THE SMALLEST GRAPHS WITH CERTAIN ADJACENCY PROPERTIES

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A graph is said to have property P_{1,n} if for every sequence of n + 1 points, there is another point adjacent only to the first point. It has previously been shown that almost all graphs have property P_{1,n}. It is easy to verify that for each n, there is a cube with this property. A more delicate question asks for the construction of the smallest graphs having property P_{1,n}. We find that this problem is intimately related with the discovery of the highly symmetric graphs known as cages, and are thereby enabled to resolve this question for 1 ≤ n ≤ 6.

1. The notation

For two points u and v of a graph G we write uAv if u and v are adjacent, and u & v if they are not. The set of points of G adjacent to a given point v is the neighborhood N(v). The subgraph induced by N(v) is the link of v in G, written link(v). The subgraph of G induced by all points neither equal to v nor adjacent to v is denoted G_v. Thus this is the subgraph of G obtained by removing the closed neighborhood N*(v). As usual, the minimum degree of G is denoted by δ(G), the maximum degree by Δ(G).

We say of two graphs G_1 and G_2 that G_1 is smaller than G_2 if p_1 < p_2 or if p_1 = p_2 and q_1 < q_2. For other graph theoretic notation and terminology we follow [3].

2. The problem

Some fascinating adjacency properties of graphs have been found in [2] to hold for almost all graphs. However, to our consternation, almost no graphs have been constructed which enjoy these properties. We generalize and then investigate a special case, not only to discover graphs with these properties but also to find the smallest such graphs.

Axiom n in Blass and Harary [2] states that for every sequence of 2n points (u_1, ..., u_n; v_1, ..., v_n), there is another point w such that wAu_i and wAv_j, for i, j = 1, ..., n. We generalize this to property P_{m,n}. A graph G has property P_{m,n} (written G ∈ P_{m,n}) if for any sequence of points (u_1, ..., u_n; v_1, ..., v_n) there is
another point \( w \) such that \( wA_{u_i} \) and \( wA_{u_j} \) for all \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \). Obviously \( G \in P_{m,n} \) implies \( \bar{G} \in P_{n,m} \).

It was shown in [2] that for each \( n \), almost all graphs satisfy Axiom \( n \), i.e., have property \( P_{n,n} \). It follows at once that for each \( m \) and \( n \) with say \( m \leq n \), almost all graphs are in \( P_{m,n} \). In other words, if we let \( G_p \) be the family of all graphs with \( p \) points then

\[
\lim_{p \to \infty} \frac{|G_p \cap P_{m,n}|}{|G_p|} = 1. \tag{1}
\]

But in spite of (1), it appears to be very difficult to construct graphs in \( P_{m,n} \) for general \( m \) and \( n \).

As a special case of this problem we concentrate on graphs with property \( P_{1,n} \). We note that for sufficiently large \( k \), \( Q_k \) (the \( k \)-cube) is in \( P_{1,n} \). This observation is made more precise in Lemma 1. Our object is to find the smallest graphs in \( P_{1,n} \).

Cubes are not the answer. We have succeeded for \( n = 1, \ldots, 6 \) and propose a conjecture related to values of \( n > 6 \) by linking the determination of such smallest graphs to the discovery of certain cages [3]. We include here proofs only for \( n = 2 \) and 3, as the other arguments are long, complicated, and analogous.

In what follows we use \((u; v_1, \ldots, v_k)\) to denote the set of points of \( G \) adjacent to \( u \), but not adjacent to any \( v_i \). If \( k \leq n \) and \( G \in P_{1,n} \), then \((u; v_1, \ldots, v_k) \neq \emptyset \), for all \((k + 1)\)-sequences. And if \( wA_{u_i} \) and \( wA_{v_j} \), for \( i = 1, \ldots, k \), we write \( w \in (u; v_1, \ldots, v_k) \), i.e., \( w \) is a point in the set \((u; v_1, \ldots, v_k)\).

3. The lemmas

It is convenient to develop six preliminary results before proving the main theorems.

