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On the Symmetric and Rees Algebra of an Ideal Generated by a d-sequence

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TO NATHAN JACOBSON ON HIS 70TH BIRTHDAY

1. INTRODUCTION

Let R be a commutative ring and I an ideal of R. In this paper, we consider the question of when the symmetric algebra of I is a domain, and hence isomorphic to the Rees algebra of I. (see Section 2 for definitions.) Several authors have studied this question (for example, [1, 4, 9, 10], or [14]). In the cases in which the symmetric algebra is a domain, other questions have been asked: Is it Cohen–Macaulay [1]? Is it factorial [15]? Is it integrally closed [1, 12]? In this paper we prove the symmetric algebra of I is a domain whenever R is a domain and I is generated by a d-sequence (see [6] or [7]). A sequence of elements $x_1, ..., x_n$ in R is said to be a d-sequence if (i) $x_i \notin (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ for i between 1 and n and (ii) if $\{i_1, ..., i_j\}$ is a subset (possibly ϕ) of $\{1, ..., n\}$ and $k, m \in \{1, ..., n\} \backslash \{i_1, ..., i_j\}$ then $((x_{i_1}, ..., x_{i_j}) : x_k x_m) = ((x_{i_1}, ..., x_{i_j}) : x_k)$. Many examples were given in [7] of d-sequences. We list some examples here.

(1) Any R-sequence which can be permuted and remain an R-sequence is a d-sequence.

(2) If $X = (x_{ij})$ is an $n \times n + 1$ matrix of indeterminates, then the maximal minors of X form a d-sequence in the ring of polynomials.

(3) If $X = (x_{ij})$ is an $r \times s$ matrix of indeterminates over a field k and I is the ideal in $R = k[x_{ij}]$ generated by all the $t \times t$ minors of X ($t \leqslant r \leqslant s$), then the images of $x_{11}, ..., x_{1s}$ in the ring R/I form a d-sequence.

(4) If A is a local Buchsbaum ring [17], then any system of parameters forms a d-sequence.

(5) Let A be a ring satisfying Serre's condition S_{n+1} and p a height n

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prime in A such that A_p is regular and p is generated by $n + 1$ elements. (Hence p is an almost complete intersection.) Then p is generated by a d-sequence.

(6) Let A, m be a regular local ring and p a Gorenstein prime. If $x_1, \ldots, x_k$ is an A-sequence such that $pA_p = (x_1, \ldots, x_k) A_p$, then the ideal $((x_1, \ldots, x_k) : p)$ is generated by a d-sequence.

(7) If

$$X = \begin{pmatrix} 0 & X_{12} & X_{12} & X_{14} & X_{15} \\ -X_{12} & 0 & X_{23} & X_{24} & X_{25} \\ -X_{13} & -X_{23} & 0 & X_{24} & X_{35} \\ -X_{14} & -X_{24} & -X_{34} & 0 & X_{45} \\ -X_{15} & -X_{25} & -X_{35} & -X_{45} & 0 \end{pmatrix}$$

and $p_1, \ldots, p_5$ are the Pffafians of order 4 [2], then $p_1, \ldots, p_5$ form a d-sequence.

(8) Any ideal in an integrally closed domain minimally generated by two elements can be generated minimally by a d-sequence.

(9) If $p \subset k[X_0, X_1, X_2, X_3]$ is the prime defining the cubic given parameterically $(\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3)$ then p is generated by a d-sequence, namely, the 2×2 minors of $\left(\begin{smallmatrix} X_1 & X_2 & X_0 \\ X_2 & X_3 & X_1 \end{smallmatrix}\right)$. On the other hand the defining ideal of the quartic $q \subseteq k[X_0, X_1, X_2, X_3]$ given parametrically by $(\lambda^4, \lambda^3\mu, \lambda\mu^3, \mu^4)$ is not generated by a d-sequence; its defining ideal is generated (not minimally) by the 2×2 minors of

$$\begin{pmatrix} X_1 & X_3 & X_2^2 & X_0X_2 \\ X_0 & X_2 & X_3X_1 & X_1^2 \end{pmatrix}.$$

(10) The prime $p \subseteq k[X, Y, Z]$ determined by any curve given parametrically by $k[t^{n_1}, t^{n_2}, t^{n_3}]$ is generated by a d-sequence. It is known this ideal is generated by three elements [5].

(11) If R is a two-dimensional local domain which is unmixed then there is an n such that for every system of parameters x, y of R, $\{x^n, y^n\}$ is a d-sequence.

