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Cartan Decompositions and Engel Subalgebra Triangulability*

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TO NATHAN JACOBSON ON HIS 70TH BIRTHDAY

1. INTRODUCTION

Motivated by the paper of Feit *et al.* [8] on *centralizer nilpotent* (abbreviated c.n.) groups, Benkart and Isaacs [1] have studied c.n. Lie algebras $\mathscr{L}$ over an algebraically closed field of arbitrary characteristic p. They show that $\mathscr{L}/\mathrm{nil}\,\mathscr{L}$ is simple where $\mathrm{nil}\,\mathscr{L}$ is the nilpotent radical of $\mathscr{L}$, and that $\mathscr{L}/\mathrm{nil}\,\mathscr{L}$ is isomorphic to $\mathscr{S}$ (which is three dimensional for all characteristics and isomorphic to $\mathscr{S}\mathscr{L}_2F$ for $p \neq 2$), $\mathscr{S}\mathscr{L}_3F/FI$ ($p = 3$) or $\mathscr{W}_pF$ (the p-dimensional Witt algebra over F, $p > 3$).

In the present paper, we use Cartan decomposition methods to study *Engel subalgebra triangulable* (abbreviated E.t.) Lie algebras $\mathscr{L}$, that is, Lie algebras $\mathscr{L}$ all of whose proper Engel subalgebras $\mathscr{E} = \mathscr{L}_0(\mathrm{ad}\,x)$ are triangulable when represented as linear Lie algebras on $\mathscr{L}$ by way of the adjoint representation. By the results of Benkart and Isaacs, every c.n. Lie algebra is E.t. In fact, the E.t. Lie algebras were conceived as generalizations of the c.n. Lie algebras.

We prove in Section 5 that *every nonsolvable E.t. Lie algebra $\mathscr{L}$ can be expressed as $\mathscr{L} = \mathrm{nil}\,\mathscr{H} + \mathscr{L}^\infty$ where $\mathscr{H} = \mathscr{H}_\infty + \mathrm{nil}\,\mathscr{H}$ is any Cartan subalgebra of $\mathscr{L}$ and* nil $\mathscr{H}$ *is ad-nilpotent on $\mathscr{L}$* (see 2.1) *and of codimension* 1 *in $\mathscr{H}$. Furthermore, the core* Core $\mathscr{L} = \mathscr{L}^\infty/\mathrm{nil}\,\mathscr{L}^\infty$ *of $\mathscr{L}$ is a simple Lie algebra,* nil $\mathscr{L}^\infty$ *being the nil radical of $\mathscr{L}^\infty$ and also being ad-nilpotent on $\mathscr{L}$.* The core Core $\mathscr{L}$ of an E.t. Lie algebra is then a simple E.t. Lie algebra.

In Section 6, we prove that *every simple E.t. Lie algebra with characteristic not* 2, 3, 5, *or* 7 *is isomorphic to $\mathscr{S}\mathscr{L}_2F$ or $\mathscr{W}_pF$, assuming conjecture* 6.5.

The structural methods (Section 5) are of a different character than those of Benkart and Isaacs [1]. The classification methods (Section 6) follow a related approach, with refinements which are needed to pass from a context with Cartan subalgebras Fx to a context with Cartan subalgebras $Fx + \mathscr{N}$ where

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$\mathscr{N}$ is ad-nilpotent on $\mathscr{L}$. The key classification device is the use of *special modules* of $\mathscr{S}\mathscr{L}_2F$ and their *integrability*—introduced and studied in Benkart and Isaacs [1]. This is used to show that if the simple (proper) subquotients of $\mathscr{L}$ are only $\mathscr{S}\mathscr{L}_2F$ and $\mathscr{W}_pF$, then $\mathscr{L}$ is toral rank 1, at which point the classification theorem of Wilson [15] is used inductively to show that simple E.t. Lie algebras of characteristic not 2, 3, 5, 7 are $\mathscr{S}\mathscr{L}_2F$ or $\mathscr{W}_pF$, assuming conjecture 6.5.

2. Triangulable and Semitriangulable Subalgebras

An element x of $\mathscr{L}$ is *nilpotent on* $\mathscr{L}$ if $\operatorname{ad}_{\mathscr{L}} x$ is a nilpotent linear transformation of $\mathscr{L}$. A subalgebra/ideal $\mathscr{N}$ of $\mathscr{L}$ is a *nil* subalgebra/ideal of $\mathscr{L}$ if every element of $\mathscr{N}$ is nilpotent on $\mathscr{L}$. A subalgebra/ideal $\mathscr{B}$ of $\mathscr{L}$ is *triangulable on* $\mathscr{L}$ if $\operatorname{ad}_{\mathscr{L}} \mathscr{B} = \{\operatorname{ad}_{\mathscr{L}} x \mid x \in \mathscr{B}\}$ is a Lie algebra of linear transformations of $\mathscr{L}$ which is triangulable over the algebraic closure of F. A *triangulable subalgebra* of $\mathscr{L}$ is a subalgebra of $\mathscr{L}$ which is triangulable on $\mathscr{L}$.

Note that the ideal nil $\mathscr{L}$ introduced in the following proposition is the unique maximal nilpotent ideal of $\mathscr{L}$, whereas nil $\mathscr{B}$ need not be the unique maximal nilpotent ideal of $\mathscr{B}$. (Elements of $\mathscr{B}$ nilpotent on $\mathscr{B}$ need not be nilpotent on $\mathscr{L}$.) Thus, it must be emphasized that nil $\mathscr{B}$ is defined relative to $\mathscr{L}$.

2.1. Proposition. *For any subalgebra $\mathscr{B}$ of $\mathscr{L}$, there is a unique maximal ideal* nil $\mathscr{B}$ *of $\mathscr{B}$ consisting of nilpotent elements of $\mathscr{L}$.*

Proof. It suffices to show that if $\mathscr{I}$ and $\mathscr{J}$ are ideals of $\mathscr{B}$ which are nil $\mathscr{L}$, then $\mathscr{I} + \mathscr{J}$ is nil on $\mathscr{L}$. But this follows from Jacobson [10], since $\operatorname{ad} \mathscr{I} \cup \operatorname{ad} \mathscr{J}$ is weakly closed. ∎

2.2. Theorem. *A subalgebra $\mathscr{B}$ of $\mathscr{L}$ is a triangulable subalgebra of $\mathscr{L}$ if and only if $\mathscr{B}^{(1)}$ is a nil subalgebra of $\mathscr{L}$ if and only if $\mathscr{B}/$*nil $\mathscr{B}$ *is Abelian.*

Proof. This follows as in Seligman [14] or Schue [12] or Winter [22] since $\operatorname{ad} \mathscr{B}^{(1)} = (\operatorname{ad} \mathscr{B})^{(1)}$ is a nil ideal of $\operatorname{ad} \mathscr{B}$ on $\mathscr{L}$. ∎

It is convenient to use the above theorem to motivate the following definition. ∎

2.3. Definition. Letting $\mathscr{B}^{\infty} = \bigcap \mathscr{B}^i$ $(i = 1, 2,...)$, we say that a subalgebra $\mathscr{B}$ of $\mathscr{L}$ is *semitriangulable* (abbreviated s.t.) *on* $\mathscr{L}$ if $\mathscr{B}^{\infty}$ is a nil subalgebra of $\mathscr{L}$. We say that $\mathscr{L}$ is *semitriangulable* if the ideal $\mathscr{L}^{\infty}$ of $\mathscr{L}$ is nilpotent.

Note that every semitriangulable subalgebra $\mathscr{B}$ of $\mathscr{L}$ is solvable. A nilpotent subalgebra of $\mathscr{L}$ is always semitriangulable on $\mathscr{L}$, but need not be triangulable on $\mathscr{L}$.

Note also that, whereas the irreducible nil-preserving representations of a triangulable subalgebra $\mathscr{B}$ of $\mathscr{L}$ are one dimensional, the irreducible nil-preserving representations of a semitriangulable subalgebra $\mathscr{B}$ are *univalued* in the sense that every representing linear transformation has only one eigenvalue. Here, a representation of $\mathscr{B}$ is *nil-preserving* if for each $x \in \mathscr{B}$ which is nilpotent on $\mathscr{L}$, the representing transformation is also nilpotent.

The following two propositions are straightforward.

2.4. PROPOSITION. *$\mathscr{B}$ is semitriangulable on $\mathscr{L}$ if and only if $\mathscr{B}/\mathrm{nil}\,\mathscr{B}$ is nilpotent.*

2.5. PROPOSITION. *If $\mathscr{H}$ is a triangulable/semitriangulable subalgebra of $\mathscr{L}$ and $\mathscr{H}$ normalizes a nil subalgebra $\mathscr{N}$ of $\mathscr{L}$, then $\mathscr{H} + \mathscr{N}$ is triangulable/semitriangulable on $\mathscr{L}$.*

2.6. COROLLARY. *A semitriangulable subalgebra $\mathscr{B}$ of $\mathscr{L}$ is triangulable on $\mathscr{L}$ if and only if some Cartan subalgebra $\mathscr{H}$ of $\mathscr{B}$ is triangulable on $\mathscr{L}$.*

Proof. If $\mathscr{H}$ is triangulable on $\mathscr{L}$, then $\mathscr{B} = \mathscr{H} + \mathscr{B}^{\infty}$ is triangulable on L by the above proposition. ∎

2.7. DEFINITION. Let $\mathscr{H}$ be a Cartan subalgebra of $\mathscr{L}$. Then $\mathscr{H}_{\infty}$ is the ideal $\mathscr{H}_{\infty} = \mathscr{H} \cap \mathscr{L}^{\infty}$ of $\mathscr{H}$.

The following theorem shows that for any Cartan subalgebra $\mathscr{H}$ of $\mathscr{L}$ with root spaces $\mathscr{L}_\alpha$ $(\alpha \in R)$ $\mathscr{L}^{\infty}$ is the ideal $\sum_{\alpha \in R} [\mathscr{L}_\alpha, \mathscr{L}_{-\alpha}] + \sum_{\alpha \in R} \mathscr{L}_\alpha$ studied in Schue [12] so that, in particular, the ideal $\sum_{\alpha \in R} [\mathscr{L}_\alpha, \mathscr{L}_{-\alpha}] + \sum_{\alpha \in R} \mathscr{L}_\alpha$ is independent of the choice of $\mathscr{H}$.

