q^2+\lambda^2-i\epsilon} + \frac{\xi q_\kappa q_\nu}{(q^2+\lambda^2-i\epsilon)^2}\right) \tag{B.1}$$

$$= \Gamma_\mu^{(1)}(p', p) + \xi\Gamma_\mu^{(2)}(p, p),$$

where

$$\Gamma_\mu^{(1)}(p', p) = \frac{ie^2}{(2\pi)^4}\frac{1}{(2\pi\mu)^{2(\omega-2)}}\int d^{2\omega}q \int_0^1 2\,dx \int_0^x dy \frac{N_\mu^{(1)}}{(q^2+F)^3} \tag{B.2}$$

with

$$F = m^2\{x^2 + \rho(1-x)(x-y) + \rho' y(1-x) + \kappa y(x-y) + \nu(1-x)\},$$

$$\rho = \frac{p^2+m^2}{m^2}, \qquad \rho' = \frac{p'^2+m^2}{m^2}, \qquad \kappa = \frac{k^2}{m^2}, \qquad \nu = \frac{\lambda^2}{m^2} \tag{B.3}$$

and

$$\Gamma_\mu^{(2)}(p', p) = \frac{ie^2}{(2\pi)^4}\frac{1}{(2\pi\mu)^{2(\omega-2)}}\int d^{2\omega}q \int_0^1 dx \int_0^x dy \frac{6(1-x)}{(q^2+F)^4} N_\mu^{(2)}. \tag{B.4}$$

The numerators in the integrands $N_\mu^{(i)}$ are

$$N_\mu^{(1)} = \gamma_\nu(-i\gamma(p'-q-px+ky)+m)\,\gamma_\mu(-i\gamma(p-q-px+ky)+m)\,\gamma_\nu \tag{B.5}$$

and

$$N_\mu^{(2)} = \gamma(q+px-ky)(-i\gamma(p'-q-px+ky)+m)$$

$$\times \gamma_\mu(-i\gamma(p-q-px+ky)+m)\,\gamma(q+px-ky). \tag{B.6}$$

Expanding

$$\Gamma_\mu^{(i)}(p', p) = L^{(i)}\gamma_\mu + \Lambda_{\mu f}^{(i)}(p', p) \qquad (i = 1, 2) \tag{B.7}$$

for the general covariant gauge and

$$\Gamma^{\mathrm{FY}}(p', p) = L_{\mathrm{FY}}\gamma_\mu + \Lambda_{\mu f}^{(\mathrm{FY})}(p', p) \tag{B.8}$$

for the Fried–Yennie gauge, we obtain the Ward identity

$$L^{(1)} = B^{(1)} = \frac{\alpha}{4\pi}(D + 4 + 2\ln\nu),$$
$$L^{(2)} = B^{(2)} = \frac{\alpha}{4\pi}(D - \ln\nu), \tag{B.9}$$
$$L_{\mathrm{FY}} = B_{\mathrm{FY}} = \frac{\alpha}{4\pi}D.$$

(ii) *Contribution to the Decay Amplitude*

The vertex corrections contribute to the decay amplitude in the following form

$$A_{b1} = \frac{-ie^2}{(2k_0 2k_0')^{1/2}} \int \mathrm{Tr}\left\{C(\epsilon'\gamma)\frac{1}{i\gamma(p_1 - k) + m}\epsilon_\mu \Lambda_{\mu f}(p_1 - k, p_1)\,\psi(p)\right\}$$
$$\times\, d^4p\,\delta^{(4)}(K - k - k') + \begin{pmatrix} k \leftrightarrow k' \\ \epsilon \leftrightarrow \epsilon' \end{pmatrix}. \tag{B.10}$$

If

$$\Lambda_{\mu f}(p_1 - k, p_1) = \gamma_\mu a + iP_\mu b + (i\gamma(p_1 - k) + m)\,\gamma_\mu c$$
$$+ (i\gamma(p_1 - k) + m)\,P_\mu d$$
$$+ X_\mu(ip_1 + m)f$$
$$+ k_\mu g, \qquad P_\mu = (2p_1 - k)_\mu, \tag{B.11}$$

we can easily conclude that only the first term contributes to the decay amplitude: First, $\epsilon k = 0$ eliminates g. Furthermore the relationships

$$(i\gamma p_1 + m)(\epsilon\gamma)\,\psi \simeq \begin{pmatrix} 0 & 0 \\ 0 & 2m \end{pmatrix}\begin{pmatrix} 0 & -i\sigma\boldsymbol{\epsilon} \\ i\sigma\boldsymbol{\epsilon} & 0 \end{pmatrix}\begin{pmatrix} \psi_{11} & 0 \\ 0 & 0 \end{pmatrix} = 0, \tag{B.12}$$

$$\mathrm{Tr}(C(\epsilon'\gamma)(\epsilon\gamma)\begin{pmatrix} \psi_{11} & 0 \\ 0 & 0 \end{pmatrix} = 0, \tag{B.13}$$

$$\boldsymbol{\epsilon}\mathbf{p}_1 = 0 \tag{B.14}$$

ensure the validity of our statement. (Here we have taken the limit $p_0 = \mathbf{p} = 0$). Hence if we find the coefficient of the γ_μ term in Eq. (B.11), we can obtain the contribution to the decay amplitude by

$$A_{b1} = -\frac{(2\pi)^4\, 2^{1/2} e^2}{2m^3(\pi a^3)^{1/2}}\,\mathbf{k}\cdot\boldsymbol{\epsilon}\times\boldsymbol{\epsilon}'\delta^{(4)}(K - k - k')\cdot a. \tag{B.15}$$

Further, the values of ρ, $\rho' = ((p_1 - k)^2 + m^2)/m^2$ in a are given by

$$\rho = 0, \qquad \rho' = 2. \tag{B.16}$$

The a terms in Eqs. (B.3) and (B.4) are obtained by replacing γp on the right and $\gamma p'$ on the left by im, and dropping the k_μ, $p_{1\mu}$ terms. Performing the q integration (see the formula in Appendix E), we obtain

$$a_\xi = a^{(1)} + \xi a^{(2)},$$

where

$$\begin{aligned} a^{(1)} = \frac{\alpha}{4\pi}\Bigg[D - 2 - 2\int_0^1 dx \int_0^x dy \ln\frac{F}{m^2} \\ - \int_0^1 dx \int_0^x dy \frac{2m^2}{F}\begin{Bmatrix}(2-2x-x^2) - \rho(1-x+y)(1-x) \\ -\rho'(1-x)(1-y) - 2x(1-x)\end{Bmatrix}\Bigg] \\ - \frac{\alpha}{4\pi}[D + 4 + 2\ln\nu] \end{aligned} \tag{B.17}$$