In [7], the basic properties of d-sequences were studied, among them the fact that any d-sequence in a local ring is analytically independent. The purpose of this note is to prove:

THEOREM 3.1. *Let R be a commutative Noetherian ring and $x_1, \ldots, x_n$ a d-sequence in R. Set $I = (x_1, \ldots, x_n)$. Then the map ϕ defined in Section 2*

$$\phi\colon S(I) \to R(I)$$

is an isomorphism.

2. Generalities

Let $I = (a_1, ..., a_n)$ be an ideal in a commutative ring A with unit. The map $A^n \to I$ given by $(b_1, ..., b_n) \to \sum_{i=1}^{s} b_i a_i$ induces an A-algebra epimorphism α: $A[X_1, ..., X_n] \to S(I)$, the symmetric algebra of I. The kernel of α, which we will henceforth denote by q is generated by all linear forms

$$\sum_{i=1}^{n} b_i X_i$$

such that

$$\sum_{i=1}^{n} b_i a_i = 0.$$

The Rees algebra $R(I)$ of I is the subring $A[a_1 T, ..., a_n T] \subset A[T]$ and we obtain a map

$$p: A[X_1, ..., X_n] \to R(I) \qquad \text{by} \quad X_i \to a_i T.$$

This map has a kernel which is generated by all forms $F(X_1, ..., X_n)$ such that $F(a_1, ..., a_n) = 0$. In particular, we may factor β through $S(I)$ and obtain the diagram

$$\begin{array}{ccc} A[X_1, ..., X_n] & \xrightarrow{\beta} & R(I) \\ {\scriptstyle\alpha}\downarrow & \nearrow{\scriptstyle\phi} & \\ S(I) & & \end{array}$$

ϕ is onto. If A is a domain, then $R(I)$ is clearly also a domain. The following proposition is proved in [9].

Proposition. *Let A be a domain and I an ideal of A. The following conditions are equivalent*:

(i) *$S(I)$ is a domain*;

(ii) *$S(I)$ is without torsion*;

(iii) *ϕ is injective (and hence an isomorphism)*.

If I is generated by an A-sequence, then the isomorphism $S(I) \simeq R(I)$ has long been known. If p is a homogeneous prime in $k[X_1, ..., X_n]$ then $S(p)$ a domain implies [9, 10] that p is generated by analytically independent elements. In addition, if $I = (a_1, ..., a_n)$ and $S(I)$ is a domain [10] then $a_1, ..., a_n$ must be a relative regular sequence in the sense of Fiorentini [4], i.e., $((a_1, ..., a_{i-1},$

$a_{i+1}, \dots, a_n) : a_i) \cap (a_1, \dots, a_n) = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$. It was shown in [7] that any d-sequence is a relative regular sequence.

Finally we list two propositions of [7] which will be used in the sequel.

PROPOSITION 2.1. *Let $x_1, \dots, x_n$ be a d-sequence. Then $(0 : x_1) \cap (x_1, \dots, x_n) = 0$.*

PROPOSITION 2.2. *Suppose I is an ideal in a ring R, and $x_1, \dots, x_n$ form a d-sequence modulo I. Then*

$$I \cap (x_1, \dots, x_n)^m \subseteq (x_1, \dots, x_n)^{m-1} I$$

3. PROOF OF THEOREM 3.1

The proof of Theorem 3.1 requires a result concerning the powers of an ideal generated by a d-sequence which generalizes Proposition 2.2 above.

PROPOSITION 3.1. *If I is an ideal in A and the images of $x_1, \dots, x_n$ are a d-sequence in A/I, then $I \cap (x_1, \dots, x_n)(x_1, \dots, x_k)^m \subseteq (x_1, \dots, x_n)(x_1, \dots, x_k)^{m-1} I$ if $0 \leqslant k \leqslant n$ and $m \geqslant 1$. Proposition 2.2 asserts $I \cap (x_1, \dots, x_n)^m \subseteq I(x_1, \dots, x_n)^{m-1}$; the content here is that the left side remains linear in $x_{k+1}, \dots, x_n$.*

Proof. We induct on $n - k$. If $n - k = 0$ then the quoted Proposition 2.2 of [7] shows the veracity of the statement. Suppose the proposition has been demonstrated for all m whenever $n - k - 1 < t$. We wish to prove the proposition for every m and $n - k - 1 = t$.

Set $x = x_1$. By induction we may assume

$$(I, x) \cap (x_2, \dots, x_n)(x_2, \dots, x_k)^m \subseteq (x_2, \dots, x_n)(x_2, \dots, x_k)^{m-1} (I, x)$$

for all $m \geqslant 1$.

