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INVITED ADDRESS

GRAPH THEORETIC MODELS

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1. Introduction

I feel particularly honored by this invitation to deliver the opening address at the annual meeting of the European Association for Theoretical Computer Science because I am not a computer scientist. Thus I must attribute my presence here to the importance of the field of graph theory as a useful mathematical model in your field. Similarly I was astounded by the invitation to give the opening lecture at the Bremen Conference on Chemistry in June 1978. At that meeting, I found empirically that graph theoretic models are quite useful in theoretical chemistry. For I presented eight topics in graph theory to the chemists on Monday morning, and during the week, for each of these topics at least one chemist informed me that his research involved that part of graph theory. It appears likely that a similar phenomenon can be expected in computer science.

As it is well-known that computer science involves graph theoretic algorithms, I shall begin with an accounting of my personal adventures and misadventures in that field.

2. My experiences with graphical algorithms

When Ross and I [22] were among the first to publish a graph theoretic algorithm, for finding all the cliques (maximal complete subgraphs) of a given graph G, we did not realize that our procedure would give not only all the cliques of G but also a few other subgraphs! We committed a similar error in our attempt [21] to locate all the cut points of G. Had these early efforts been correct, we would have been pioneers in this dynamic contemporary area of computer science.

My first correct algorithm [12] was developed as genuine applied mathematics when a physicist friend at the Institute for Advanced Study in Princeton, Larry

Wilets, asked me to help him and his colleagues to find eigenvalues of a given sparse square matrix using graph theoretic methods. Soon thereafter [13], I pointed out that a similar approach should be helpful in the inversion of sparse matrices. To my pleasant surprise, my colleagues at the University of Michigan recently told me that this algorithm is now taught in courses on numerical analysis. The idea of the procedure, which is described in my book [15, p. 205] is this:

(1) Given a sparse matrix $M = [m_{ij}]$, construct its binary matrix $A = A(M) = [a_{ij}]$ by defining a_{ij} with $i \neq j$ to be 1 if $m_{ij} \neq 0$ and $a_{ij} = 0$ otherwise.

(2) Consider the digraph D whose adjacency matrix is A and find its reachability matrix $R = R(D) = [r_{ij}]$ defined by $r_{ij} = 1$ whenever $i = j$ or there is a directed path in D from the point v_i to v_j, and $r_{ij} = 0$ otherwise. This is done by taking boolean $(1+1=1)$ powers of $B = A + I$ until arriving for the first time at two consecutive equal powers: $R = B^m = B^{m+1}$ of B, so that m is the maximum distance from one point of D to another. (More quickly one can work with $B, B^2, B^4, \ldots$. This is also presented in [19].)

(3) Now find the strong components of D using matrix R. This is done by noting that the strong component S_1, containing the first point v_1 of D, consists of all points mutually reachable with v_1. Then delete from R all rows and columns of the points in S_1, and repeat to find the remaining strong components $S_2, S_3, \ldots$ of D.

(4) As the condensed digraph D^* of D, with the S_j as its points, is acyclic, it has a point of indegree 0, call it T_1 and delete this point from D^*. Continue by finding strong components $T_2, \ldots$.

(5) The adjacency matrix of this relabeling of D^* with points T_j is thus upper triangular. This induces a permutation matrix P such that PMP^{-1} is block-upper-triangular and every diagonal block is square and irreducible.

Remark. Dulmage and Mendelsohn [6] developed a more powerful algorithm for determining *two* permutation matrices P and Q such that PMQ is full reduced. Subsequently Alan Hoffman and Phil Wolfe independently rediscovered precisely the same algorithm.

A *signed graph* has lines which are either positive or negative. It is *balanced* if every cycle is positive. Balanced signed graphs have proved applicable in social psychology; see [19, Chapter 9]. Up until very recently, there has been no useful algorithm for testing a given signed graph for balance. Now Kabell and I [18] have developed a linear algorithm for this purpose, which is correct.

3. Properties of almost all graphs

It is known that almost all graphs are connected. This means that if g_p is the total number of graphs with p points and c_p is the number of connected ones, then $\lim_{p\to\infty} c_p/g_p = 1$.

Blass and I [2] considered the special case $m = n$ of the following:

Adjacency property $A(m, n)$ **of graphs.** For each sequence of $m+n$ points $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ of graph G, there is another point w such that for all u_i, wAu_i (adjacent) and for all v_j, $w\bar{A}v_j$ (not adjacent).