$$\begin{aligned} a^{(2)} = \frac{\alpha}{4\pi}\Bigg\{D - \frac{5}{6} - 6\int_0^1 dx\int_0^x dy(1-x)\ln\frac{F}{m^2} \\ - 6\int_0^1 dx \int_0^x dy(1-x)\frac{m^2}{6F}\begin{Bmatrix}2 - 12x + 6x^2 + \rho(-1 + 6x - 6x^2 + y(6x-3)) \\ + \rho'(-1 + 3x + 3y(1-2x))\end{Bmatrix} \\ - 6\int_0^1 dx\int_0^x dy(1-x)\frac{m^4}{6F^2}\begin{pmatrix}-x^2(2-x)^2 - \rho^2 x(1-x)(x-y)(1-x+y) \\ -\rho'^2 xy(1-x)(1-y) - \rho\rho' x(1-x) \\ \times (x - 2y(x-y)) + \rho x(2-x)[2x(1-x) \\ - y(1-2x)] + \rho' x(2-x)(x + y(1-2x))\end{pmatrix} \\ - \frac{\alpha}{4\pi}(D - \ln\nu) \end{aligned} \tag{B.18}$$

for the general covariant gauge and

$$a_{\mathrm{FY}} = a^{(1)} + 2a^{(2)} + \alpha/\pi \tag{B.19}$$

for the Fried–Yennie gauge. The last terms of Eqs. (B.17) and (B.18) are to subtract the renormalization constant and the last term of Eq. (B.19) is to take care of the discrepancy between the renormalization constants in the general covariant gauge ($\xi = 2$) and the Fried–Yennie gauge [11].

By putting $\rho = 0$ and $\rho' = 2$, $\kappa = 0$ we obtain

$$\begin{aligned} a^{(1)} &= \frac{\alpha}{4\pi}\Bigg[D - 2 - 2\int_0^1 dx\int_0^x \ln(x^2 + 2y(1-x)) \\ &\quad - 2\int_0^1 dy\int_0^x dy\frac{2y(1-x) - x^2}{x^2 + 2y(1-x)} - D - 4 - 2\ln\nu\Bigg] \\ &= \frac{\alpha}{4\pi}\left[-4 + \frac{\pi^2}{4} - 2\ln\nu - 4\ln 2\right] \end{aligned} \tag{B.20}$$

and

$$a^{(2)} = \frac{\alpha}{4\pi}\Big[D - \frac{5}{6} - 6\int_0^1 dx \int_0^x dy(1-x)\ln(x^2 + 2y(1-x))$$
$$- \int_0^1 dx \int_0^x dy(1-x)\Big\{\frac{6(-x + x^2 + y(1-2x))}{x^2 + 2y(1-x)}$$
$$+ \frac{(2x^3 - x^4 + 2xy(-3x + 2x^2 + 2y(1-x)))}{(x^2 + 2y(1-x))^2}\Big\} - D + \ln\nu\Big]$$
$$= \frac{\alpha}{4\pi}(2 + \ln\nu - 2\ln 2). \tag{B.21}$$

and

$$a_{\rm FY} = \frac{\alpha}{4\pi}\left(4 + \frac{\pi^2}{4} - 8\ln 2\right). \tag{B.22}$$

Twice $a^{(1)} + \xi a^{(2)}$ and $a_{\rm FY}$ are the entries (b) in Table I (there are two vertex corrections).

APPENDIX C: THE BINDING DIAGRAM

The binding diagram (Fig. 1c) gives the decay amplitude

$$A_c = \frac{e^4}{(2k_0 2k_0')^{1/2}}$$
$$\times \int\Big\{C\gamma_\mu \frac{1}{i\gamma(q+p_2)+m}(\epsilon'\gamma)\frac{1}{i\gamma(p_1-q-k)+m}(\epsilon\gamma)\frac{1}{i\gamma(p_1-q)+m}\gamma_\nu \cdot$$
$$\times \left(\frac{\delta_{\mu\nu}}{q^2+\lambda^2-i\epsilon} + \frac{\xi q_\mu q_\nu}{(q^2+\lambda^2-i\epsilon)^2}\right)\frac{d^4q}{(2\pi)^4}\Big\}_{\alpha\beta}\psi_{\beta\alpha}(p)\,d^4p\,\delta^{(4)}(K-k-k')$$
$$+ \begin{pmatrix} k \leftrightarrow k' \\ \epsilon \leftrightarrow \epsilon' \end{pmatrix}$$
$$= -i\frac{e^4}{(2k_0 2k_0')^{1/2}}\int \mathrm{Tr}(CG_{\mu\nu}\psi(p))\,d^4p\,\delta^{(4)}(K-k-k')\,\epsilon'_\mu\epsilon_\nu$$
$$+ \begin{pmatrix} k \leftrightarrow k' \\ \epsilon \leftrightarrow \epsilon' \end{pmatrix},$$

where

$$G_{\mu\nu} = G_{\mu\nu}^{(1)} + \xi G_{\mu\nu}^{(2)} \tag{C.2}$$

$$G_{\mu\nu}^{(1)} = i\int\frac{d^4q}{(2\pi)^4}\,6\int_0^1 dx\int_0^x dy\int_0^y dz\,\frac{q^2 M_{\mu\nu} + L_{\mu\nu}}{(q^2+H)^4}$$
$$= -\frac{1}{16\pi^2}\int_0^1 dx\int_0^x dy\int_0^y dz\left(\frac{2M_{\mu\nu}}{H} + \frac{1}{H^2}L_{\mu\nu}\right), \tag{C.3}$$

where

$$H = (p_1{}^2 + m^2)(y - z) + (p_2{}^2 + m^2)(1 - x) + ((p_1 - k)^2 + m^2)(x - y) + \lambda^2 z$$
$$- (p_1(z - x) + P_2(1 - x) + k(x - y))^2, \tag{C.4}$$

$$M_{\mu\nu} = i((\gamma Q_3)\,\gamma_\nu\gamma_\mu + \gamma_\mu(\gamma Q_2)\,\gamma_\nu - \gamma_\nu\gamma_\mu\gamma Q_1), \tag{C.5}$$

$$\begin{aligned} L_{\mu\nu} = {} & 4m^3\delta_{\mu\nu} \\ & + 2im^2\{-\gamma_\nu\gamma_\mu(\gamma Q_1) + \gamma_\nu(\gamma Q_2)\,\gamma_\mu + (\gamma Q_3)\,\gamma_\nu\gamma_\mu\} \\ & + 2m\{\gamma_\mu(\gamma Q_2)\,\gamma_\nu(\gamma Q_1) + (\gamma Q_1)\,\gamma_\nu(\gamma Q_2)\,\gamma_\mu + \gamma_\mu\gamma_\nu(\gamma Q_3)(\gamma Q_1) \\ & + (\gamma Q_1)(\gamma Q_3)\,\gamma_\nu\gamma_\mu - (\gamma Q_2)\,\gamma_\nu(\gamma Q_3)\,\gamma_\mu \\ & - \gamma_\mu(\gamma Q_3)\,\gamma_\nu(\gamma Q_2)\} \\ & + 2i\{-\gamma_\mu(\gamma Q_2)\,\gamma_\nu(\gamma Q_3)(\gamma Q_1) + (\gamma Q_1)(\gamma Q_2)\,\gamma_\nu(\gamma Q_3)\,\gamma_\mu \\ & + (\gamma Q_1)\,\gamma_\mu(\gamma Q_3)\,\gamma_\nu(\gamma Q_2)\} \end{aligned} \tag{C.6}$$