**Lemma 1.** The cube \( Q_{2n+1} \) is in \( P_{1,n} \).

**Proof.** Each point \( v \) of \( Q_{2n+1} \) has degree \( 2n + 1 \). Let \( u_1, \ldots, u_{2n+1} \) be the points adjacent to \( v \). It follows at once from the definition of a cube that \( u_iA_{u_j} \), for \( i, j = 1, \ldots, 2n + 1 \). Also no point of \( Q_{2n+1} \), except \( u_i \), is adjacent to more than two of the \( u_i \). Thus for each set \( X \) of \( n \) points of this cube, there is a point \( u \) such that \( uAv \) and for all \( x \in X \), \( uA_{x} \), hence \( Q_{2n+1} \in P_{1,n} \). \( \square \)

We note in passing that \( Q_{2n} \notin P_{1,n} \).

**Lemma 2.** If \( G \in P_{1,n} \), then \( \delta(G) \geq n + 1 \).
Proof. Assume the contrary and suppose \( u \) is a point of \( G \) with \( \deg u = k \leq n \). Let \( v_1, \ldots, v_k \) be the points of link \((u)\). Then there is no point \( w \) such that \( w \in (u; v_1, \ldots, v_k) \). This contradiction proves the lemma.

Lemma 3. If \( G \in P_{1,n} \) and \( \deg u = n + 1 \), then \( u \) is on no 3- or 4-cycles.

Proof. Let \( v_1, \ldots, v_{n+1} \) be the points of link \((u)\). If \( u \) is on a 3-cycle then the other two points on this 3-cycle are two of the \( v_i \), say \( v_1 \) and \( v_2 \). But then there is no point \( w \) in \((u; v_2, \ldots, v_{n+1})\). So \( u \) is not on a 3-cycle. If \( u \) is on a 4-cycle, we can with no loss of generality suppose that \( v_1 \) and \( v_2 \) are on the 4-cycle as well. Let \( x \) be the point opposite \( u \) on the 4-cycle. Then there is no point \( w \) in \((u; x, v_3, \ldots, v_{n+1})\). Hence \( u \) is not on a 4-cycle.

Lemma 4. For any sequence \( u, v_1, \ldots, v_k \) in \( V(G) \), \(|(u; v_1, \ldots, v_k)| \geq n - k + 1\).

Proof. If \(|(u; v_1, \ldots, v_k)| \leq n - k \), let \( w_1, \ldots, w_m \) be all the points in \((u; v_1, \ldots, v_k)\), where \( m \leq n - k \). Then there is no point \( x \) in \((u; v_1, \ldots, v_k, w_1, \ldots, w_m)\), a contradiction.

Lemma 5. If \( G \in P_{1,n} \) and \( w \) is any point of \( G \), then \( G_w \in P_{1,n-1} \).

Proof. Let \( u, v_1, \ldots, v_{n-1} \) be any sequence of \( n \) points of \( G \). Then some point \( x \) of \( G \) is in \((u; w, v_1, \ldots, v_{n-1})\). But \( x \neq w \) and \( x^-w \), so \( x \in G_w \); hence \( G_w \in P_{1,n-1} \).

Lemma 6. If every graph in \( P_{1,n} \) has at least \( r \) points and if \( G \in P_{1,n} \) has order \( p \) and maximum degree \( \Delta \), then \( p \geq 1 + N + r \).

Proof. It is sufficient to show that for all points \( v \) of \( G \), \( p \geq 1 + \deg v + r \). But this follows at once from the facts that \(|V| = |\{v\}| + |N(v)| + |V(G_v)|\), and that \( G_v \in P_{1,m-1} \) by Lemma 5.

The girth \( g \) of a graph \( G \) (which is not a forest) is the smallest cycle length in \( G \).

Lemma 7. If \( G \in P_{1,n} \) and \( g = 5 \) and if \( G \) is not an \((n + 1)\)-regular graph of girth 5, then \( p \geq n^2 + 3n + 2 \).

Proof. If \( G \) is regular of degree \( n + 1 \), the result follows from a theorem in Biggs [1, p. 153], which states that if a \( k \)-regular graph has odd girth \( g \), then

\[
p \geq 1 + k + k(k-1) + \cdots + k(k-1)^{g-3} + k(k-1)^{g-3/2} \tag{1}
\]

and if it has even girth,

\[
p \geq 1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{g-2} + (k-1)^{g-2/2} \tag{2}
\]
If $G$ is not $(n + 1)$-regular, then by Lemma 2, $G$ contains a point of degree $d \geq n + 2$. If $v_1, \ldots, v_d$ are the points of $G$ adjacent to $u$, then