Theorem. *For every positive integer n, almost all graphs have property* $A(n, n)$.

It follows at once that the same statement holds for every property $A(m, n)$. Using this result, we obtained the next observation.

Corollary. *For every first order sentence S about graphs, either almost all graphs satisfy S or almost no graphs do.*

The sentence, *a graph is connected*, is not first order but that *it has diameter 2* is first order! It follows from the Corollary that almost all graphs have diameter 2. Hence they are connected, as already noted in [20, p. 205] by a simple counting argument. There are many other applications of this Corollary which was earlier independently discovered by Fagin [9] and has recently been strengthened by Bollobás [3].

However at the time of the lecture in Udine, although it was known that almost all graphs satisfy $A(n, n)$ for each n, the only known example of such graphs was the pentagon C_5 for $n = 1$. Now Exoo and I have reported in [8] the construction of a graph with 61 points in the class $A(2, 2)$ and we believe this is the smallest such graph. The discovery of such a graph for $n \geq 3$ and the determination of the smallest graphs is a difficult open question.

Exoo and I [7] have also studied graphs in the classes $A(1, n)$ and we found that the smallest such graphs are 'cages' in the sense of [15, p. 174]. In particular, the smallest graph in $A(1, 2)$ is shown in Fig. 1; it is the well-known Petersen graph. This graph is the subject of an expository article [17].

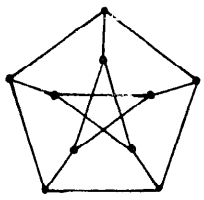

Fig. 1. The Petersen graph.

3. The Reconstruction Conjecture and the Graph Isomorphism Problem

In its present form, this conjecture was formulated in [14] as follows:

RC (Reconstruction Conjecture). If a graph G has $p \geq 3$ points v_i, and $G_i = G - v_i$ constitute the *deck* (of point-deleted unlabeled subgraphs) of G, then the deck of G determines G uniquely up to isomorphism.

This is perhaps the most outstanding unsolved problem in the theory of (finite) graphs. However it has been shown that the RC is false for infinite graphs. The counterexample which is easiest to follow [10] is provided by taking G_1 as the infinite tree in which every point has countable degree and G_2 as two copies of G_1. Then each point-deleted subgraph of either G_1 or G_2 consists of $\aleph_0$ copies of G_1 although G_1 and G_2 are not isomorphic as G_1 is connected and G_2 is not.

Legitimate Deck Problem. Given a sequence $H_1, H_2, \ldots, H_r$ of graphs, do they constitute the deck of some graph?

Graph Isomorphism Problem. Given two graphs G and H, determine whether or not they are isomorphic.

Statman and I recently observed in [23] that the Legitimate Deck Problem for regular graphs and the Graph Isomorphism problem are computationally equivalent.

A frequently rediscovered exponential algorithm for the Graph Isomorphism Problem was discussed in [16]. This procedure involves the determination of a canonical form for the adjacency matrix of a graph or digraph, and it involves testing all the $p!$ permutations which can be used for labeling a p-point graph.

The *spectrum* of a graph G is defined as the nondecreasing sequence of eigenvalues of its adjacency matrix A. (As A is real and symmetric, the eigenvalues of A are real.) The spectra of all the trees with $p \leq 8$ points were calculated by Collatz and Sinogowitz [4] without computer assistance in their pioneering paper which introduced this important contemporary branch of graph theory.

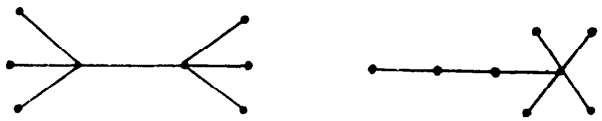

Fig. 2. The two smallest cospectral trees.

There was an outstanding conjecture in the chemical literature that if two graphs are cospectral (have the same spectrum), then they are isomorphic. However, it was already noted in [4] that the two trees of Fig. 2 are not isomorphic but are cospectral. This was explicitly pointed out in [1] in order to dispel this conjecture. Of course, had the conjecture been true, it would have provided an easy answer to the Graph Isomophism Problem which is of much interest to chemists as it is a step in the process of recognizing the similarity of two chemical compounds. Annotated bibliographies for the Graph Isomorphism Problem are given in [5, 11].

In fact, Schwenk [24] has proved the strong result that as $p \to \infty$, almost all trees with p points have a non-isomorphic cospectral mate.

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