with

$$\begin{aligned} Q_1 &= p_1(x - z) + p_2 x + k(y - x) = p_2 - Q, \\ Q_2 &= p_1(1 - x + z) + p_2(1 - x) - k(1 - x + y) = p_1 - k + Q, \\ Q_3 &= p_1(1 - x + z) + p_2(1 - x) + k(x - y) = p_1 + Q, \\ Q &= p_1(z - x) + p_2(1 - x) + k(x - y) \end{aligned} \tag{C.7}$$

and

$$\begin{aligned} G_{\mu\nu}^{(2)} = {} & i\int \frac{d^4q}{(2\pi)^4}\int_0^1 dx\int_0^x dy\int_0^y dz\,\frac{24z}{(q^2 + H)^5} \\ & \cdot(\gamma q - \gamma Q)(i\gamma q + i\gamma Q_1 + m)\,\gamma_\mu(i\gamma q - i\gamma Q_2 + m)\,\gamma_\nu(i\gamma q - i\gamma Q_3 + m)(\gamma q - \gamma Q) \\ = {} & -\frac{1}{16\pi^2}\int_0^1 dx\int_0^x dy\int_0^y dz\left(\frac{2}{H^3}\,N_{\mu\nu}^{(0)} + \frac{1}{2H^2}\,N_{\mu\nu}^{(2)} + \frac{3}{4H}\,N_{\mu\nu}^{(4)}\right), \end{aligned} \tag{C.8}$$

where

$$N_{\mu\nu}^{(0)} = \gamma Q(i\gamma Q_1 + m)\,\gamma_\mu(-i\gamma Q_2 + m)\,\gamma_\nu(-i\gamma Q_3 + m)(\gamma Q),$$

$$\begin{aligned} N_{\mu\nu}^{(2)} = {} & 4i\gamma_\mu(-i\gamma Q_2 + m)\,\gamma_\nu(-i\gamma Q_3 + m)(-\gamma Q) \\ & + i\gamma_\rho(i\gamma Q_1 + m)\,\gamma_\mu\gamma_\rho\gamma_\nu(-i\gamma Q_3 + m)(-\gamma Q) \\ & + i\gamma_\rho(i\gamma Q_1 + m)\,\gamma_\mu(-i\gamma Q_2 + m)\,\gamma_\nu\gamma_\rho(-\gamma Q) \\ & + \gamma_\rho(i\gamma Q_1 + m)\,\gamma_\mu(-i\gamma Q_2 + m)\,\gamma_\nu(-i\gamma Q_3 + m)\,\gamma_\rho \\ & + 2(\gamma Q)\,\gamma_\mu\gamma_\nu(-i\gamma Q_3 + m)(\gamma Q) \end{aligned}$$

$$
\begin{aligned}
&- \gamma Q\gamma_\rho\gamma_\mu(-i\gamma Q_2 + m)\,\gamma_\nu\gamma_\rho(\gamma Q) \\
&- i\gamma Q\gamma_\rho\gamma_\mu(-i\gamma Q_2 + m)\,\gamma_\nu(-i\gamma Q_3 + m)\,\gamma_\rho \\
&+ 2(\gamma Q)(i\gamma Q_1 + m)\,\gamma_\mu\gamma_\nu(\gamma Q) \\
&- i\gamma Q(i\gamma Q_1 + m)\,\gamma_\mu\gamma_\rho\gamma_\nu(-i\gamma Q_3 + m)\,\gamma_\rho \\
&- 4i\gamma Q(i\gamma Q_1 + m)\,\gamma_\mu(-i\gamma Q_2 + m)\,\gamma_\nu\,, \qquad \text{(C.9)}
\end{aligned}
$$

$$
\begin{aligned}
N_{\mu\nu}^{(f)} = &- 2\gamma_\mu\gamma_\nu(i\gamma Q) + \gamma_\mu(+iQ_3{}^\nu + 2m\gamma_\nu) \\
&- 4\gamma_\mu(-i\gamma Q_2 + m)\,\gamma_\nu + (-4iQ_1{}^\mu + 2m\gamma_\mu)\,\gamma_\nu \\
&- 2i\gamma Q\gamma_\mu\gamma_\nu\,.
\end{aligned}
$$

Using the expressions

$$
\begin{aligned}
p_1 &= \frac{K}{2} + p = \left(\mathbf{p}, i\frac{K_0}{2}\right), \qquad \text{(C.10)} \\
p_2 &= \frac{K}{2} - p = \left(-\mathbf{p}, i\frac{K_0}{2}\right), \\
k &= \left(\mathbf{k}, i\frac{K_0}{2}\right), \qquad |\,k\,| = \frac{K_0}{2} = m - \frac{\gamma^2}{2m}\,,
\end{aligned}
$$

we get

$$
\begin{aligned}
\bar{H} \equiv \frac{H}{m^2} &= \rho z(1 - z) - 2\rho^{1/2}czt + 2t - t^2 + (1 - s + z)^2 \\
&+ \sigma(1 - 2t - z + t^2 - (1 - s + z)^2) + \nu z, \qquad \text{(C.11)}
\end{aligned}
$$

where

$$
\mathbf{p}^2 = m^2\rho, \qquad \gamma^2 = m^2\sigma, \qquad c = \cos(\widehat{\mathbf{kp}}), \qquad \text{(C.12)}
$$

$$
t = x - y, \qquad s = x + y. \qquad \text{(C.13)}
$$

Here we can set $c = 0$, since the answer does not depend on the direction of $\mathbf{k}$. The $\bar{H}$ can be rewritten as

$$
\bar{H} = \hbar + (s - 1 - z)^2, \qquad \text{(C.14)}
$$

where

$$
\hbar = \rho z(1 - z) + 2t - t^2 + \nu z + \sigma((1 - t)^2 - z). \qquad \text{(C.15)}
$$

In the numerator, we may set $\mathbf{p} = 0$. Thus

$$
\begin{aligned}
Q_1 &= (-t\mathbf{k}, \text{im}(s - z)), \\
Q_2 &= (-(1 - t)\,\mathbf{k}, \text{im}(1 - s + z)), \\
Q_3 &= (t\mathbf{k}, \text{im}(2 - s + z), \qquad \text{(C.16)} \\
Q &= (t\mathbf{k}, \text{im}(1 - s + z)).
\end{aligned}
$$