2.8. THEOREM. *Let $\mathscr{H}$ be a Cartan subalgebra of $\mathscr{L}$ with root spaces $\mathscr{L}_\alpha$ $(\alpha \in R)$. Then $\mathscr{L} = \mathscr{H} + \mathscr{L}^{\infty}$, $\mathscr{L} = \mathscr{H}_{\infty} \oplus \sum_{\alpha \in R} \mathscr{L}_\alpha$, and $\mathscr{H}_{\infty} = \sum_{\alpha \in R} [\mathscr{L}_\alpha, \mathscr{L}_{-\alpha}]$.*

Proof. It is clear that $\mathscr{I} = \sum_{\alpha \in R} [\mathscr{L}_\alpha, \mathscr{L}_{-\alpha}] + \sum_{\alpha \in R} \mathscr{L}_\alpha$ is the subalgebra of L generated by $\sum_{\alpha \in R} \mathscr{L}_\alpha$, and it is an ideal since it is normalized by itself and by $\mathscr{H}$. Since $[\mathscr{H}, \mathscr{L}_\alpha] = \mathscr{L}_\alpha$ for all α, $\sum_{\alpha \in R} \mathscr{L}_\alpha$ is contained in $\mathscr{L}^{\infty}$. Thus, $\mathscr{I} \subset \mathscr{L}^{\infty}$. Since $\mathscr{L} = \mathscr{H} + \mathscr{I}$, we see that $\mathscr{L} = \mathscr{H} + \mathscr{L}^{\infty}$; and also that $\mathscr{L}/\mathscr{I}$ is nilpotent. The nilpotency of $\mathscr{L}/\mathscr{I}$ implies that $\mathscr{L}^{\infty} \subset \mathscr{I}$. Thus, $\mathscr{L}^{\infty} = \mathscr{I}$. It now is apparent that $\mathscr{H}_{\infty} = \mathscr{H} \cap \mathscr{L}^{\infty} = \sum_{\alpha \in R} [\mathscr{L}_\alpha, \mathscr{L}_{-\alpha}]$. ∎

2.9. PROPOSITION. *Let $\mathscr{H}$ be a Cartan subalgebra of $\mathscr{L}$ root spaces $\mathscr{L}_\alpha$ $(\alpha \in R)$. Then $\mathscr{H}_{\infty}$ is nil on $\mathscr{L}$ if and only if the elements of $[\mathscr{L}_\alpha, \mathscr{L}_{-\alpha}]$ are nil on $\mathscr{L}$ for all α.*

Proof. This is a consequence of Jacobson [10] since $\sum_{\alpha \in R} \mathrm{ad}[\mathscr{L}_\alpha, \mathscr{L}_{-\alpha}]$ is a weakly closed set. ∎

The next theorem is related to work of Schue [12] on semirestricted Lie algebras $\mathscr{L}$.

2.10. THEOREM. *Let $\mathscr{H}$ be a Cartan subalgebra of $\mathscr{L}$ with root spaces $\mathscr{L}_\alpha$ ($\alpha \in R$). Then $\mathscr{L}$ is semitriangulable on $\mathscr{L}$ if and only if the elements of $[\mathscr{L}_\alpha, \mathscr{L}_{-\alpha}]$ and $\mathscr{L}_\alpha$ are nilpotent on $\mathscr{L}$ for $\alpha \in R$.*

Proof. $\mathscr{L}^\infty$ is nil iff the weakly closed set $\bigcup_{\alpha \in R} (\operatorname{ad}[\mathscr{L}_\alpha, \mathscr{L}_{-\alpha}] \cup \operatorname{ad} \mathscr{L}_\alpha)$ consists of nilpotent transformations, by Jacobson [10]. ∎

2.11. COROLLARY. *Let H be a Cartan subalgebra of L with root spaces $\mathscr{L}_\alpha$ ($\alpha \in R$). Then $\mathscr{L}$ is semitriangulable on $\mathscr{L}$ if and only if the subalgebras $\mathscr{L}^{(\alpha)} = \mathscr{H} + \sum_{i=1}^{p-1} \mathscr{L}_{\alpha_i}$ are semitriangulable on $\mathscr{L}$ for all $\alpha \in R$.*

2.12. *Open question.* Can "semitriangulable on L" be replaced by "solvable" in 2.11?

The answer to 2.12 is "yes" for $\mathscr{L}$ of characteristic 0, by Lie's theorem and Corollary 2.11. The answer to 2.12 is "yes" in general if the answer to 2.12 is "yes" for Lie algebras $\bar{\mathscr{L}}$ which are semirestricted with respect to the given Cartan subalgebra $\bar{\mathscr{H}}$. (See Section 7.)

2.13. EXAMPLE. Let a vector space $\mathscr{L} = \mathscr{H}_* + \mathscr{L}_0 + \cdots + \mathscr{L}_{p-1}$ be a direct sum of subspaces $\mathscr{H}_*, \mathscr{L}_0, \ldots, \mathscr{L}_{p-1}$ where $\mathscr{H}_*$ is spanned by a vector $h \neq 0$, $\mathscr{L}_1$ is two dimensional with basis x, y_1, and $\mathscr{L}_i$ is spanned by a vector $y_i \neq 0$ for $i = 0, 2, 3, \ldots, p-1$. Here, we regard the indices i as elements of the cyclic group $\pi = \mathbb{Z}_p = \{0, \ldots, p-1\}$. Make the span $\mathscr{A}$ of $y_0, y_1, \ldots, y_{p-1}$ into an Abelian Lie algebra. Make x into the derivation $xy_i = y_{i+1}$, $0 \leqslant i \leqslant p-2$, $xy_{p-1} = y_0$ of $\mathscr{A}$. Form the split extension Lie algebra $\mathscr{B} = Fx + \mathscr{A} = \mathscr{L}_0 + \cdots + \mathscr{L}_{p-1}$, and note that $\mathscr{B}$ is a π-graded Lie algebra since $[\mathscr{L}_i, \mathscr{L}_j] \subset \mathscr{L}_{i+j}$ for i, j. Let h be the derivation of $\mathscr{B}$ such that $h|_{\mathscr{L}_i}$ is i times the identity transformation on $\mathscr{L}_i$. Since $\mathscr{B}$ is a graded Lie algebra, h, so defined, is a derivation. Finally, let L be the split extension $\mathscr{L} = Fh + \mathscr{B} = \mathscr{H}_* + \sum_{i=0}^{p-1} \mathscr{L}_i = \mathscr{H} \oplus \sum_{i=1}^{p-1} \mathscr{L}_i$. Then $\mathscr{L}$ is a Lie algebra of toral rank 1 with respect to the Cartan subalgebra $\mathscr{H}$ (see Section 3), $\mathscr{H}_r = \mathscr{L}_0$ is nil on $\mathscr{L}$ and the element $x \in \mathscr{L}_1$ is not nil on $\mathscr{L}_1$.

The above example shows that *there exist Lie algebras of toral rank 1 which are solvable and which have a triangulable Cartan subalgebra H, but which are not semitriangulable.*

In passing, we note the following theorem.

2.14. THEOREM. *Let $\mathscr{B}$ be a subalgebra which is maximal such that $\mathscr{B}$ is semitriangulable on $\mathscr{L}$. Let $\mathscr{H}$ be a Cartan subalgebra of $\mathscr{B}$. Then $\mathscr{H}$ is a maximal nilpotent subalgebra of the normalizer $\mathscr{N}(\mathscr{B}^\infty) = \{x \in \mathscr{L} / [x, \mathscr{B}^\infty] \subset \mathscr{B}^\infty\}$ of $\mathscr{B}^\infty$ in $\mathscr{L}$.*

Proof. Let $\mathscr{N}(\mathscr{B}^{\infty}) \supset \mathscr{H}' \supset \mathscr{H}$ with $\mathscr{H}'$ nilpotent. Then $\mathscr{H}' + \mathscr{B}^{\infty}$ is semitriangulable, by an earlier proposition, and contains $\mathscr{B} = \mathscr{H} + \mathscr{B}^{\infty}$. Thus, $\mathscr{H}' + \mathscr{B}^{\infty} = \mathscr{B} = \mathscr{H} + \mathscr{B}^{\infty}$, by the maximality of $\mathscr{B}$. It then follows that $\mathscr{H}' = \mathscr{H}$. ∎

3. Subalgebras of Toral Rank 1 and an Application

We first recall a result of Winter [18] needed for the section.

3.1. Theorem. *Let $\mathscr{L}$ be a finite-dimensional Lie algebra graded by a group A and let $\mathscr{H}$ be a Cartan subalgebra of the* identity subalgebra $\mathscr{L}_1$ (1 *is the identity of $\mathscr{A}$). Then $\mathscr{L}_0(\operatorname{ad}\mathscr{H})$ is a Cartan subalgebra of $\mathscr{L}$ if either $\mathscr{A}$ is torsion free or if* $(\operatorname{ad}\mathscr{L})^p \subset \operatorname{ad}\mathscr{L}$ *and A is a p-group; and $\mathscr{L}_0(\operatorname{ad}\mathscr{H})$ is solvable if A is cyclic. In particular, if $\mathscr{L}_1 = \{0\}$, then $\mathscr{L}$ is nilpotent for A torsion free or for* $(\operatorname{ad}\mathscr{L})^p \subset \operatorname{ad}\mathscr{L}$ *and A a p-group; and $\mathscr{L}$ is solvable for A cyclic.*

The above theorem applies to the study of a Lie algebra $\mathscr{L}$ having a Cartan decomposition of the form $\mathscr{L} = \mathscr{H} + \sum_{i=1}^{p-1} \mathscr{L}_{\alpha i}$, since $\mathscr{L}^{\infty} = \mathscr{H}_{\infty} + \sum_{i=1}^{p-1} \mathscr{L}_{\alpha i}$ is then graded by the cyclic group $(\mathbb{Z}_p, +) = \{\bar{0}, \bar{1}, \ldots, \overline{p-1}\}$ of order p, $\mathscr{H}_{\infty}$ being the identity subalgebra $\mathscr{H}_{\infty} = (\mathscr{L}^{\infty})_{\bar{0}}$ and $(\mathscr{L}^{\infty})_{\bar{i}}$ being $\mathscr{L}_{\alpha i}$ for $1 \leqslant i \leqslant p-1$.

We use the above theorem to establish the following theorem, which is related to a result of Schue [12].