Let $J_u = Ax^{m+1} + x^m(x_2, \dots, x_n) + x^{m-1}(x_2, \dots, x_k)(x_2, \dots, x_n) + \cdots + x^{m+1-u}(x_2, \dots, x_k)^{u-1}(x_2, \dots, x_n)$. Then we claim $J_u \cap I \subseteq I(x_2, \dots, x_k)^{u-2} \times (x_2, \dots, x_n)\, x^{m+1-u} + J_{u-1} \cap I$. Here $1 \leqslant u \leqslant m + 1$. For $u = m + 1$, $J_u = J_m + (x_2, \dots, x_k)^m(x_2, \dots, x_n)$. Hence if $r \in J_m$ and $s \in (x_2, \dots, x_k)^m(x_2, \dots, x_n)$ such that $r + s \in I$, then as $J_m \subseteq (I, x)$ we see $s \in (x_2, \dots, x_k)^m(x_2, \dots, x_n) \cap (I, x)$ which by the induction is contained in $(x_2, \dots, x_k)^{m-1}(x_2, \dots, x_n)(I, x) \subseteq J_m + I(x_2, \dots, x_k)^{m-1}(x_2, \dots, x_n)$. Hence, $r + s \in J_m \cap I + I(x_2, \dots, x_k)^{m-1}(x_2, \dots, x_n)$. Assume $1 < u < m + 1$, and write $J_u = J_{u-1} + x^{m+1-u}(x_2, \dots, x_k)^{u-1} \times (x_2, \dots, x_n)$. Suppose $y \in J_{u-1}$, $z \in x^{m+1-u}(x_2, \dots, x_k)^{u-1}(x_2, \dots, x_n)$ are such that $y + z \in I$. We may write $y = x^{m+2-u}w$ and $z = x^{m+1-u}v$ where

$$v \in (x_2, \dots, x_k)^{u-1}(x_2, \dots, x_n).$$

Then, $x^{m+1-u}(v + xw) \in I$ and hence $x(v + xw) \in I$ since $(I : x) = (I : x^2)$ by definition of a d-sequence.

Thus $v + xw \in (I : x) \cap (x, x_2, \ldots, x_n, I) = I$ by Proposition 2.1. This implies $v \in (x_2, \ldots, x_k)^{u-1}(x_2, \ldots, x_n) \cap (I, x)$. Thus, by the induction,

$$\begin{aligned} z = x^{m+1-u}v \in x^{m+2-u}(x_2, \ldots, x_k)^{u-2}(x_2, \ldots, x_n) \\ + I(x_2, \ldots, x_k)^{u-2}(x_2, \ldots, x_u)\, x^{m+1-u} \in J_{u-1} \\ + I(x_2, \ldots, x_k)^{u-2}(x_2, \ldots, x_n)\, x^{m+1-u}. \end{aligned}$$

Then $y + z \in J_{u-1} \cap I + I(x_2, \ldots, x_k)^{u-2}(x_2, \ldots, x_n)\, x^{m+1-u}$ as required. Consider $J_1 = Ax^{m+1} + x^m(x_2, \ldots, x_n)$. Then if $rx^{m+1} + sx^m J_1 \cap I$ with $s \in (x_2, \ldots, x_n)$, then $x^m(s + rx) \in I$ implies as above that $s + rx \in I$ and hence $s \in (I, x)$; thus $x^m s \in Ix^m + (x^{m+1})$ and $rx^{m+1} + sx^m \in Ax^{m+1} \cap I + Ix^m \subseteq Ix^m$ by Proposition 2.2.

Now $J_{m+1} = (x, x_2, \ldots, x_n)(x, x_2, \ldots, x_k)^m$. Hence $J_{m+1} \cap I = (x_1, x_2, \ldots, x_n) \times (x_1, x_2, \ldots, x_k)^m \cap I \subseteq J_m \cap I + I(x_2, \ldots, x_k)^{m-1}(x_2, \ldots, x_n) \subseteq J_{m-1} \cap I + I(x_2, \ldots, x_k)^{m-1}(x_2, \ldots, x_n) + I(x_2, \ldots, x_k)^{m-2}(x_2, \ldots, x_n)x \subseteq \cdots \subseteq J_1 \cap I + I(x_2, \ldots, x_k)^{m-1}(x_2, \ldots, x_n)x + \cdots + I(x_2, \ldots, x_n)\, x^{m-1} \subseteq x^m I + I(x_2, \ldots, x_k)^{m-1} \times (x_2, \ldots, x_n) + \cdots + I(x_2, \ldots, x_n)\, x^{m-1} \subseteq I(x, x_2, \ldots, x_n)(x, x_2, \ldots, x_k)^{m-1}$ which proves the proposition.