Performing the trace calculation in Eq. (C.2) with

$$\psi = \begin{pmatrix} \psi_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

we obtain

$$A_c = -\frac{(2\pi)^4\, 2^{1/2} e^2}{2m^3(\pi a^3)^{1/2}}\, \mathbf{k} \cdot \boldsymbol{\epsilon} \times \boldsymbol{\epsilon}' \delta^{(4)}(K - k - k') \cdot (I^{(1)} + \xi I^{(2)}), \tag{C.17}$$

where

$$I^{(1)} = \frac{\alpha}{\pi} \int_0^1 dx \int_0^x dy \int_0^y dz \left(\frac{1 - 3t}{\overline{H}} + \frac{2 + t^2 - t^3 + (1 + t)(s - z - 1)^2}{\overline{H}^2} \right) \tag{C.18}$$

and

$$I^{(2)} = \frac{\alpha}{\pi} \int_0^1 dx \int_0^x dy \int_0^y dz\, 12z \Big(\frac{1 - 2t}{4\overline{H}} + \frac{1}{12\overline{H}^2} \{1 - 3(1 - s + z)^2$$
$$+ t(-1 + 4(1 - s + z)^2) + 3t^2 - 4t^3\} + \frac{1 - t}{12\overline{H}^3} \{(1 - s + z)^4 - 2(1 - s + z)^2$$
$$\times (2 + t^2) + t^4\} \Big). \tag{C.19}$$

With the change of variables

$$\int_0^1 dx \int_0^x dy \int_0^y dz = \frac{1}{2} \int_0^1 dz \int_0^{1-z} dt \int_{t+2z}^{2-t} ds \tag{C.20}$$

and performing the s integration, we obtain

$$I^{(1)} = \frac{\alpha}{2\pi} \int_0^1 dz \int_0^{1-z} dt \Big\{ \Big[3 - 5t + \frac{2 + t^2 - t^3}{h} \Big] \frac{1}{h^{1/2}} \tan^{-1} \frac{1 - z - t}{h^{1/2}}$$
$$+ \frac{2 + t^2 - t^3 - (1 + t) h}{h} \frac{1 - z - t}{(1 - z - t)^2 + h} \Big\} \tag{C.21}$$

and

$$I^{(2)} = \frac{\alpha}{2\pi} \int_0^1 z\, dz \int_0^{1-z} dt \Big\{ \Big[\frac{15 - 35t}{4h^{1/2}} + \frac{5t^2 - 7t^3}{2h^{3/2}} + \frac{3(1 - t)\, t^4}{4h^{5/2}} \Big] \tan^{-1} \frac{1 - t - z}{h^{1/2}}$$
$$+ \frac{1 - t - z}{(1 - t - z)^2 + h} \Big(\frac{7 - 11t}{4} + \frac{5t^2 - 7t^3}{2h} + \frac{3}{4h^2} (1 - t)\, t^4 \Big)$$
$$+ \frac{1 - t - z}{((1 - t - z)^2 + h)^2} \Big\{ \Big(2 - 2t + t^2 - t^3 + \frac{h(1 - t)}{2} + \frac{(1 - t)\, t^4}{2h} \Big) \Big\}. \tag{C22}$$

Note that

$$h + (1 - t - z)^2 = \rho z(1 - z) + (1 - z)^2 + 2tz + \nu z + \sigma(1 - z).$$

In order to evaluate the integral in Eq. (C.21), we perform a partial integration of the first term, obtaining

$$
\begin{aligned}
I^{(1)} = \frac{\alpha}{2\pi}\int_0^1 dz \Big\{\Big[&-5(\rho z(1-z)+\nu z+\sigma(1-z))^{1/2} \\
&+\frac{2}{(\rho z(1-z)+\sigma(1-z)+\nu z)^{1/2}}\Big]\cdot\tan^{-1}\frac{1-z}{(\rho z(1-z)+\sigma(1-z)+\nu z)^{1/2}} \\
&+\int_0^{1-z} dt\,\Big\{\Big(-\frac{t^2}{\hbar}+3+\frac{2t-2}{\hbar}\Big)(1-z-zt)+\frac{2+t^2-t^3}{\hbar} \\
&-(1+t))(1-z-t)\Big\}\frac{1}{\hbar+(1-z-t)^2}\Big\}. \qquad \text{(C.23)}
\end{aligned}
$$

For $\rho \to 0$, $\sigma \to 0$, the single integral in Eq. (C.23) gives

$$\frac{\alpha}{2\pi}\int_0^1 \frac{2}{(\nu z)^{1/2}}\tan^{-1}\frac{1-z}{(\nu z)^{1/2}} = \frac{\alpha}{2\pi}\left(\frac{2\pi}{\nu^{1/2}}+\ln\nu-2\right), \qquad \text{(C.24)}$$

while the double integral becomes zero (we can put $\nu = 0$ in this integral):

$$\frac{\alpha}{2\pi}\int_0^1 dz\int_0^{1-z} dt\left\{\frac{2-4z+4tz}{2zt+(1-z)^2}\right\} = \frac{\alpha}{2\pi}\int_0^1 dz\left(2(1-z)-z\ln\frac{1+z}{1-z}\right)=0, \qquad \text{(C.25)}$$

A similar calculation for $I^{(2)}$ gives

$$
\begin{aligned}
I^{(2)} = \frac{\alpha}{2\pi}\int_0^1 z\,dz\Big\{&-\frac{15}{4}(\rho z(1-z)+\sigma(1-z)+\nu z)^{1/2}\tan^{-1}\frac{1-z}{(\rho z(1-z)+\sigma(1-z)+\nu z)^{1/2}} \\
&+\int_0^{1-z} dt\,\frac{1}{(1-t-z)^2+\hbar}\Big[\Big(\frac{15}{4}\hbar-\frac{1}{4}\frac{t^4}{\hbar}-\frac{5t^2}{2}\Big)\Big(1+\frac{(1-t-z)(1-t)}{\hbar}\Big) \\
&+\Big(\frac{7-11t}{4}+\frac{5t^2-7t^3}{2\hbar}+\frac{3}{4\hbar^2}(1-t)\,t^4\Big)(1-t-z)\Big] \qquad \text{(C.26)} \\
&+\int_0^{1-z} dt\,\frac{1-t-z}{((1-t-z)^2+\hbar)^2}\Big[2-2t+t^2-t^3+\frac{\hbar(1-t)}{2}+\frac{(1-t)\,t^4}{2\hbar}\Big]\Big\}.
\end{aligned}
$$