3.2. Theorem. *Let $\mathscr{L}$ be a Lie algebra of toral rank* 1, *so that $\mathscr{L}$ has a Cartan decomposition $\mathscr{L} = \mathscr{H} + \sum_{i=1}^{p-1} \mathscr{L}_{\alpha i}$ for some α. Then the following conditions are equivalent*:

(1) *$\mathscr{L}$ is solvable*;

(2) $\alpha([\mathscr{L}_{\alpha i}, \mathscr{L}_{-\alpha i}]) = 0$ *for* $1 \leqslant i \leqslant p-1$;

(3) *the ideal $\mathscr{H}_{\infty}$ of $\mathscr{H}$ is nil on $\mathscr{L}$.*

Proof. As in Schue [12], (1) implies (2). For if $\mathscr{L}$ is solvable and if $x \in [\mathscr{L}_{\alpha i}, \mathscr{L}_{-\alpha i}]$ with $\alpha(x) \neq 0$, then $x \in \mathscr{L}^{(2)}$; and $x \in \mathscr{L}^{(k)}$ implies that $x \in \mathscr{L}^{(k+1)}$ for all k (which is impossible, being in contradiction with the solvability of $\mathscr{L}$) since $\mathscr{L}_{\alpha i} = [\mathscr{L}_{\alpha i}, x] \subset \mathscr{L}^{(k)}$, $\mathscr{L}_{-\alpha i} = [\mathscr{L}_{-\alpha i}, x] \subset \mathscr{L}^{(k)}$ imply $x \in [\mathscr{L}_{\alpha i}, \mathscr{L}_{-\alpha i}] \subset \mathscr{L}^{(k+1)}$. To see that (2) implies (3), simply observe that $\alpha(x) = 0$ implies that $\operatorname{ad} x$ is nil on $\mathscr{L}$, so that $\bigcup_{i=1}^{p-1} \operatorname{ad}[\mathscr{L}_{\alpha i}, \mathscr{L}_{-\alpha i}]$ is a weakly closed set of nilpotent linear transformations, so that (3) follows from Jacobson [10]. Finally, (3) implies that $\mathscr{H}_{\infty} + \sum_{i=1}^{p-1} \mathscr{L}_{\alpha i}$ is solvable, by the preceding theorem, when $\mathscr{H} + \mathscr{H}_{\infty} + \sum_{i=1}^{p-1} \mathscr{L}_{\alpha i} = \mathscr{H} + \sum_{i=1}^{p-1} \mathscr{L}_{\alpha i}$ is certainly also solvable. ∎

Schue's theorem reads the same as the above theorem, except that $\mathscr{L}$ is assumed to be semirestricted and condition (3) is replaced by the following condition (3'):

(3') $\mathscr{W} = \sum_{i=1}^{p-1} [\mathscr{L}_{\alpha i}, \mathscr{L}_{-\alpha i}] + \sum_{i=1}^{p-1} \mathscr{L}_{\alpha i}$ is nilpotent.

(In our language, $\mathscr{W}$ is $\mathscr{L}^{\infty}$.) In the following example, a semirestricted Lie algebra $\bar{\mathscr{L}}$ is constructed whose existence shows that Schue's theorem cannot be generalized from toral rank 1 to *toral rank* 2 (meaning, in Example 3.3, that the additive group generated by roots has rank 2).

3.3. EXAMPLE. Let $\mathscr{L} = \mathscr{H} + \sum_{i=1}^{p-1} \mathscr{L}_i$, $\mathscr{H} = \mathscr{H}_* + \mathscr{L}_0$ be the Lie algebra $\mathscr{L}$ with Cartan subalgebra $\mathscr{H}$ of Example 2.13. Let t be the derivation $t = (\operatorname{ad} x)^p$ of $\mathscr{L}$, so that $ty_i = y_i$ for $0 \leqslant i \leqslant p-1$, $tx = 0$, $th = 0$. Form the split extension $\bar{\mathscr{L}} = Ft + \mathscr{L}$, which has Cartan decomposition $\bar{\mathscr{L}} = \bar{\mathscr{H}} + \mathscr{L}_{\alpha_0} + \mathscr{L}_{\alpha_1} + \cdots + \mathscr{L}_{\alpha_{p-1}} + \mathscr{L}_\beta$ where $\bar{\mathscr{H}} = Ft + \mathscr{H}_* = Ft + Fh$ and $\mathscr{L}_{\alpha_i} = Fy_i$, $\alpha_i(h) = i$, $\alpha_i(t) = 1$ for $0 \leqslant i \leqslant p-1$; and where $\mathscr{L}_\infty = Fx$, $\beta(h) = 1$, $\beta(t) = 0$. Since $\alpha_i = i\beta + \alpha_0$ for $0 \leqslant i \leqslant p-1$, $\bar{\mathscr{L}}$ has toral rank 2. Furthermore, $\bar{\mathscr{L}}$ is solvable and $\bar{\mathscr{L}}$ is semirestricted with respect to $\bar{\mathscr{H}}$. However, the above condition (3') that $\mathscr{W} = \bar{\mathscr{L}}^\infty$ be nilpotent is not satisfied, since $\bar{\mathscr{L}}^\infty$ contains $\mathscr{L}_\beta = Fx$ and $\operatorname{ad} x$ is not nilpotent on $\bar{\mathscr{L}}^\infty$.

3.4. *Open question.* If we drop the condition that $\mathscr{L}$ be toral rank 1 over $\mathscr{H}$, is it still true that *$\mathscr{L}$ is solvable if $\mathscr{H}_\infty$ is nil on $\mathscr{L}$*, $\mathscr{H}$ being a specified Cartan subalgebra of $\mathscr{L}$?

The answer to 3.4 is "yes" if the answer to 2.12 is "yes," by Theorem 3.2. In particular, the answer is "yes" in characteristic 0. The answer is also "yes" if $\mathscr{L}$ is s.s.s.t. (see Theorem 3.6 below), as is the case for the *Engel subalgebra triangulable* Lie algebras studied in Section 5. Finally, we show in Section 7 that the answer to 3.4 is "yes" for every Lie algebra $\mathscr{L}$ and triangulable Cartan subalgebra $\mathscr{H}$ if and only if the answer to 3.4 is "yes" for every Lie algebra $\bar{\mathscr{L}}$ which is semirestricted with respect to a given triangulable Cartan subalgebra $\bar{\mathscr{H}}$ of $\bar{\mathscr{L}}$.

3.4. DEFINITION. A Lie algebra $\mathscr{L}$ is *solvable subalgebra triangulable* (abbreviated s.s.t.) if every solvable subalgebra of $\mathscr{L}$ is triangulable on $\mathscr{L}$. And $\mathscr{L}$ is *solvable subalgebra semitriangulable* (abbreviated s.s.s.t.) if every solvable subalgebra of $\mathscr{L}$ is semitriangulable. (Recall that s.t. abbreviates semitriangulable.)

3.5. THEOREM. *Let $\mathscr{L}$ be s.s.s.t. and let $\mathscr{H}$ be a Cartan subalgebra of $\mathscr{L}$. Then $\mathscr{L}$ is solvable if and only if $\mathscr{H}_\infty$ is nil on $\mathscr{L}$.*

Proof. If $\mathscr{L}$ is solvable, it is semitriangulable on $\mathscr{L}$ (since $\mathscr{L}$ is s.s.s.t.) and $\mathscr{L}^\infty$ is nil on $\mathscr{L}$, whence $\mathscr{H}_\infty$ is nil on $\mathscr{L}$. Suppose, conversely, that $\mathscr{H}_\infty$

is nil on $\mathscr{L}$. Then the subalgebra $\mathscr{L}^{(\alpha)} = \mathscr{H} + \sum_{i=1}^{p-1} \mathscr{L}_{\alpha i}$ is solvable by Theorem 3.2, since $\mathscr{H}_{\infty}^{(\alpha)}$ (that is, $\mathscr{H}_{\infty}$ as defined relative to $\mathscr{L}^{(\alpha)}$) is nil on $\mathscr{L}^{(\alpha)}$, for all $\alpha \in R$. Thus, $\mathscr{L}^{(\alpha)}$ is semitriangulable on $\mathscr{L}$ (since $\mathscr{L}$ is s.s.s.t.) for all $\alpha \in R$, whence $\mathscr{L}$ is semitriangulable by Corollary 2.11. ∎

4. Engel Subalgebras of a Lie Algebra L

4.1. Definition. We recall that subalgebras $\mathscr{E}$ of $\mathscr{L}$ of the form $\mathscr{E} = \mathscr{L}_0(\operatorname{ad} x)$ for some $x \in \mathscr{L}$ are called *Engel subalgebras* of $\mathscr{L}$, and that *an Engel subalgebra $\mathscr{E}$ of $\mathscr{L}$ is a Cartan subalgebra of $\mathscr{L}$ if and only if $\mathscr{E}$ is nilpotent*. In this paper, we say that an element $x \in \mathscr{L}$ is *regular* if $\mathscr{L}_0(\operatorname{ad} x)$ is nilpotent (and therefore a Cartan subalgebra of $\mathscr{L}$).

The set $\mathscr{L}_{\text{reg}}$ regular elements of $\mathscr{L}$ contains a Zariski dense open subset of $\mathscr{L}$. (*We do not assert that $\mathscr{L}_{\text{reg}}$ is open*, with the present notion of regular.) It is easily shown that $\mathscr{E}_{\text{reg}} \cap \mathscr{L}_{\text{reg}}$ contains a dense open subset $\mathscr{W}$ of $\mathscr{E}$ for any Engel subalgebra $\mathscr{E} = \mathscr{L}_0(\operatorname{ad} x)$ of $\mathscr{L}$. For we can take $\mathscr{W} = \mathscr{U} \cap \mathscr{V}$ where $\mathscr{V}$ is the nonempty and therefore dense open set of those y in $\mathscr{E}$ which, like x, have nonsingular adjoint action on $\mathscr{L}/\mathscr{E}$; and $\mathscr{U}$ is any dense open subset of $\mathscr{E}_{\text{reg}}$. We state these observations for reference as follows.

4.2. Theorem. *Every Engel subalgebra $\mathscr{E}$ of a Lie algebra $\mathscr{L}$ has a Zariski dense open subset which consists of regular elements of $\mathscr{L}$.*

In passing we note the following consequence of Theorem 4.2.

4.3. Corollary. *If $\mathscr{L}$ has characteristic* 0 *and $\mathscr{E}$ is an Engel subalgebra of $\mathscr{L}$, then every Cartan subalgebra of $\mathscr{E}$ is a Cartan subalgebra of $\mathscr{L}$.*

Proof. The proof is based on *strongly nilpotent* elements $x \in \mathscr{L}$, that is, elements $x \in \mathscr{L}$ which are contained in $\mathscr{L}_\alpha(\operatorname{ad} y)$ for some y and some nonzero root α. For each such x, $\operatorname{ad}_{\mathscr{L}} x$ is certainly nilpotent. The group $\operatorname{Ad}_{\mathscr{L}} \mathscr{E}$ of automorphisms of $\mathscr{L}$ generated by $\{\exp \operatorname{ad}_{\mathscr{L}} x / x \in \mathscr{E},\ x \text{ is strongly nilpotent}\}$ acts transitively on the set of Cartan subalgebras of $\mathscr{E}$ by Winter [21, pp. 92–98]. Since one Cartan subalgebra of $\mathscr{E}$ is a Cartan subalgebra of $\mathscr{L}$ by Theorem 4.2, all Cartan subalgebras of $\mathscr{E}$ are therefore Cartan subalgebras of $\mathscr{L}$. ∎

5. E.t. Lie Algebras and Their Structure

5.1. Definition. We say that a Lie algebra L is *Engel subalgebra triangulable* (abbreviated E.t.) if every proper Engel subalgebra $\mathscr{E}$ of $\mathscr{L}$ is triangulable on $\mathscr{L}$. Here, $\mathscr{E}$ is *proper* if $\mathscr{E} \subsetneq \mathscr{L}$.