THEOREM 3.1. *Suppose $I = (z_1, \ldots, z_n)$ is generated by a d-sequence. Then the map ϕ: $S(I) \to R(I)$ is an isomorphism.*

Proof. We need to show if $H(X_1, \ldots, X_n)$ is a homogeneous polynomial such that $H(z_1, \ldots, z_n) = 0$, then $H(X_1, \ldots, X_n) \in q = \ker(\alpha)$ where

$$\alpha\colon A[X_1, \ldots, X_n] \to S(I) \to 0$$

is defined as in Section 2.

First we show this if $H(X_1, \ldots, X_n)$ is linear in every $X_1, \ldots, X_n$. Let H have degree d. If only one monomial appears in H then $H(X_1, \ldots, X_n) = aX_{i_1} \cdots X_{i_d}$. But then as $H(z_1, \ldots, z_n) = az_{i_1} \cdots z_{i_d} = 0$ the definition of a d-sequence shows $a \in (0 : z_{i_1} \cdots z_{i_d}) = (0 : z_{i_1})$ so $az_{i_1} = 0$. Let

$$F(X_1, \ldots, X_n) = aX_{i_1}.$$

Then $F(z_1, \ldots, z_n) = az_{i_1} = 0$ so $F \in q$. But $H = X_{i_2} \cdots X_{i_q} F$ so $H \in q$.

Now lexicographically order the monomials appearing in H by

$$X_{i_1} \cdots X_{i_d} < X_{j_1} \cdots X_{j_d}$$

if and only if $i_d = j_d, i_{d-1} = j_{d-1}, \ldots, i_{k+1} = j_{k+1}, i_k < j_k$ for some $1 \leqslant k \leqslant d$, and induct on the greatest monomial appearing in H. Let $aX_{i_1} \cdots X_{i_d}$ be the maximal monomial appearing in $H(X_1, \ldots, X_n)$ under this order. Put $J =$ (the ideal generated by z_k for $k \neq i_1 \ldots i_d$, $k < i_d$).

Now $H(z_1, \ldots, z_n) = 0$ shows $az_{i_1} \cdots z_{i_d} \in J$ as every other monomial has at least one z_k which appears in J. Then as $z_{i_1} \cdots z_{i_d}$ form a d-sequence modulo J, we see

$$a \in (J : z_{i_1} \cdots z_{i_d}) = (J : z_{i_d})$$

and so $az_{i_d} \in J$. Hence there is an equation $az_{i_d} = \sum_k b_k z_k$ where $z_k \in J$. Then the polynomial $F(X_1 \ldots X_n) = aX_{i_d} - \sum_k b_k X_k$ is in q. Hence $X_{i_1} \cdots X_{i_{d-1}} \times F \in q$ and so it is enough to show

$$H - X_{i_1} \cdots X_{i_{d-1}} \quad F \in q.$$

But $H - X_{i_1} \cdots X_{i_{d-1}} F$ only has monomials which are strictly less that $X_{i_1} \cdots X_{i_d}$. The induction now shows that $H - X_{i_1} \cdots X_{i_{d-1}} F \in q$ which shows the theorem if $H(X_1, \ldots, X_n)$ is linear in all variables.

We proceed to "linearize" H; induct on the degree of H to show $H \in q$. Now suppose $\deg H = d$ and $H(z_1, \ldots, z_n) = 0$, with H linear in $X_n, \ldots, X_{i+1}$. Write $H(X_1, \ldots, X_n) = X_i F(X_1, \ldots, X_n) + G(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$ where F and G are linear in $X_n, \ldots, X_{i+1}$, and degree $F = d - 1$, degree $G = d$. Since $H(z_1, \ldots, z_n) = 0$ we see that $w = G(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) \in (z_i)$ and so $w \in (z_i) \cap (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)(z_1, \ldots, z_{i-1})^{d-1}$. By Proposition 3.1 this is contained in

$$z_i(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)(z_1, \ldots, z_{i-1})^{d-2}.$$

Hence there is a polynomial $F'(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$, linear in $X_n, \ldots, X_{i+1}$ so that

$$w = z_i F'(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n).$$

Now this shows that

$$z_i F(z_1, \ldots, z_n) + z_i F'(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) = 0$$

so that $(F + F')(z_1, \ldots, z_n)$ is in $(0 : z_i)$. By Proposition 2.1 of [7], we see this implies $(F + F')(z_1, \ldots, z_n) = 0$. Now $\deg(F + F') < d$ so the induction shows $F + F' \in q$; hence $X_i F + X_i F' \in q$ and it is enough to show $G - X_i F' = (X_i F + G) - (X_i F + X_i F') \in q$. But G is a polynomial in $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n$ linear in $X_n, \ldots, X_{i+1}$ and so $G - X_i F'$ is linear in $X_n, \ldots, X_{i+1}, X_i$. Continuing, we may clearly completely linearize and apply the above work to finish the proof.