Assume that $\rho = \sigma = 0$. The single integral of (C.26) vanishes in the limit $\nu \to 0$. The only singular term comes from the first term of the last bracket,

$$
\begin{aligned}
J &= \frac{\alpha}{2\pi}\int_0^1 z\,dz\int_0^{1-z}\frac{2(1-z)}{((1-z)^2+\nu z+2tz)^2}\,dt \\
&= \frac{\alpha}{2\pi}\int_0^1 dz(1-z)\left(\frac{1}{(1-z)^2+\nu z}-\frac{1}{(1-z^2)+\nu z}\right). \qquad \text{(C.27)} \\
&= \frac{\alpha}{2\pi}\left(-\frac{1}{2}\ln\nu-\ln 2\right).
\end{aligned}
$$

For the rest of the integral, the limit $\nu \to 0$ can be taken

$$I^{(2)} - J = \frac{\alpha}{2\pi} \int_0^1 z\,dz \int_0^{1-z} dt \left\{ \frac{1}{(1-z)^2 + 2zt} \left[-\frac{15}{4}(2t - t^2) - \frac{5}{2} t^2 - \frac{1}{4}\frac{t^3}{2-t} + (1-t-z)\left(\frac{7-11t}{4} - \frac{t^2}{2-t} + \frac{(1-t)\,t^2}{2(2-t)^2} \right) \right] + \frac{1}{((1-z)^2 + 2zt)^2} \times \left[(1-t-z)\left(-2t + t^2 - t^3 + \frac{(2t-t^2)(1-t)}{2} + \frac{(1-t)\,t^4}{2(1-t)} \right) - 2t \right] \right\}. \tag{C.28}$$

After a lengthy and tedious calculation, we obtain

$$I^{(2)} - J = (\alpha/2\pi)(-1 + 2\ln 2). \tag{C.29}$$

Collecting the results of Eqs. (C.24), (C.25), (C.27), and (C.29), we obtain the entry (c) for the general gauge in Table I.

For the Fried–Yennie gauge, we have to take the limit $\nu \to 0$ first; then we have

$$I_{\text{FY}} = \frac{\alpha}{2\pi} \int_0^1 dz \frac{2}{(\rho z(1-z) + \sigma(1-z))^{1/2}} \tan^{-1} \frac{1-z}{(\rho z(1-z) + \sigma(1-z))^{1/2}} + 2 \cdot \left\{ \frac{\alpha}{2\pi} \int_0^1 z\,dz \int_0^{1-z} dt \frac{2(1-z)}{((1-z)^2 + \rho z(1-z) - \sigma z + \sigma(1-t)^2 + 2tz)^2} + \frac{\alpha}{2\pi}(-1 + 2\ln 2) \right\} \tag{C.30}$$

The single integral of Eq. (C.30) is estimated in Appendix E, and is given by

$$\frac{\alpha}{2\pi} \left(\frac{2\pi}{\rho^{1/2}} \tan^{-1} \left(\frac{\rho}{\sigma} \right)^{1/2} - 4 + 2\ln(\rho + \sigma) + O(\sigma) \right) \tag{C.31}$$

for small σ but for arbitrary values of $\rho/\sigma = p/\gamma$, while the double integral is easily evaluated,

$$\frac{\alpha}{2\pi} \int_0^1 dz \left(-\frac{1}{1+z} + \frac{1}{1-z+\rho z+\sigma} \right) = \frac{\alpha}{2\pi} \{ -\ln 2 - \ln(\rho + \sigma) + O(\sigma, \gamma, \rho) \}. \tag{C.32}$$

Hence, we obtain

$$\begin{aligned} I_{\text{FY}} &= \frac{\alpha}{2\pi} \left(\frac{2\pi}{\rho^{1/2}} \tan^{-1} \left(\frac{\rho}{\sigma} \right)^{1/2} - 6 + 2\ln 2 \right) \\ &= \frac{2\gamma}{p} \tan^{-1} \frac{p}{\gamma} + \frac{\alpha}{2\pi}(-6 + 2\ln 2), \end{aligned} \tag{C.33}$$

which is the entry for case (c) of the Fried–Yennie gauge in Table I.

APPENDIX D: THE BINDING DIAGRAM FOR THE COULOMB KERNEL

$$A_d = -i\frac{e^4}{(2k_0 2k_0')^{1/2}} \int \mathrm{Tr}(C\tilde{G}_{\mu\nu}\psi(p))\, d^4p\,\delta^{(4)}(K-k-k')\,\epsilon_\mu'\epsilon_\nu + \begin{pmatrix} k \leftrightarrow k' \\ \epsilon \leftrightarrow \epsilon' \end{pmatrix}, \tag{D.1}$$

where

$$\begin{aligned} \tilde{G}_{\mu\nu} &= i\int \frac{d^4q}{(2\pi)^4}\gamma_4 \frac{1}{i\gamma(q+p_2)+m}\gamma_\mu \frac{1}{i\gamma(p_1-q-k)+m}\gamma_\nu \frac{1}{i\gamma(p_1-q)+m} \\ &\quad \times \gamma_4 \frac{1}{(\mathbf{q}^2+\lambda^2)} \\ &= 6i\int \frac{d^4q}{(2\pi)^4}\int_0^1 dx \int_0^x dy \int_0^y dz \frac{1}{(1-z)^{1/2}}\frac{1}{(q^2+\tilde{H})} \\ &\quad \cdot \gamma_4(i\gamma(\tilde{q}+\hat{Q}_1)+m)\,\gamma_\mu(-i\gamma(-\tilde{q}+\hat{Q}_2)+m)\,\gamma_\nu(-i\gamma(-\tilde{q}+\hat{Q}_3)+m)\,\gamma_4 . \end{aligned} \tag{D.2}$$

Here we introduce the notation

$$\tilde{q} = \left(\mathbf{q}, \frac{iq_0}{(1-z)^{1/2}}\right), \qquad \hat{Q}_i = \left(\mathbf{Q}_i, i\frac{Q_i^0}{1-z}\right) \tag{D.3}$$

and

$$\begin{aligned} \frac{\tilde{H}}{m^2} &= \rho z(1-z) + 2t - t^2 + \frac{(1-s+z)^2}{1-z} \\ &\quad + \sigma\left((1-t)^2 - z - \frac{(1-s+z)^2}{1-z}\right) - 2\rho^{1/2}czt + \nu z. \end{aligned} \tag{D.4}$$