If I is an ideal of an E.t. Lie algebra $\mathscr{L}$, and if $(\mathscr{L}_0(\operatorname{ad} x) + \mathscr{I})/\mathscr{I}$ is a typical proper Engel subalgebra of $\mathscr{L}/\mathscr{I}$, then $\mathscr{L}_0(\operatorname{ad} x)$ is a proper Engel subalgebra of $\mathscr{L}$ and therefore is triangulable on $\mathscr{L}$. Similarly, any proper Engel subalgebra of a subalgebra of an E.t. is certainly triangulable on $\mathscr{L}$. We formulate these observations as follows.

5.2. Proposition. *If $\mathscr{L}$ is an E.t. Lie algebra, then subalgebras and quotients of $\mathscr{L}$ are also.*

Note that Cartan subalgebras of $\mathscr{L}$, being proper Engel subalgebras of $\mathscr{L}$, are triangulable on $\mathscr{L}$ for any nonnilpotent E.t. Lie algebra $\mathscr{L}$. Thus, by Theorem 2.6, a nonnilpotent Engel triangulable Lie algebra $\mathscr{L}$ is triangulable if and only if it is semitriangular—which we state as follows.

5.3. Proposition. *If $\mathscr{L}$ is E.t., then $\mathscr{L}$ is semitriangulable if and only if $\mathscr{L}$ is nilpotent or $\mathscr{L}$ is triangulable.*

Henceforth, we are interested only in nonsemitriangulable Engel subalgebras.

5.4. Theorem. *Let $\mathscr{L}$ be E.t. but not s.t. and let $\mathscr{H}$ be a Cartan subalgebra of $\mathscr{L}$. Then* nil $\mathscr{H} = \{x \in h \mid \operatorname{ad} x$ *is nilpotent on* $\mathscr{L}\}$ *is an ideal of $\mathscr{H}$ of codimension 1. Thus, $\mathscr{H}$ is a triangulable subalgebra of $\mathscr{L}$ of the form $\mathscr{H} = kx + \mathscr{N}$ where $\mathscr{N} =$* nil $\mathscr{H}$ *is a nil subalgebra of $\mathscr{L}$ normalized by x.*

Proof. Since every Cartan subalgebra of $\mathscr{L}$ is an Engel subalgebra of $\mathscr{L}$, $\mathscr{H}$ is a triangulable subalgebra of $\mathscr{L}$. In the Cartan decomposition $\mathscr{L} = \mathscr{H} \oplus \sum_{\alpha \in R} \mathscr{L}_\alpha$, choose for each $\alpha \in R$ a nonzero element x_α of $\mathscr{L}_\alpha$ such that $[\mathscr{H}, x_\alpha] = kx_\alpha$. Then kx_α is a one-dimensional $\mathscr{H}$-module, so that $\mathscr{H}_\alpha = \{h \in \mathscr{H} \mid [h, x_\alpha] = 0\}$ is an ideal of codimension one in $\mathscr{H}$. Since nil $\mathscr{H}$ is not of codimension one in $\mathscr{H}$, $\mathscr{H}_\alpha$ contains an element h_α which is not nilpotent on $\mathscr{L}$. Since $[h_\alpha, x_\alpha] = 0$, we have $\alpha(h_\alpha) = 0$. Thus, the proper Engel subalgebra $\mathscr{B} = \mathscr{L}_0(\operatorname{ad} h_\alpha)$ contains $\mathscr{L}_\alpha$, $\mathscr{L}_{-\alpha}$, $\mathscr{H}$. Since $\mathscr{B}$ is triangulable, $\mathscr{B}^{(1)}$ is a nil subalgebra of $\mathscr{L}$ and contains $[\mathscr{H}, \mathscr{L}_\alpha] = \mathscr{L}_\alpha$, $[\mathscr{H}, \mathscr{L}_{-\alpha}] = \mathscr{L}_{-\alpha}$ and $[\mathscr{L}_\alpha, \mathscr{L}_{-\alpha}]$, whence their elements are nilpotent on $\mathscr{L}$. This is true for all α. Thus, $\mathscr{L}$ is semitriangulable by Theorem 2.10, a contradiction. ∎

A general principle emerges from the above proof. If the Lie algebra $\mathscr{L}$ has a triangulable Cartan subalgebra H, then for every root α and every nonnilpotent $h_\alpha \in \mathscr{H}_\alpha$, the proper Engel subalgebra $\mathscr{E}_\alpha = \mathscr{L}_0(\operatorname{ad} h_\alpha)$ contains the root subalgebra $\mathscr{L}^{(\alpha)} = \mathscr{H} + \sum_{i=1}^{p-1} \mathscr{L}_{\alpha i}$. It is for this reason that our conditions on the Engel subalgebras evoke conditions on these $\mathscr{L}^{(\alpha)}$ and therefore ultimately on $\mathscr{L}$.

What is suggested here is that one can study certain aspects of $\mathscr{L}$ by passing to the study of the finite family $\mathscr{E}_\alpha$ (α a root) of subalgebras $\mathscr{E}_\alpha = \mathscr{L}_0(\operatorname{ad} h_\alpha) = \mathscr{H} \oplus \sum_{\beta \in R_\alpha} \mathscr{L}_\beta$. Here, $R_\alpha = \{\beta \mid \beta$ is a root and $\beta(h_\alpha) = 0\}$ and R_α contains the root string $\pm\alpha$, $\pm 2\alpha$, etc. The subalgebras $\mathscr{E}_\alpha$ are analogous to the reductive subgroups $\{g \in G \mid [g, T_\alpha] = 0\}$ corresponding to the tori T_α defined as connected kernals of roots α of a semisimple algebraic group G with maximal torus T. Thus, for semisimple Lie algebras of characteristic 0, the $\mathscr{E}_\alpha$ are reductive, that is, $\mathscr{E}_\alpha = \mathscr{T} \oplus \mathscr{E}_\alpha^{(1)}$ where $\mathscr{E}_\alpha^{(1)}$ is semisimple and $\mathscr{T}$ is ad-diagonalizable on $\mathscr{L}$.

5.5. THEOREM. *Suppose that $\mathscr{L}$ is not semitriangulable. Then every proper Engel subalgebra of $\mathscr{L}$ is a Cartan subalgebra of $\mathscr{L}$.*

Proof. Let $\mathscr{B} = \mathscr{L}_0(\operatorname{ad} x)$ be a proper Engel subalgebra of $\mathscr{L}$. Then $\mathscr{B}$ contains a regular element y of $\mathscr{L}$ by Theorem 4.2. The preceding Theorem 5.4 then shows that $\mathscr{L}_0(\operatorname{ad} y) = ky + \mathscr{N}$ where $\mathscr{N}$ is a nil subalgebra of $\mathscr{L}$ normalized by y. Consequently, $\mathscr{B} = \mathscr{B}_0(\operatorname{ad} y) + \mathscr{B}^\infty = ky + \mathscr{N}_0 + \mathscr{B}^\alpha = ky + \mathscr{M}$ where $\mathscr{N}_0 = \mathscr{B} \cap \mathscr{N}$ and where $\mathscr{M} = \mathscr{N}_0 + \mathscr{B}^\infty$ is a nil subalgebra of $\mathscr{L}$ normalized by y. Here, we are invoking the triangulability of $\mathscr{B}$, which implies that $\mathscr{B}^\infty$ is indeed nil.

We may change y be a constant factor so that $x = y + m$ with $m \in M$. Then $B = kx + \mathscr{M}$. Since ad x is nilpotent on $\mathscr{B}$, it follows that $\mathscr{B}$ is nilpotent. But then $\mathscr{B}$ is a nilpotent Engel subalgebra of $\mathscr{L}$, hence a Cartan subalgebra of $\mathscr{L}$. ▮

5.6. COROLLARY. *Suppose that $\mathscr{L}$ is not semitriangulable. Then the following conditions are equivalent:*

(1) *$\mathscr{L}$ is E.t.;*

(2) *every nonnilpotent element x of $\mathscr{L}$ is regular;*

(3) *every nonnilpotent element x of $\mathscr{L}$ satisfies the condition that $\mathscr{L}_0(\operatorname{ad} x) = kx + \mathscr{N}$ where $\mathscr{N}$ is a nil subalgebra of $\mathscr{L}$ normalized by x.*

In Benkart and Isaacs [1], Lie algebras $\mathscr{L}$ are studied which are *centralizer nilpotent* (abbreviated c.n.) in the sense that the centralizer of every nonzero element is nilpotent.

5.7. COROLLARY. *Every centralizer nilpotent Lie algebra is E.t.*

Proof. It is shown in Benkart–Isaacs [1] that a Lie algebra $\mathscr{L}$ is centralizer nilpotent if and only if every nonnilpotent element x is regular with $\mathscr{L}_0(\operatorname{ad} x) = Fx$. ▮

5.8. EXAMPLES. We prove in Section 6 that for $p > 7$, the only simple E.t. Lie algebra are the centralizer nilpotent algebras $\mathscr{S}\mathscr{L}_2F$ and $\mathscr{W}_pF$ (the p-dimensional Witt algebra). It is shown later in this section, assuming Conjecture 6.5, that an arbitrary E.t. Lie algebra $\mathscr{L}$ can be written as $\mathscr{L} =$

nil $\mathscr{H} + \mathscr{L}^\infty$ where $\mathscr{L}^\infty/\mathscr{N}$ is a simple E.t. Lie algebra (and therefore is $\mathscr{S}\mathscr{L}_2F$ or $\mathscr{W}_pF$ if $p > 7$) and where $\mathscr{N}$ is nil on $\mathscr{L}$. Conversely, starting with an E.t. Lie algebra $\bar{\mathscr{L}}$, we can construct the E.t. Lie algebra $\mathscr{L} = \mathscr{V} \oplus \bar{\mathscr{L}}$ (direct) where $\mathscr{V}$ is any nilpotent Lie algebra. (We could, presumably, also let $\mathscr{L} = \mathscr{V} \oplus \bar{\mathscr{L}}$ (semidirect) for certain suitable nonzero subalgebras $\mathscr{V}$ of $\mathscr{D}er\, \bar{\mathscr{L}}$ which are nil on $\bar{\mathscr{L}}$.) Another construction involves taking any nil-preserving module $\mathscr{N}$ for an E.t. Lie algebra $\bar{\mathscr{L}}$ (see Definition 6.1), making $\mathscr{N}$ into an Abelian Lie algebra so that $\bar{\mathscr{L}}$ acts on $\mathscr{N}$ by derivations, and letting $\mathscr{L} = \bar{\mathscr{L}} \oplus \mathscr{N}$ (semidirect). Then $\mathscr{N}$ is a nil ideal of $\mathscr{L}$ and we can show that $\mathscr{L}$ is E.t. because every $\mathscr{L}_0(\operatorname{ad} x)$ is $\bar{\mathscr{L}}_0(\operatorname{ad} x) + \mathscr{N}_0(\operatorname{ad} x)$. If $x \in \bar{\mathscr{L}}$ is nilpotent, then $\mathscr{L} = \mathscr{L}_0(\operatorname{ad} x)$. If $x \in \bar{\mathscr{L}}$ is not nilpotent, $\bar{\mathscr{L}}_0(\operatorname{ad} x)$ is triangulable, so that $\mathscr{L}_0(\operatorname{ad} x)$ is triangulable. Finally, if $y = x + n$ with $x \in \bar{\mathscr{L}}$, $n \in \mathscr{N}$, $\operatorname{ad}_{\mathscr{L}} y$ is nilpotent if x is nilpotent, by an application of Jacobson [10] to $Fx \cup \mathscr{N}$; and if x is not nilpotent, $(\mathscr{L}/\mathscr{N})_0(\operatorname{ad} y) = (\mathscr{L}/\mathscr{N})_0(\operatorname{ad} x) = (\mathscr{L}_0(\operatorname{ad} x) + \mathscr{N})/\mathscr{N}$ shows that $\mathscr{L}_0(\operatorname{ad} y) + \mathscr{N} = \mathscr{L}_0(\operatorname{ad} x) + \mathscr{N}$ is triangulable, hence $\mathscr{L}_0(\operatorname{ad} y)$ is triangulable.