4. Applications

Theorem 3.1 can be used effectively to compute the graded ring of an ideal generated by a d-sequence. We illustrate this in the case of Example 1 of the Introduction; where I is the ideal generated by the maximal minors of a generic $n \times n+1$ matrix X.

First, we recall some isomorphisms. If $I = (a_1, \ldots, a_n)$ then the Rees algebra $R(I)$ is the subring $A[a_1T, \ldots, a_nT] \subseteq A[T]$. Adjoin T^{-1} to this ring; set $B = A[a_1T, \ldots, a_nT, T^{-1}]$. Then it is easy to see $B/BT^{-1} \simeq gr_I(A) = A/I \oplus I^2/I^3 \oplus \cdots$.

Now let A be a domain, $a, b \in A$. Consider the ring $B = A[a/b]$. It is immediate to check that if the kernel of the map $A[T] \to A[a/b]$ is generated by linear polynomials then $B/Ba/b \simeq A/(a : b)$. If $(a : b^2) = (a : b)$ then this is indeed the case (see Ratliff [11]).

Now consider the example above. Let $X = (x_{ij})$ be a $n \times n+1$ matrix of indeterminates. It is well known that the linear relations on the maximal minors $\Delta_1, \ldots, \Delta_{n+1}$ of X are generated by the relations

$$\sum_{j=1}^{n+1} x_{ij} \Delta_j = 0.$$

Thus if $I = (\Delta_1, \ldots, \Delta_{n+1})$, $S(I) = A[T_1, \ldots, T_{n+1}]/J$ where J is the ideal generated by $(\sum_{j=1}^{n+1} x_{ij}T_j)_{i=1}^{n}$ and $A = k[x_{ij}]$.

By Theorem 3.1, $S(I) \simeq R(I)$. $R(I) = A[\Delta_1T, \ldots, \Delta_{n+1}T] \subseteq A[T]$ and $T^{-1} = \Delta_1/\Delta_1T$. Now the map φ from $S(I) \to R(I)$ sends $T_i \to \Delta_iT$. Hence $T^{-1} = \Delta_1/\Delta_1T = \Delta_1/T_1$. Set $B = S(I)[\Delta_1/T_1]$: to find $B/B(\Delta_1/T_1)$ it is enough to find $(\Delta_1 : T_1)$. But $T_1\Delta_j = \Delta_1T_j$ follows from the relations $\sum_{j=1}^{n+1} x_{ij}T_j = 0$ in $S(I)$. As $(\Delta_1 : T_1^2) = (\Delta_1 : T_1)$, $gr_I(A) \simeq k[x_{ij}, T_1, \ldots, T_{n+1}]/(\sum_{j=1}^{n+1} x_{ij}T_j, \Delta_1, \ldots, \Delta_{n+1})$. Now in [6] the following result is shown.

Theorem. *Let $x = (x_{ij})$ be an $r \times s$ matrix of indeterminates and $Y = (y_{jk})$ an $s \times t$ matrix of indeterminates. Let k be a field, and let J be the ideal in $k[x_{ij}, y_{jk}]$ generated by the entries of the product matrix XY, all $a+1 \times a+1$ minors of X and all $b+1 \times b+1$ minors of Y. If $a+b \leqslant s$, then J is prime and $k[x_{ij}, y_{ik}]/J$ is Cohen–Macaulay and integrally closed.*

We apply this result with $X = (x_{ij})$ an $nx(n+1)$ matrix and

$$Y = \begin{pmatrix} T_1 \\ \vdots \\ T_{n+1} \end{pmatrix}$$

a $(n+1) \times 1$ matrix. The ideal J defining the graded algebra of I is given by the entries of XY and all $n \times n$ minors of X. Since $(n-1)+1 \leqslant n+1$, we can conclude $gr_I(A)$ is Cohen–Macaulay and integrally closed.

In characteristic zero, this result has been shown by Hochster (unpublished) by representing $gr_I(R)$ as a ring of invariants of a reductive algebraic group. Recently, DeConcini, Eisenbud, and Procesi have derived this result without restriction on the characteristic [3]. We also note that Theorem 3.1 for the d-sequence of maximal minors follows from the above theorem, as the ideal J generated by the entries of

$$X\begin{pmatrix} T_1 \\ \vdots \\ T_{n+1} \end{pmatrix}$$

is prime by the quoted result, and hence $S(I) \simeq R(I)$.

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