As we have done before, we can drop $\sigma(1-s+z)^2/(1-z) + 2\rho^{1/2}czt$ from Eq. (D.4). The factor $1/(1-z)^{1/2}$ is introduced by the change of integration variable $q_0 \Rightarrow q_0/(1-z)^{1/2}$. Using

$$\tilde{\gamma} = (\boldsymbol{\gamma}, \gamma_4/(1-z)^{1/2})$$

and

$$\gamma\tilde{q} = \tilde{\gamma}q \tag{D.5}$$

we can perform the q integration in Eq. (D.2)

$$\begin{aligned} \tilde{G}_{\mu\nu} &= \frac{6i}{(2\pi)^4}\int_0^1 dx\int_0^x dy\int_0^y \frac{dz}{(1-z)^{1/2}}\left\{\frac{i\pi^2}{12\tilde{H}}[-\tilde{\gamma}_\alpha\gamma_\mu\tilde{\gamma}_\alpha\gamma_\nu(-i\gamma\hat{Q}_3+m)\right. \\ &\quad -(i\gamma\hat{Q}_1+m)\,\gamma_\mu(\tilde{\gamma}_\alpha\gamma_\nu\tilde{\gamma}_\alpha) - \tilde{\gamma}_\alpha\tilde{\gamma}_\mu(-i\gamma\hat{Q}_2+m)\,\gamma_\nu\tilde{\gamma}_\alpha] \\ &\quad \left. + \frac{i\pi^2}{6\tilde{H}^2}[\gamma_4((i\gamma\hat{Q}_1+m)\,\gamma_\mu(-i\gamma\hat{Q}_2+m)\,\gamma_\nu(-i\gamma\hat{Q}_3+m)\,\gamma_4]\right\}. \end{aligned} \tag{D.6}$$

The repeated use of

$$\tilde{\gamma}_\alpha \gamma_k \tilde{\gamma}_\alpha = -\gamma_k - \gamma_k/(1-z)$$

and

$$\tilde{\gamma}_\alpha \gamma_4 \tilde{\gamma}_\alpha = (-3 + 1/(1-z))\,\gamma_4 \tag{D.7}$$

and the trace calculation of Eq. (D.1) enable us to write down the corresponding amplitude in the following way,

$$A_d = -\frac{(2\pi)^4\, 2^{1/2} e^2}{2m^3(\pi a^3)^{1/2}}\,\mathbf{k}\cdot\boldsymbol{\epsilon}\times\boldsymbol{\epsilon}'\delta^{(4)}(K-k-k')\,I_{\text{coul}}\,, \tag{D.8}$$

where

$$I_{\text{coul}} = \frac{\alpha}{4\pi}\int_0^1 dz \int_0^{1-z} dt \int_{t+2z}^{2-t} ds\,\frac{1}{(1-z)^{1/2}}\left\{\frac{m^2}{2\tilde{H}}\left(5t-3+\frac{1+t}{1-z}\right)+\frac{m^4}{(1-z)^2\,\tilde{H}^2}\right.$$
$$\left.\times\,[(1-t)(1-z)^2-(1+t)(s-1-z)^2+(1-z)^2\,(3+t-t^2+t^3)]\right\}. \tag{D.9}$$

Since

$$(1-z)\,\tilde{H} = (1-s+z)^2 + \tilde{\hbar}, \tag{D.10}$$

where

$$\tilde{\hbar} = (1-z)\,h,$$

the integration of the s variable can be done easily:

$$I_{\text{coul}} = \frac{\alpha}{4\pi}\int_0^1 dz\int_0^{1-z} dt\left\{\left[\frac{5t-3}{\hbar^{1/2}}+\frac{4-t^2+t^3}{\hbar^{3/2}}\right]\tan^{-1}\frac{1-t-z}{((1-z)\,\hbar)^{1/2}}\right.$$
$$\left.+\left(\frac{(1-z)^{1/2}\,(4-t^2+t^3)}{\hbar}+\frac{1+t}{(1-z)^{1/2}}\right)\frac{1-t-z}{(1-t-z)^2+\hbar(1-z)}\right\}. \tag{D.11}$$

Denoting

$$\hbar_0 = \rho z(1-z) + \nu z + \sigma(1-z)$$

we perform a partial integration in Eq. (D.11),

$$I_{\text{coul}} = \frac{\alpha}{4\pi}\int_0^1 dz\left\{\left[-3(2t-t^2+\hbar_0)^{1/2}\right.\right.$$
$$\left.-\frac{4(1-t)-t^2}{(2t-t^2+\hbar_0)^{1/2}}\right]\tan^{-1}\frac{1-t-z}{((1-z)(2t-t^2+\hbar_0))^{1/2}}\Bigg|_0^{1-z}$$
$$+\int_0^{1-z} dt\left[\frac{(1-z)^{1/2}}{\hbar}(-8tz+4zt^2+2t^3-t^4)\right.$$

$$+ (1-z)^{1/2}\left(-2+t+3z(1-t)-\frac{t(1+t)}{(1-z)^{1/2}}\right]\frac{1}{zt^2+(1-z)^2+\hbar_0}\Big\}$$

$$= \frac{\alpha}{4\pi}\int_0^1 dz\left(3\hbar_0^{1/2}+\frac{4}{\hbar_0^{1/2}}\right)\tan^{-1}\left(\frac{1-z}{\hbar_0}\right)^{1/2}$$

$$+ \text{(nonsingular term).} \tag{D.12}$$

The singular part of I_{coul} is given by

$$J(\rho,\sigma,\nu) = \frac{\alpha}{\pi}\int_0^1 dt\,\frac{1}{(\rho z(1-z)+\nu z+\sigma(1-z))^{1/2}}\tan^{-1}$$

$$\times\left(\frac{1-z}{\rho z(1-z)+\nu z+\sigma(1-z)}\right)^{1/2}. \tag{D.13}$$

As is shown in Appendix E,

$$J(\rho,\sigma,0)-J(0,0,\nu) = \frac{2\gamma}{p}\tan^{-1}\frac{p}{\gamma}-\frac{\alpha}{\nu^{1/2}}. \tag{D.14}$$

This is the entry for (−d) + (d′) in Table I.

APPENDIX E: FORMULAS AND INTEGRALS

We list some of useful formulae and integrals which were used in the text.

$$\{\gamma_\mu\gamma_\nu\} = 2\delta_{\mu\nu}$$

$$\gamma_\mu\gamma_\mu = 2\omega$$

$$\gamma_\mu\gamma_\lambda\gamma_\mu = -2(\omega-1)\,\gamma_\lambda$$

$$\gamma_\nu\gamma_\lambda\gamma_\mu\gamma_\nu = 4\delta_{\mu\lambda}+2(\omega-2)\,\gamma_\lambda\gamma_\mu$$

$$\gamma_\nu\gamma_\lambda\gamma_\mu\gamma_\rho\gamma_\nu = -2\gamma_\rho\gamma_\mu\gamma_\lambda-2(\omega-2)\,\gamma_\lambda\gamma_\mu\gamma_\rho$$

$$\gamma_\nu\gamma_\lambda\gamma_\mu\gamma_\rho\gamma_\sigma\gamma_\nu = 2\gamma_\lambda\gamma_\sigma\gamma_\rho\gamma_\mu+2\gamma_\mu\gamma_\rho\gamma_\sigma\gamma_\lambda+2(\omega-2)\,\gamma_\lambda\gamma_\mu\gamma_\rho\gamma_\sigma$$