5.9. THEOREM. *Let $\mathscr{L}$ be a nonsemitriangulable E.t. Lie algebra. Then every solvable subalgebra $\mathscr{B}$ of $\mathscr{L}$ is triangulable with nil radical* nil $\mathscr{B}$ *of codimension* 0 *or* 1 *in $\mathscr{B}$.*

Proof. Let $\mathscr{M} = \operatorname{nil} \mathscr{B}$ and suppose that $\mathscr{B} \supsetneqq \mathscr{M}$. Then $\mathscr{B}/\mathscr{M}$ contains a maximal abelian ideal $\mathscr{A}/\mathscr{M}$, which is nonzero. Since $\mathscr{A}$ is not nil, we can take a nonnilpotent element a from $\mathscr{A} - \mathscr{M}$. Since $[\mathscr{B}, a] \subset \mathscr{A}$ and therefore $[[\mathscr{B}, a], a] \subset \mathscr{M}$ by the commutativity of $\mathscr{A}/\mathscr{M}$, we have $\mathscr{B}_*(\operatorname{ad} a) \subset \mathscr{M}$. Thus, $\mathscr{B} = \mathscr{B}_0(\operatorname{ad} a) + \mathscr{M} = ka + \mathscr{N} + \mathscr{M} = ka + \operatorname{nil} \mathscr{B}$ by Corollary 5.6. ▌

5.10. COROLLARY. *Let $\mathscr{L}$ be E.t. Then*

(1) *$\mathscr{L}$ is solvable if and only if $\mathscr{L}$ is semitriangulable;*

(2) *if $\mathscr{L}$ is not nilpotent $\mathscr{L}$ is solvable if and only if $\mathscr{L}$ is triangulable.*

5.11. COROLLARY. *Let $\mathscr{L}$ be E.t. but not semitriangulable. Then for $x, y \in \mathscr{L}$ such that $[y, x] = \alpha x$ where α is a nonzero element of F, x is nilpotent on $\mathscr{L}$.*

Proof. $\mathscr{B} = Fy + Fx$ is solvable, therefore, triangulable. Thus, $\mathscr{B}^{(1)} = Fx$ is a nil subalgebra of $\mathscr{L}$. ▌

5.12. THEOREM. *Let $\mathscr{L}$ be E.t. but not semitriangulable, and let $\mathscr{B}$ be a subnormal subalgebra of $\mathscr{L}$. Then either $\mathscr{B}$ is a nil subalgebra of $\mathscr{L}$ or $\mathscr{B}$ contains $\mathscr{L}^\infty$.*

Proof. Suppose that $\mathscr{B}$ is not nil and therefore contains a nonnilpotent element x of $\mathscr{L}$. Then $\mathscr{L} = \mathscr{L}_0(\operatorname{ad} x) + \mathscr{L}_*(\operatorname{ad} x) = \mathscr{L}_0(\operatorname{ad} x) + \mathscr{B}_*(\operatorname{ad} x) = \mathscr{H} + \mathscr{B}_*(\operatorname{ad} x)$ where $\mathscr{H}$ is the Cartan subalgebra $\mathscr{H} = \mathscr{L}_0(\operatorname{ad} x)$ of $\mathscr{L}$. (To verify the second equality, observe that $(\operatorname{ad} x)^i(\mathscr{L}) \subset \mathscr{B}$ for some i since $\mathscr{B}$ is subnormal.) But then $\mathscr{L}^\infty = \mathscr{B}_*(\operatorname{ad} x) \subset \mathscr{B}$. ▮

5.13. Corollary. *Let $\mathscr{L}$ be E.t. but not semitriangulable. Then $\mathscr{L}$ has a unique minimal nonnilpotent ideal, namely, $\mathscr{L}^\infty$, and $\mathscr{L}/\operatorname{nil} \mathscr{L}$ has a unique minimal ideal, namely, $(\mathscr{L}^\infty + \operatorname{nil} \mathscr{L})/\operatorname{nil} \mathscr{L}$.*

5.14. Definition. The *core* of a Lie algebra $\mathscr{L}$ is the Lie algebra Core $\mathscr{L} = \mathscr{L}^\infty/\operatorname{nil} \mathscr{L}^\infty$. (Recall that nil $\mathscr{L}^\infty$ is nil on $\mathscr{L}$ by convention.)

5.15. Theorem. *Let $\mathscr{L}$ be E.t. but not semitriangulable. Then* nil $\mathscr{L}^\infty$ *is the radical of $\mathscr{L}^\infty$,* Core $\mathscr{L} = \mathscr{L}^\infty/\operatorname{nil} \mathscr{L}^\infty$ *is simple, and* Core $\mathscr{L}$ *is isomorphic to $\mathscr{S}/(\mathscr{L}^\infty \cap \operatorname{nil} \mathscr{L})$ for some subalgebra $\mathscr{S}$ of $\mathscr{L}^\infty$ containing $\mathscr{L}^\infty \cap \operatorname{nil} \mathscr{L}$ such that $\mathscr{L}^\infty/(\mathscr{L}^\infty \cap \operatorname{nil} \mathscr{L}) = \mathscr{S}/(\mathscr{L}^\infty \cap \operatorname{nil} \mathscr{L}) \oplus \operatorname{nil} \mathscr{L}^\infty/(\mathscr{L}^\infty \cap \operatorname{nil} \mathscr{L})$.*

Proof. We know that $\mathscr{L}^\infty$ is the unique minimal nonnil ideal of $\mathscr{L}$. Let $\mathscr{I}$ be maximal among the ideals of $\mathscr{L}$ which are properly contained in $\mathscr{L}^\infty$. Then $\mathscr{I}$ is nil on $\mathscr{L}$ by the minimality of $\mathscr{L}^\infty$. Thus, $\mathscr{I} = \mathscr{L}^\infty \cap \operatorname{nil} \mathscr{L}$. Furthermore, $\mathscr{L}^\infty/\mathscr{I}$ is *differentiably simple*, by the maximality of $\mathscr{I}$, so that $\mathscr{L}^\infty/\mathscr{I}$ has a Levi decomposition $\mathscr{L}^\infty/\mathscr{I} = \mathscr{S}/\mathscr{I} \oplus \mathscr{N}/\mathscr{I}$ (semidirect) where $\mathscr{S}/\mathscr{I}$ is simple and $\mathscr{N}/\mathscr{I}$ is the nilpotent radical of $\mathscr{L}^\infty/\mathscr{I}$—by Block [4]. Let $n \in \mathscr{N}$. Then $\mathscr{L}^\infty \subset \mathscr{L}_0(\operatorname{ad} n) + \mathscr{I}$. Since $\mathscr{L}$ is not semitriangulable, it follows that $\mathscr{L}$ is not solvable, therefore $\mathscr{L}^\infty$ is not solvable, therefore $\mathscr{L}_0(\operatorname{ad} n) + \mathscr{I}$ is not solvable, therefore $\mathscr{L}_0(\operatorname{ad} n) = \mathscr{L}$. Thus, $\mathscr{N}$ is nil on $\mathscr{L}$. We conclude that $\mathscr{N} = \operatorname{nil} \mathscr{L}^\infty$, so that Core $\mathscr{L} = \mathscr{L}^\infty/\operatorname{nil} \mathscr{L}^\infty$ is isomorphic to $\mathscr{S}/\mathscr{I}$ and is therefore simple. The other assertions have been proved along the way. ▮

Whereas the above results concern the simple Lie algebra Core $\mathscr{L}$ and the structure of the unique minimal nonnilpotent ideal $\mathscr{L}^\infty$ of $\mathscr{L}$, the following theorem concerns the structure of $\mathscr{L}$ in terms of $\mathscr{L}^\infty$.

5.16. Theorem. *For any Cartan subalgebra $\mathscr{H}$ of a nonsemitriangulable E.t. Lie algebra $\mathscr{L}$, $\mathscr{L} = \operatorname{nil} \mathscr{H} + \mathscr{L}^\infty$ and $\mathscr{H} = \mathscr{H}_\infty + \operatorname{nil} \mathscr{H}$.*

Proof. We show first that $\mathscr{H} \cap \mathscr{L}^\infty$ has a nonnilpotent element x. For suppose not. Then $\mathscr{H}_\infty = \mathscr{H} \cap \mathscr{L}^\infty$ is nil on $\mathscr{L}$. But $\mathscr{L}$ is s.s.s.t. by Theorem 5.7, so that the fact that $\mathscr{H}_\infty$ is nil on $\mathscr{L}$ implies that $\mathscr{L}$ is solvable, by Theorem 3.5. But $\mathscr{L}$ is not solvable (since $\mathscr{L}$ is s.s.s.t. and $\mathscr{L}$ is not semitriangulable), so that we may conclude that $\mathscr{H}_\infty = \mathscr{H} \cap \mathscr{L}^\infty$ has some nonnilpotent element x. It follows that $\mathscr{H} = \mathscr{L}_0(\operatorname{ad} x) = Fx + \operatorname{nil} \mathscr{H}$. But then $\mathscr{L} = \mathscr{H} + \mathscr{L}^\infty = \operatorname{nil} \mathscr{H} + \mathscr{L}^\infty$ and $\mathscr{H} = \mathscr{H}_\infty + \operatorname{nil} \mathscr{H}$ since $x \in \mathscr{L}^\infty$ and $x \in \mathscr{H}^\infty = \mathscr{H} \cap \mathscr{L}^\infty$. ▮

6. Classification of E.t. Lie Algebras

In classifying c.n. Lie algebras, Benkart and Isaacs (1979) introduce the concept of special module (see below) and study the special modules of $\mathscr{S}\mathscr{L}_2F$ and $\mathscr{W}_pF$. Since the nonnilpotent elements x of a c.n. Lie algebra L correspond to the subalgebras $\mathscr{H} = Fx$ of $\mathscr{L}$, the condition that the nonnilpotent elements x of $\mathscr{L}$ be nonsingular on a special module $\mathscr{M}$ can also be given by the condition that $\mathscr{M}_0(\mathscr{H}) = \{0\}$ for every Cartan subalgebra $\mathscr{H}$ of $\mathscr{L}$.