$$\gamma_\nu\gamma_\lambda\gamma_\mu\gamma_\rho\gamma_\sigma\gamma_\epsilon\gamma_\nu = -2\gamma_\lambda\gamma_\mu\gamma_\epsilon\gamma_\sigma\gamma_\rho-2\gamma_\lambda\gamma_\rho\gamma_\sigma\gamma_\epsilon\gamma_\mu+2\gamma_\mu\delta_\rho\gamma_\sigma\gamma_\epsilon\gamma_\lambda-2(\omega-2)\,\gamma_\lambda\gamma_\mu\gamma_\rho\gamma_\sigma\gamma_\epsilon$$

$$\frac{1}{AB} = \int_0^1 dx\,\frac{1}{(A(1-x)+Bx)^2}$$

$$\frac{1}{ABC} = 2\int_0^1 dx\int_0^x dy\,\frac{1}{(A(1-x)+B(x-y)+Cy)^3}$$

$$\frac{1}{ABCD} = 6\int_0^1 dx\int_0^x dy\int_0^y dz\,\frac{1}{[A(1-x)+B(x-y)+C(y-z)+Dz]^4}$$

$$\int \frac{d^{2\omega}q}{(q^2+H)^\alpha} = i\pi^\omega H^{\omega-\alpha}\,\frac{\Gamma(\alpha-\omega)}{\Gamma(\alpha)}$$

$$\int \frac{q_\mu q_\nu d^{2\omega}q}{(q^2+H)^\alpha} = \delta_{\mu\nu}\,\frac{i\pi^\omega}{2}\,H^{\omega-\alpha+1}\,\frac{\Gamma(\alpha-\omega-1)}{\Gamma(\alpha)}$$

$$\int \frac{q^2 d^{2\omega}q}{(q^2+H)^\alpha} = i\omega\pi^\omega H^{\omega-\alpha+1}\,\frac{\Gamma(\alpha-\omega-1)}{\Gamma(\alpha)}$$

$$\int \frac{q_\mu q_\nu q_\lambda q_\rho d^{2\omega}q}{(q^2+H)^\alpha} = (\delta_{\mu\nu}\delta_{\lambda\rho}+\delta_{\mu\lambda}\delta_{\nu\rho}+\delta_{\mu\rho}\delta_{\nu\lambda})\,\frac{i\pi^\omega}{4}\,H^{\omega-\alpha+2}\,\frac{(\alpha-\omega-2)}{\Gamma(\alpha)}$$

$$\int \frac{q^2 q_\mu q_\nu d^{2\omega}q}{(q^2+H)^\alpha} = \delta_{\mu\nu}\,\frac{i(\omega+1)\,\pi^\omega}{2}\,H^{\omega-\alpha+2}\,\frac{\Gamma(\alpha-\omega-2)}{\Gamma(\alpha)}$$

$$\int \frac{q^4 d^{2\omega}q}{(q^2+H)^\alpha} = i\omega(\omega+1)\,\pi^\omega H^{\omega-\alpha+2}\,\frac{\Gamma(\alpha-\omega-2)}{\Gamma(\alpha)}$$

$$i(p+p')_\mu = \sigma_{\mu\nu}(p-p')_\nu + (i\gamma p'+m)\,\gamma_\mu + \gamma_\mu(i\gamma p+m) - 2m\gamma_\mu$$

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12}$$

$$\int_0^1 \frac{\ln(1-x)}{x}\,dx = -\frac{\pi^2}{6}$$

$$\int_0^1 \frac{1}{x}\ln\frac{1+x}{1-x}\,dx = \frac{\pi^2}{4}$$

$$\int_0^1 \ln\frac{1+x}{1-x}\,dx = 2\ln 2$$

$$\int_0^1 x\ln\frac{1+x}{1-x}\,dx = 1$$

$$\int_0^1 x^2\ln\frac{1+x}{1-x}\,dx = \frac{1}{3}+\frac{2}{3}\ln 2$$

$$\Gamma(2-\omega) = \frac{1}{2-\omega} + \psi(1)$$

$$R = a + bx + cx^2 \qquad \Delta = 4ac - b^2$$

$$\begin{aligned}\int \frac{dx}{R^{1/2}} &= \frac{1}{c^{1/2}}\ln(2+(cR)^{1/2}+2cx+b) && c>0\\ &= \frac{1}{c^{1/2}}\,\text{arc sinh}\,\frac{2cx+b}{\Delta^{1/2}} && c>0,\ \Delta>0\\ &= \frac{-1}{(-c)^{1/2}}\sin^{-1}\frac{2cx+b}{(-\Delta)^{1/2}} && c<0,\ \Delta<0\\ &= \frac{1}{c^{1/2}}\ln(2cx+b) && c>0,\ \Delta=0\end{aligned}$$

$$\int \frac{x\,dx}{R^{1/2}} = \frac{R^{1/2}}{c} - \frac{b}{2c}\int\frac{dx}{R^{1/2}}$$

$$\int \frac{dx}{R^{3/2}} = \frac{2(2cx+b)}{\Delta R^{1/2}}$$

$$\int \frac{x\,dx}{R^{3/2}} = \frac{-2(2a+bx)}{\Delta R^{1/2}}$$

$$\int \frac{dx}{(1-x^2)^{3/2}} = \frac{x}{(1-x^2)^{1/2}}$$

$$\int_0^1 \frac{dx}{x+\rho(1-x)} = -\frac{1}{1-\rho}\ln\rho$$

$$\int_0^1 \frac{x}{x+\rho(1-x)} = \frac{1}{1-\rho} + \frac{\rho}{(1-\rho)^2}\ln\rho$$

$$\int_0^1 \frac{x^2}{x+\rho(1-x)}\,dx = \frac{1}{2}\frac{1}{1-\rho} - \frac{\rho}{(1-\rho)^2} - \frac{\rho^2}{(1-\rho)^3}\ln\rho$$

$$\int_0^1 \frac{x^3}{x+\rho(1-x)}\,dx = \frac{1}{3}\frac{1}{1-\rho} - \frac{1}{2}\frac{\rho}{(1-\rho)^2} + \frac{\rho^2}{(1-\rho)^3} + \frac{\rho^3}{(1-\rho)^4}\ln\rho$$

$$\int_0^1 \frac{x^4\,dx}{x+\rho(1-x)} = \frac{1}{4}\frac{1}{1-\rho} - \frac{1}{3}\frac{\rho}{(1-\rho)^2} + \frac{1}{2}\frac{\rho^2}{(1-\rho)^3} - \frac{\rho^3}{(1-\rho)^4} - \frac{\rho^4}{(1-\rho)^5} \times \ln\rho$$

$$\int_0^1 \ln(x+\rho(1-x))\,dx = -1 - \frac{\rho}{1-\rho}\ln\rho$$

$$\int_0^1 x\ln(x+\rho(1-x))\,dx = -\frac{1}{2}\left(\frac{1}{2} - \frac{\rho}{1-\rho} - \frac{\rho^2}{(1-\rho)^2}\ln\rho\right)$$

$$\int_0^1 x^2\ln(x+\rho(1-x))\,dx = -\frac{1}{3}\left(\frac{1}{3} - \frac{1}{2}\frac{\rho}{1-\rho} + \frac{\rho^2}{(1-\rho)^2} + \frac{\rho^3}{(1-\rho)^3}\ln\rho\right)$$