For an arbitrary Lie algebra $\mathscr{L}$ with Cartan subalgebra $\mathscr{H}$ and module $\mathscr{M}$, the condition $\mathscr{M}_0(\mathscr{H}) = \{0\}$ ensures that $\mathscr{H}$ remains a Cartan subalgebra in the split extension $\mathscr{L} \oplus \mathscr{M} = \mathscr{H} + \sum_{\alpha\neq 0} \mathscr{L}_\alpha + \sum_{\beta\neq 0} \mathscr{M}_\beta$, $\mathscr{M}$ being given the structure of Abelian Lie algebra. Furthermore, in characteristic 0 with $\mathscr{L}$ semisimple, if one Cartan subalgebra $\mathscr{H}$ of $\mathscr{L} \oplus \mathscr{M}$ is contained in $\mathscr{L}$, then every Cartan subalgebra of $\mathscr{L} \oplus \mathscr{M}$ has trivial intersection with $\mathscr{M}$, from which it follows easily that $\mathscr{M}_0(\mathscr{H}^\sigma) = \{0\}$ for every Cartan subalgebra $\mathscr{H}^\sigma$ of $\mathscr{L}$. Finally, we are studying a Lie algebra $\mathscr{L}$ in terms of the nilpotency of its elements, so that it is essential, for this study, that the modules which we consider preserve this nilpotency. This serves as motivation for the introduction and use of the following concepts.

6.1. Definition. An $\mathscr{L}$-module $\mathscr{M}$ is *nil-preserving* if every nilpotent element x of $\mathscr{L}$ is nilpotent as a linear transformation of $\mathscr{M}$.

An $\mathscr{L}$-module $\mathscr{M}$ is *$\mathscr{H}$-nonsingular* if $\mathscr{M}_0(\mathscr{H}) = \{0\}$, $\mathscr{H}$ being a Cartan subalgebra of $\mathscr{L}$. An $\mathscr{L}$-module $\mathscr{M}$ is *nonsingular* if $\mathscr{M}$ is $\mathscr{H}$-nonsingular for every Cartan subalgebra $\mathscr{H}$. An $\mathscr{L}$-module $\mathscr{M}$ is *special* if it is nil-preserving and $\mathscr{M}_0(x) = \{0\}$ for every nonnilpotent element x of $\mathscr{L}$. An $\mathscr{L}$-module $\mathscr{M}$ is *integral* if every element x of $\mathscr{L}$ which is *integral* on $\mathscr{L}$ in the sense that the eigenvalues of $\operatorname{ad}_{\mathscr{L}} x$ are in the prime ring is also integral on $\mathscr{M}$ in the sense that the eigenvalues of x on $\mathscr{M}$ are in the prime ring.

Benkart and Isaacs [1] state for $p \geqslant 3$ that an irreducible module $\mathscr{M}$ for $\mathscr{S} = \mathscr{S}\mathscr{L}_2F$ is special if and only if $\mathscr{M}$ is even dimensional and therefore integral). They prove the "only if" direction of this assertion by taking a basis e_{-1}, e_0, e_1 for $\mathscr{S}$ such that $[e_{-1}, e_0] = e_{-1}$, $[e_{-1}, e_1] = e_0$ and $[e_0, e_1] = e_1$, pointing out that $e_{-1} - e_1$ is not nilpotent on $\mathscr{L}$ and therefore is nonsingular on the given special module $\mathscr{M}$. They then show that the nonsingularity of $e_{-1} - e_1$ on $\mathscr{M}$ implies that $\mathscr{M}$ is even dimensional. We use this device to sharpen their result as follows.

6.2. Theorem. *Let $p \geqslant 3$ and let $\mathscr{M}$ be an irreducible nil-preserving module for $\mathscr{S} = \mathscr{S}\mathscr{L}_2F$ which is $\mathscr{H}$-nonsingular for some Cartan subalgebra $\mathscr{H}$ of $\mathscr{S}$. Then $\mathscr{M}$ is even dimensional and therefore integral.*

Proof. Choose $\mathscr{H}$ so that $\mathscr{M}_0(\operatorname{ad} \mathscr{H}) = \{0\}$, and take the basis e_{-1}, e_0, e_1 for $\mathscr{S}$ described above. Then $e_{-1} - e_1$ is not nilpotent on $\mathscr{S}$, whence $F(e_{-1} - e_1)$

is a Cartan subalgebra of $\mathcal{S}$ (e.g., since $\mathcal{S}$ is c.n.). But the Cartan subalgebras of $\mathcal{S}$ are conjugate under Aut $\mathcal{S}$, so that $F(\sigma(e_{-1}) - \sigma(e_1)) = \mathcal{H}$ and $\sigma(e_{-1}) - \sigma(e_1)$ is nonsingular on $\mathcal{M}$ for some automorphism σ of $\mathcal{S}$. By applying the method of Benkart and Isaacs [1] to the basis $\sigma(e_{-1})$, $\sigma(e_0)$, $\sigma(e_1)$, where we now know that $\sigma(e_{-1}) - \sigma(e_1)$ is nonsingular on $\mathcal{M}$, we find that $\mathcal{M}$ is even dimensional and therefore integral. ∎

The "if" direction of the above Benkart and Isaacs assertion is easily verified and is used in the following corollary.

6.3. Corollary. *Let $p \geqslant 3$ and let $\mathcal{M}$ be an irreducible module for $\mathcal{S} = \mathcal{S}\mathcal{L}_2F$. Then the following conditions are equivalent*:

(1) *$\mathcal{M}$ is nil preserving and nonsingular*;
(2) *$\mathcal{M}$ is nil preserving and H-nonsingular for some Cartan subalgebra $\mathcal{H}$*;
(3) *$\mathcal{M}$ is even dimensional* (*and therefore integral*);
(4) *$\mathcal{M}$ is special.*

Proof. Condition (4) implies (1) implies (2) implies (3); and (3) implies (4) is the assertion of Benkart and Isaacs. ∎

We note in passing that Wilson [15, p. 290] shows that an irreducible restricted module $\mathcal{V}$ for $\mathcal{W}_pF$ such that $\mathcal{V}_0(e_0) = \{0\}$ has $\dim \mathcal{V} = p - 1$ and basis v_i $(1 \leqslant i \leqslant p-1)$ such that $v_ie_j = iv_{i+j}$ where $1 \leqslant i + j \leqslant p - 1$ and $v_ie_j = 0$ otherwise, $e_{-1}, e_0, \dots, e_{p-2}$ being a preassigned basis for $\mathcal{W}_pF$ such that $[e_i, e_j] = (i-j)\, e_{i+j}$. Moreover, $\mathcal{V}$ remains irreducible when viewed as $\mathcal{S}\mathcal{L}_2F$-module, $\mathcal{S}\mathcal{L}_2F$ being the subalgebra $Fe_{-1} + Fe_0 + Fe_1$. Since every nonnilpotent element x of $\mathcal{W}_pF$ is regular (e.g., since $\mathcal{W}_pF$ is c.n.) and since the Cartan subalgebras of $\mathcal{W}_pF$ are conjugate by Brown [6], there exists an automorphism σ of $\mathcal{W}_pF$ such that $\sigma(e_0) = cx$ for some $c \in F$. Replacing x by $(1/c)x$ so that $\sigma(e_0) = x$, we see that x can be put as e_0' in a basis $e_i' = \sigma(e_i)$ of the same form. Since V has only one restricted irreducible representation for $\mathcal{W}_pF$ of dimension $p - 1$, by Chang [7], it follows that $x = e_0'$ is also nonsingular on $\mathcal{V}$. Thus, as observed in Benkart and Isaacs [1] with indications of proof without full detail, the unique $p - 1$ dimensional restricted irreducible $\mathcal{W}_pF$-module $\mathcal{V}$ is special. Furthermore, it is clear from the above discussion that this $\mathcal{V}$ is the only restricted $\mathcal{W}_pF$-module which is H-nonsingular for some $\mathcal{H}$. This discussion therefore shows that *for $p > 3$ the preceding Corollary* 6.3 *is valid for restricted modules $\mathcal{M}$ with "$\mathcal{W}_pF$" in place of "$\mathcal{S}\mathcal{L}_2F$" and* "$\dim \mathcal{M} = p - 1$" *in place of "$\mathcal{M}$ is even dimensional."*

A *subquotient* of $\mathcal{L}$ is any quotient $\mathcal{B}/\mathcal{C}$ where $\mathcal{B}$ is a proper subalgebra of $\mathcal{L}$ and $\mathcal{C}$ is an ideal of $\mathcal{B}$.

6.4. Theorem. *Let $\mathcal{L}$ be a nonsemitriangulable E.t. Lie algebra of characteristic $p \geqslant 3$ such that every simple subquotient of $\mathcal{L}$ is isomorphic to $\mathcal{S}\mathcal{L}_2F$ or $\mathcal{W}_pF$. Then $\mathcal{L}$ is toral rank* 1.

Proof. Let $\mathscr{H}$ be a Cartan subalgebra of $\mathscr{L}$ and consider the root subalgebras $\mathscr{L}^{(\alpha)} = \mathscr{H} + \sum_{i=1}^{p-1} \mathscr{L}_{\alpha i}$. Since $\mathscr{L}$ is not semitriangulable, it follows from Theorem 2.11 that some $\mathscr{L}^{(\alpha)}$ is not semitriangulable, hence not solvable. Let $\mathscr{B}$ denote such an $\mathscr{L}^{(\alpha)}$. We may assume that $\mathscr{L} \supsetneq \mathscr{B}$, for otherwise $\mathscr{L}$ is toral rank 1. Thus, Core $\mathscr{B} = \mathscr{B}^{\infty}/\mathscr{N}$ with $\mathscr{N}$ being the radical of $\mathscr{B}$ is a simple subquotient of $\mathscr{B}$, so that Core $\mathscr{B}$ is isomorphic to $\mathscr{S}\mathscr{L}_2F$ or $\mathscr{W}_pF$. We next observe that $\mathscr{N}$ is nil on $\mathscr{L}$. For if $n \in \mathscr{N}$, then $\mathscr{L}_0(\mathrm{ad}\, n) + \mathscr{N} \supset \mathscr{B}^{\infty}$. Since $\mathscr{B}^{\infty}$ is not solvable, $\mathscr{L}_0(\mathrm{ad}\, n)$ is not solvable and therefore ad n is nilpotent on $\mathscr{L}$. It follows that any irreducible subquotient $\mathscr{M}$ of the $\mathscr{B}^{\infty}$-module $\mathscr{L}/\mathscr{B}$ can be regarded as an irreducible $\mathscr{B}^{\infty}/\mathscr{N}$-module, since $\mathscr{N}$, being nil on $\mathscr{M}$, must be in the kernal of the representation of $\mathscr{B}^{\infty}$ on $\mathscr{M}$. The $\mathscr{B}^{\infty}/\mathscr{N}$-module $\mathscr{M}$ is nil-preserving. For suppose that $y + \mathscr{N}$ is a nilpotent element of $\mathscr{B}^{\infty}/\mathscr{N}$. Then $\mathscr{L}_0(\mathrm{ad}\, y) + \mathscr{N} \supset \mathscr{B}^{\infty}$, and the nonsolvability of $\mathscr{B}^{\infty}$ implies that ad y is nilpotent on $\mathscr{L}$ (as in a similar argument above), therefore on $\mathscr{M}$. Next, we note that $\mathscr{H} \cap \mathscr{B}^{\infty}$ has a regular element x of $\mathscr{L}$; for otherwise, $\mathscr{H} \cap \mathscr{B}^{\infty}$ is nil on $\mathscr{L}$ and $\mathscr{B}$ is solvable by Theorem 3.5, a contradiction. Since x is regular and $x \in \mathscr{H}$, we have $\mathscr{H} = \mathscr{L}_0(\mathrm{ad}\, x)$ (by the maximal nilpotency property of Cartan subalgebras $\mathscr{H}$, and the fact that $\mathscr{L}_0(\mathrm{ad}\, x)$ is nilpotent). Since $\mathscr{L}^{(\alpha)}$ has toral rank 1, we may replace x by $[1/\alpha(x)]x$ and assume that x is integral on $\mathscr{B}$ (cf. Definition 6.1). Furthermore, viewing x as a linear transformation of $\mathscr{L}/\mathscr{B}$, x is nonsingular, since $\mathscr{L}_0(\mathrm{ad}\, x) = \mathscr{H} \subset \mathscr{B}$. It follows that the element $\bar{x} = x + \mathscr{N}$ of $\mathscr{B}^{\infty}/\mathscr{N}$ is nonsingular on $\mathscr{M}$, that is, $\mathscr{M}_0(\bar{x}) = \{0\}$. Since $\mathscr{H} = \mathscr{L}_0(\mathrm{ad}\, x)$ is a Cartan subalgebra of $\mathscr{B}$ and since $x \in \mathscr{B}^{\infty}$, $\mathscr{B}_0^{\infty}(\mathrm{ad}\, x)$ is a Cartan subalgebra of $\mathscr{B}^{\infty}$ and $(\mathscr{B}^{\infty}/\mathscr{N})_0(\mathrm{ad}\, \bar{x}) = \bar{\mathscr{H}}$ is a Cartan subalgebra of $\mathscr{B}^{\infty}/\mathscr{N}$. Since $\bar{x} \in \bar{\mathscr{H}}$, what we have shown above is that $\mathscr{M}_0(\bar{\mathscr{H}}) = \{0\}$, so that $\mathscr{M}$ is $\bar{\mathscr{H}}$-nonsingular. Since $\mathscr{B}^{\infty}/\mathscr{N}$ is isomorphic to $\mathscr{S}\mathscr{L}_2F$ or $\mathscr{W}_pF$, $\bar{\mathscr{H}}$ must be one dimensional and $\bar{\mathscr{H}} = F\bar{x}$. Furthermore, in each case, we can find elements e_{-1}, e_1 in $\mathscr{B}^{\infty}/\mathscr{N}$ such that the span $\mathscr{S}$ of e_{-1}, $\bar{x}$, e_1 is isomorphic to $\mathscr{S}\mathscr{L}_2F$. The upshot of what we have done is now that *for every irreducible subquotient $\mathscr{M}$ of $\mathscr{L}/\mathscr{B}$ as $\mathscr{S}$-module, $\mathscr{M}$ is nil-preserving and $\mathscr{M}$ is $F\bar{x}$-nonsingular.* (Note that our condition on $\mathscr{M}$ now is that of irreducibility over $\mathscr{S}$, not over $\mathscr{B}^{\infty}/\mathscr{N}$.) It therefore follows from Theorem 6.2 that $\mathscr{M}$ is even dimensional (and therefore integral). Since $\bar{x}$ is integral on $\mathscr{S}$ (which follows from the fact that x is integral on $\mathscr{B}$), we can finally conclude that $\bar{x}$ is integral on $\mathscr{M}$! This being true for all irreducible subquotients $\mathscr{M}$ of the S-module $\mathscr{L}/\mathscr{B}$ (in particular, for those defined by a composition series), it follows that x is integral on $\mathscr{L}/\mathscr{B}$. But x was chosen integral on $\mathscr{B}$. We conclude that x is integral on $\mathscr{L}$. Since x is also regular in $\mathscr{L}$, it follows that $\mathscr{L} = \mathscr{L}_0(\mathrm{ad}\, x) + \sum_{i=1}^{p-1} \mathscr{L}_i(\mathrm{ad}\, x) = \mathscr{H} + \sum_{i=1}^{p-1} {}_{\alpha}\mathscr{L}_i = L^{(\alpha)}$, so that $\mathscr{L}$ is toral rank 1. ∎

6.5. *Conjecture.* For $p > 7$, the only simple toral rank 1 Lie algebras $\mathscr{L}$ such that every solvable subalgebra of $\mathscr{L}$ is triangulable on L are isomorphic to $\mathscr{S}\mathscr{L}_2F$ or $\mathscr{W}_pF$.

The only simple toral rank 1 Lie algebras ($p > 7$) other than $\mathcal{S}\mathcal{L}_2F$ and $\mathcal{W}_pF$ are the Albert–Zassenhaus algebras $\mathcal{L}(G, \theta)$ (see below) and the toral rank 1 Block algebras $\mathcal{L}(G, f)$ of type $G = G_0$ and $\mathcal{L}(G, f, \delta)$ of type $G = G_1$ described in Block [5]. This is proved in Wilson [15, 17]. If $\mathcal{L}$ is a Zassenhaus algebra $\mathcal{L} = \mathcal{L}(G, \theta)$, then G is a finite subgroup of $(F, +)$ of order p^n ($n \geqslant 2$), θ is a homomorphism from G to $(F, +)$, and $\mathcal{L}$ has basis χ_α ($\alpha \in G$) over F and product $[\chi_\alpha, \chi_\beta] = (\alpha(\theta(\beta) + 1) - \beta(\theta(\alpha) + 1))\,\chi_{\alpha+\beta}$. It is clear that $F\chi_0$ is a Cartan subalgebra and the corresponding root spaces are the $F\chi_\alpha$ ($\alpha \in G$). We claim that $\Delta(\alpha, \beta) \not\equiv \beta$ modulo the span of α over the prime field π of F for some nonzero $\alpha, \beta \in G$, where $\Delta(\alpha, \beta) = \alpha\theta(\beta) - \beta\theta(\alpha)$. To see this, take $\alpha, \beta \in G$, to be linearly independent over the prime field π. If $\Delta(\alpha, \beta) \equiv \beta$ (modulo $\pi\alpha$), replace α by 2α and note that $\Delta(2\alpha, \beta) = 2\Delta(\alpha, \beta) \equiv 2\beta$ (modulo $\pi\alpha$), whence $\Delta(2\alpha, \beta) \not\equiv \beta$ (modulo $\pi\alpha$). Thus, there exist $\alpha, \beta \in G$ with $\Delta(\alpha, \beta) \not\equiv \beta$ (modulo $\pi\alpha$) (e.g., the $2\alpha, \beta \in G$ described above). For such α and β, the cofeficient $\alpha(\theta(\beta + i\alpha) + 1) - (\beta + i\alpha)(\theta(\alpha) + 1) = \alpha\theta(\beta) - \beta\theta(\alpha) - \beta - (i - 1)\alpha$ of $\chi_\alpha + (\beta + \alpha i)$ in the product $[\chi_\alpha, \chi_{\beta+i\alpha}]$ cannot be zero for any i ($0 \leqslant i \leqslant p - 1$); for otherwise $0 = \Delta(\alpha, \beta) - \beta - (i - 1)\alpha$ and $\Delta(\alpha, \beta) \equiv \beta$ (modulo $\pi\alpha$). It follows easily that ad χ_α is not nilpotent on $F\chi_\beta + \cdots + F\chi_{\beta+(p-1)\alpha}$. On the other hand, $\mathcal{B} = F\chi_0 + F\chi_\alpha$ is a solvable subalgebra of $\mathcal{L}$, which cannot be triangulable on $\mathcal{L}$ since ad χ_α is not nilpotent on $\mathcal{L}$. We conclude that *the Zassenhaus algebra $\mathcal{L}(G, \theta)$ is not s.s.t. and is therefore not E.t.*

It remains only to prove Conjecture 6.5 for the toral rank 1 Block algebras $\mathcal{L}(G, f)$ of type $G = G_0$ and $\mathcal{L}(G, f, \delta)$ of type $G = G_1$. One can reduce proving the conjecture to the case $|G| = p^2$, so that $\dim L(G, f) = p^2 - 1$ and $\dim \mathcal{L}(G, f, \delta) = p^2 - 2$.

If Conjecture 6.5 is true, we would have the following theorem.