(a) A singular integral for the binding diagram.

$$I(\rho,\sigma,\nu) = \int_0^1 \frac{1}{(\rho z(1-z)+\sigma(1-z)+\nu z)^{1/2}}\tan^{-1}\frac{1-z}{(\rho z(1-z)+\sigma(1-z)+\nu z)^{1/2}}$$

$$\begin{aligned} I(0,0,\nu) &= 2\left(\frac{z}{\nu}\right)^{1/2}\tan^{-1}\frac{1-z}{(\nu z)^{1/2}}\bigg|_0^1 + \int_0^1 \frac{1+z}{(1-z)^2+\nu z}\,dz \\ &= \frac{1}{2}\ln\nu + \frac{2}{\nu^{1/2}}\tan^{-1}\frac{\nu^{1/2}}{2} + \frac{2}{\nu^{1/2}}\tan^{-1}\frac{1}{\nu^{1/2}} \\ &= \frac{1}{2}\ln\nu + \frac{\pi}{\nu^{1/2}} - 1 + O(\nu). \end{aligned}$$

$$I(\rho,\sigma,0) = 2\int_0^{\sigma^{-1/2}} \frac{\tan^{-1}u}{1+u^2\rho}\,du,$$

where the change of variable

$$(1 - z)^{1/2}/(\rho z + \sigma)^{1/2} = u$$

is used.

Using the notation

$$a^2 = \rho/\sigma, \qquad \lambda = 1/\sigma^{1/2},$$

$$\begin{aligned} I(\rho, \sigma, 0) &= 2\lambda \int_0^1 \frac{\tan^{-1} \lambda t}{1 + a^2 t^2}\, dt \\ &= 2\lambda \left(\frac{1}{a} \tan^{-1} \mathrm{a} \tan^{-1} \lambda - \frac{1}{a} J(a)\right), \end{aligned}$$

where

$$J(a) = \int_0^1 (\tan^{-1})\, at \frac{\lambda}{1 + \lambda^2 t^2}\, dt$$

Then $J(0) = 0$ and

$$\begin{aligned} J'(a) &= \int_0^1 \frac{t}{1 + a^2 t^2} \frac{\lambda}{1 + \lambda^2 t^2}\, dt \\ &= \frac{\lambda}{2(a^2 - \lambda^2)} \ln\left(\frac{1 + a^2}{1 + \lambda^2}\right) \\ &= \frac{-1}{2\lambda} \ln(1 + a^2) + \frac{1}{2\lambda} \ln \lambda^2 + \cdots O\left(\frac{\ln \lambda}{\lambda^3}\right), \end{aligned}$$

where an expansion in the large parameter λ is performed.

$$J(a) = -\frac{1}{2\lambda}\, [a \ln(1 + a^2) - 2a + 2 \tan^{-1} a] + \frac{\ln \lambda}{\lambda} a + \cdots.$$

Hence

$$\begin{aligned} I(\rho, \sigma, 0) &= 2\left[\frac{\pi}{2\rho^{1/2}} \tan^{-1}\left(\frac{\rho}{\sigma}\right)^{1/2} - 1 + \frac{1}{2} \ln\left(1 + \frac{\rho}{\sigma}\right) - \ln \frac{1}{\sigma^{1/2}}\right] + O(\sigma \ln \sigma) \\ &= \frac{\pi}{\rho^{1/2}} \tan^{-1}\left(\frac{\rho}{\sigma}\right)^{1/2} - 2 + \ln(\rho + \sigma) + O(\sigma \ln \sigma). \end{aligned}$$

(b) Singular part of the binding diagram with the Coulomb kernel

$$\begin{aligned} J(\rho, \sigma, \nu) &= \frac{\alpha}{\pi} \int_0^1 dz \frac{1}{(\rho z(1 - z) + \nu z + \sigma(1 - z))^{1/2}} \\ &\quad \times \tan^{-1}\left(\frac{1 - z}{\rho z(1 - z) + \nu z + \sigma(1 - z)}\right)^{1/2}, \end{aligned}$$

$$J(0, 0, \nu) = \frac{\alpha}{\pi} \int_0^1 dz \frac{1}{(\nu z)^{1/2}} \tan^{-1} \left(\frac{1-z}{\nu z}\right)^{1/2}$$

$$= \frac{\alpha}{\pi} \left(2 \left(\frac{z}{\nu}\right)^{1/2} \tan^{-1} \left(\frac{1-z}{\nu z}\right)^{1/2} \Big|_0^1 + \int_0^1 dz \frac{1}{1-(1-\nu) z} \frac{1}{(1-z)^{1/2}}\right)$$

$$= \frac{2\alpha}{\pi} \int_0^1 \frac{du}{\nu + (1-\nu) u^2} = \frac{\alpha}{\pi} \frac{2}{\nu^{1/2}} \tan^{-1} \frac{1}{\nu^{1/2}}$$

$$= \frac{\alpha}{\pi} \left(\frac{\pi}{\nu^{1/2}} - 2\right)$$

and

$$J(\rho, \sigma, 0) = \frac{\alpha}{\pi} \int_0^1 dz \frac{1}{((1-z)(\rho z + \sigma))^{1/2}} \tan^{-1} \frac{1}{(\rho z + \sigma)^{1/2}}$$

$$= \frac{\alpha}{\pi} \int_0^1 dz \frac{1}{((1-z)(\nu z + \sigma))^{1/2}} \left(\frac{\pi}{2} - \tan^{-1}(\rho z + \sigma)^{1/2}\right)$$

$$= \frac{\alpha}{\pi} \left(\frac{\pi}{2} \int_0^{\sigma^{-1/2}} \frac{2\, du}{1 + \rho u^2} - \int_0^1 \frac{dz}{(1-z)^{1/2}}\right)$$

$$= \frac{\alpha}{\pi} \left(\frac{\pi}{\rho^{1/2}} \tan^{-1} \left(\frac{\rho}{\sigma}\right)^{1/2} - 2\right)$$

$$= \frac{2\gamma}{p} \tan^{-1} \frac{p}{\gamma} - 2 \frac{\alpha}{\pi}.$$

Acknowledgments

Precision measurements of positronium decay of the Michigan group has been a constant source of stimulation. The content of this work was obtained in 1977. The measurement of parapositronium decay which is in progress at the University of Michigan enabled the author to decide to write up this material. It is a great pleasure to thank Arthur Rich, David Gidley and David Williams for stimulating and useful discussions. Thanks are also due to David Williams for reading the manuscript.

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