6.6. Theorem (assuming 6.5). *For $p > 7$, the only simply toral rank 1 E.t. Lie algebras $\mathcal{L}$ are isomorphic to $\mathcal{S}\mathcal{L}_2F$ or $\mathcal{W}_pF$.*

Proof. Suppose not, and let $\mathcal{L}$ be a counterexample of minimal dimension. Then every (proper) simple subquotient is E.t. (cf. Proposition 5.2) of lower dimension than that of L and is therefore isomorphic to $\mathcal{S}\mathcal{L}_2F$ or $\mathcal{W}_pF$ by the minimality condition on the dimension of $\mathcal{L}$. Thus, $\mathcal{L}$ is toral rank 1. Since $\mathcal{L}$ is E.t. and therefore every solvable subalgebra of $\mathcal{L}$ is triangulable, it follows (assuming 6.5) that $\mathcal{L}$ is isomorphic to $\mathcal{S}\mathcal{L}_2F$ or $\mathcal{W}_pF$. ∎

7. Restricted and Semirestricted Closures Applied to Questions of Solvability

In this section, we assume that $\mathcal{L}$ is a subalgebra of a restricted Lie algebra $\mathcal{M}$. The *restricted closure* $\bar{\mathcal{L}}$ of $\mathcal{L}$ is the intersection all restricted subalgebras of $\mathcal{M}$ containing $\mathcal{L}$. Since the normalizer $\mathcal{N}(\mathcal{L})$ of $\mathcal{L}$ in $\mathcal{M}$ is restricted,

$\mathscr{L}$ is an ideal of $\bar{\mathscr{L}}$. Since $[\mathscr{N}(\mathscr{L}), \mathscr{L}] \subset \mathscr{L}$, it follows easily that $[\mathscr{N}(\mathscr{L}), \bar{\mathscr{L}}] \subset \mathscr{L}$ and, in particular, that $[\bar{\mathscr{L}}, \bar{\mathscr{L}}] \subset \mathscr{L}$. Thus $\bar{\mathscr{L}}/\mathscr{L}$ is Abelian. In particular, *$\mathscr{L}$ is solvable if and only if $\bar{\mathscr{L}}$ is solvable.* Moreover, if $\mathscr{L} = \mathscr{L}_1 \supset \mathscr{L}_2 \supset \cdots$ is a series of ideals of $\mathscr{L}$ such that $[\mathscr{L}_i, \mathscr{L}_j] \subset \mathscr{L}_{i+j}$ for all i, j, then one sees successively that $[\bar{\mathscr{L}}_i, \mathscr{L}_j] \subset \mathscr{L}_{i+j}$ and $[\bar{\mathscr{L}}_i, \bar{\mathscr{L}}_j] \subset \mathscr{L}_{i+j} \subset \bar{\mathscr{L}}_{i+j}$. It follows easily that *$\bar{\mathscr{L}}$ is nilpotent in the closed sense* that *the series $\bar{\mathscr{L}}^1 = \bar{\mathscr{L}}$, $\bar{\mathscr{L}}^{i+1} = [\bar{\mathscr{L}}, \bar{\mathscr{L}}^i]$ terminates with $\{0\}$ if and only if L is nilpotent in the usual sense.* Using Jacobson's theorem on weakly closed sets [11], we also see easily that *$\bar{\mathscr{L}}$ is nil* (that is, every element of $\bar{\mathscr{L}}$ is nilpotent in $\mathscr{M}$) *if $\mathscr{L}$ is nil*, from which it easily follows that *$\bar{\mathscr{L}}$ is triangulable if $\mathscr{L}$ is triangulable*, $\mathscr{L}$ being *triangulable* if $\mathscr{L}^{(1)}$ is nil (cf. [19]).

The following theorem is taken from Winter [19].

7.1. THEOREM. *Let $\hat{\mathscr{H}}$ be a Cartan subalgebra of $\bar{\mathscr{L}}$. Then $\bar{\mathscr{L}} = \hat{\mathscr{H}} + \mathscr{L}$.*

Proof. For $x \in \mathscr{L}_\alpha(\operatorname{ad} \hat{\mathscr{H}})$, $\operatorname{ad} x^p$ stabilizes all of the $\mathscr{L}_\alpha(\operatorname{ad} \hat{\mathscr{H}})$, so that $0 = [\operatorname{ad} \mathscr{T}, \operatorname{ad} x^p] = \operatorname{ad}[\mathscr{T}, x^p]$ where $\mathscr{T}$ is the maximal torus of $\hat{\mathscr{H}}$. Thus, $0 = [\mathscr{T}, [\mathscr{T}, x^p]]$, whence $x^p \in \mathscr{L}_0(\operatorname{ad} \mathscr{T}) = \hat{\mathscr{H}}$. Since $\hat{\mathscr{H}}^p \subset \hat{\mathscr{H}}$, it follows that $\hat{\mathscr{H}} + \mathscr{L}$ has a basis of elements whose pth powers lie in $\hat{\mathscr{H}} + \mathscr{L}$. Thus, $\hat{\mathscr{H}} + \mathscr{L}$ is a restricted subalgebra of $\mathscr{M}$ by Jacobson [10]. ∎

The above theorem and proof show that $\bar{\mathscr{L}}$ can be constructed by taking any Cartan subalgebra $\mathscr{H}$ of $\mathscr{L}$, taking any maximal torus $\mathscr{T}$ of $\bar{\mathscr{L}}$ containing $\mathscr{H}_s$ (the set of semisimple elements of $\mathscr{H}$), and letting $\hat{\mathscr{H}}$ be the Lie p-algebra generated by $\mathscr{T}$ and $\{x^p \mid x \in \mathscr{L}_\alpha(\mathscr{T})$ for some $\alpha\}$. Then $\bar{\mathscr{L}} = \hat{\mathscr{H}} + \mathscr{L}$, and $\hat{\mathscr{H}}$ is a Cartan subalgebra of $\bar{\mathscr{L}}$ which can also be described as the centralizer in $\bar{\mathscr{L}}$ of $\mathscr{T}$ by [20]. In passing from $\mathscr{H}$ to $\hat{\mathscr{H}}$, the root space decomposition $\sum \mathscr{L}_\alpha(\mathscr{H})$ is refined: $\mathscr{L}_\alpha(\mathscr{H}) = \sum_{\beta \underset{\sim}{\in} \alpha} \mathscr{L}_\beta(\mathscr{T})$ and $\mathscr{H} = \mathscr{L}_0(\operatorname{ad} \mathscr{H}) = \sum_{\beta \underset{\sim}{\in} 0} \mathscr{L}_\beta(\operatorname{ad} \mathscr{T})$, sometimes nontrivially (e.g., see Examples 2.13 and 3.3 where $\mathscr{H}_0(\operatorname{ad} \mathscr{T}) \subsetneqq \mathscr{H}$. Regard this Lie algebra $\mathscr{L}$ as imbedded in the Lie p-algebra $\operatorname{Der} \mathscr{L}$ via ad). Here, $\beta \underset{\sim}{\in} \alpha$ means that $\beta(h_s)$ is the scalar through which h_s acts on $\mathscr{L}_\alpha(\operatorname{ad} \mathscr{H})$ for all $h \in \mathscr{H}$.

$\mathscr{T}$ being as taken above, *$\hat{\mathscr{L}} = \mathscr{T} + \bar{\mathscr{H}} + \mathscr{L}$ is semirestricted with respect to its Cartan subalgebra $\hat{\mathscr{H}} = \mathscr{T} + \bar{\mathscr{H}}_0(\mathscr{T})$ in the sense of Schue* [12]: $\hat{\mathscr{H}}$ is restricted, $(\operatorname{ad} h)^p x = [h^p, x]$ for all $h \in \hat{\mathscr{H}}$, $x \in \mathscr{L}$ and $(\operatorname{ad} x)_s \in \operatorname{ad} \mathscr{T}$ for all α and all $x \in \mathscr{L}_\alpha(\mathscr{T})$. Here, $\bar{\mathscr{H}}$ is the restricted closure of $\mathscr{H}$. We call $\hat{\mathscr{L}}$, $\hat{\mathscr{H}}$ a *semirestricted closure* of $\mathscr{L}$, $\mathscr{H}$.

We now direct these considerations to the problem of showing that an arbitrary Lie algebra $\mathscr{L}$ with Cartan subalgebra $\mathscr{H}$ is solvable if $\mathscr{H}_\infty$ is nil on $\mathscr{L}$ or if $\mathscr{L}^{(\alpha)} = \mathscr{H} + \sum_{i=1}^{p-1} \mathscr{L}_{\alpha i}$ is solvable for all α. (See questions 2.12, 3.4.) For this, *we may assume with no loss of generality that $\mathscr{L}$ has center* 0, by methods such as those in [20]. We then imbed $\mathscr{L}$ in the restricted Lie algebra $\operatorname{Der} \mathscr{L}$ via ad. That is, we may assume that *$\mathscr{L}$ is contained in a restricted Lie algebra* *in such a way that* ad x *is nilpotent on* $\mathscr{L}$ *if and only if* x *is nil in the restricted Lie algebra containing* $\mathscr{L}$.

We claim that if $\mathscr{H}$ is triangulable on $\mathscr{L}$ and $\mathscr{H}_\infty$ is nil, then $\hat{\mathscr{H}}$ is triangulable on $\hat{\mathscr{L}}$ and $\hat{\mathscr{H}}_\infty$ is nil on $\hat{\mathscr{L}}$, $\hat{\mathscr{L}}, \hat{\mathscr{H}}$ being a semirestricted closure of $\mathscr{L}, \mathscr{H}$. This shows that *the problem of showing that "$\mathscr{H}$ triangulable on $\mathscr{L}$ and $\mathscr{H}_\infty$ is nil on $\mathscr{L}$ implies that $\mathscr{L}$ is solvable" reduces to the same problem for semirestricted Lie algebras*. Since $\hat{\mathscr{H}} = \mathscr{T} + \bar{\mathscr{H}}_0(\mathscr{T})$ where $\bar{\mathscr{H}}_0(\mathscr{T})$ is certainly triangulable, $\hat{\mathscr{H}}$ is triangulable since $\mathscr{T}$ centralizes $\bar{\mathscr{H}}_0(\mathscr{T})$. And $\hat{\mathscr{H}}_\infty = \sum_{\beta \subseteq 0} [\mathscr{H}_\beta \mathscr{H}_{-\beta}] + \sum_\alpha \sum_{\beta \in \alpha} [\mathscr{L}_\beta(\mathscr{T}), \mathscr{L}_{-\beta}(\mathscr{T})]$, the subspaces $[\mathscr{H}_\beta, \mathscr{H}_{-\beta}]$, $\mathscr{L}_\beta(\mathscr{T})$ being nil on $\mathscr{L}$ by the conditions $[\mathscr{H}, \mathscr{H}]$ nil, $\mathscr{H}_\infty$ nil, respectively. Thus, $\hat{\mathscr{H}}_\infty$ is nil by the theorem of Jacobson [7] on weakly closed sets.

Using the semirestricted closure $\hat{\mathscr{L}}, \hat{\mathscr{H}}$, we also see that *the problem of showing that "if $\mathscr{L}$ is a Lie algebra with Cartan subalgebra H such that $\mathscr{L}^{(\alpha)} = \mathscr{H} + \sum_{i=1}^{p-1} \mathscr{L}_{\alpha i}$ is solvable for all α, then $\mathscr{L}$ is solvable" to the same problem for semirestricted Lie algebras*. To see this, observe that $\hat{\mathscr{L}}^{(\beta)} = \hat{\mathscr{H}} + \sum_{i=1}^{p-1} \mathscr{L}_{\beta i}(\mathscr{T})$ is contained in $\mathscr{T} + \mathscr{H} + \sum_{i=1}^{p-1} \mathscr{L}_{\alpha i}(\mathscr{T})$ for $\beta \in \alpha$; so that if all $\mathscr{L}^{(\alpha)}$ are solvable, then all $\mathscr{T} + \mathscr{L}^{(\alpha)}$ are solvable and therefore all $\hat{\mathscr{L}}^{(\beta)}$ are solvable.

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