$$p \geq 1 + d + \sum_{i=1}^{d} |(u; v_i)| \geq 1 + (n + 2) + n(n + 2) > n^2 + 3n + 2,$$

as required. $\square$

**Lemma 8.** If $G$ has girth $g \geq 5$ and $\delta(G) \geq n + 1$, then $G \in P_{1,n}$.

**Proof.** For the purposes of this proof we shall write $uBv$ if $u \in N^*(v)$, i.e., $uAv$ or $u = v$. Let $u$ be any point of $G$ and let $v_1, \ldots, v_{n+1}$ be the points of link $u$. Then for any point $w \neq u$, $wBv_i$ for at most one value of $i$. Thus for any $w_1, \ldots, w_n$ we have $(u; w_1, \ldots, w_n) \neq \emptyset$. $\square$

4. Some solutions

We now proceed to indicate the smallest graphs in $P_{1,n}$, $n = 1, \ldots, 6$. One easily verifies that $C_5$ is the smallest graph in $P_{1,1}$. An $(m, n)$-cage is defined as a smallest $m$-regular graph of girth $n$. Note that $C_5$ is the $(2, 5)$-cage and in general $C_p$ is the $(2, p)$-cage.

![Petersen graph](image)

Fig. 1. The Petersen graph.

**Theorem 1.** The smallest graph in $P_{1,2}$ is the Petersen graph $P$ which has order 10 and is the $(3, 5)$-cage. Every other graph in $P_{1,2}$ has at least 12 points.

**Proof.** If $G \in P_{1,2}$ then $\delta(G) \geq 3$. Since $C_5$ is the smallest graph in $P_{1,1}$, Lemma 6 implies that $p \geq 1 + 3 + 5 = 9$. But if $p = 9$, then $G$ must be 3-regular, which is impossible. So $p \geq 10$ and $\delta(G) \geq 3$, which with Lemma 8 means that $P$ is the smallest graph in $P_{1,2}$.

To show that no graph in $P_{1,2}$ has 11 points, we first observe that if $G \in P_{1,2}$ and $p = 11$, then the degree set of $G$ is a subset of $\{3, 4, 5\}$, since for any point $v$, $\deg v \geq 3$ by Lemma 2 and $\deg v \leq 5$ by Lemma 6. Since not all points of an 11-point graph can have odd degree, there is a point $v$ of $G$ having degree 4. Let $u_1$ to $u$ be the points of link $v$. Now observe that the induced subgraph, link $v$,
contains at most one line, and that no point outside link v other than v is adjacent to more than two points of link v, or \( P_{1,2} \) will be violated. Let \( A \) be the set of points, not in the link, adjacent to just one point of link v; let \( B \) contain the points adjacent to exactly two of them.

If link v contains no lines, then either some point of link v is adjacent to every point of \( B \) or some point of link v is adjacent to no points of \( B \). To show that one of these must hold, suppose that no point of link v is adjacent to every point of \( B \). If \( B \) is empty, the second situation prevails, so let us say that \( x_0 \in B \) is adjacent to both \( u_1 \) and \( u_2 \). Then there exist points \( x_1 \) and \( x_2 \) in \( B \) with \( x_1 \bar{u}_1 \) and \( x_2 \bar{u}_2 \), or we would have a point in link v adjacent to every point of \( B \). If \( x_1 = x_2 \), then \( x_1Au_3 \) and \( x_1Au_4 \), which means \( P_{1,2} \) is violated. So we have \( x_1 \neq x_2 \) and neither is adjacent to both \( u_3 \) and \( u_4 \), or else one of the sets \((v; x_0, x_1)\) and \((v; x_0, x_2)\) is empty. So with no loss in generality we have \( x_0Au_1, u_2; x_1Au_1, u_3; \) and \( x_2Au_2, u_3 \). But then no point of \( B \) can be adjacent to \( u_4 \), and we have the second case. So one of the two cases must hold.

We now show that in either of these cases \( p \geq 12 \). In the first case let \( u_1 \) be the indicated point. Then for clarity we write

\[
p \geq |\{v\}| + \deg v + |\{u_1; v\}| + \sum_{i=2}^{4} |\{u_i; u_1\}| \geq 1 + 4 + 2 + 6 = 13.
\]

In the second case let \( u_4 \) be the indicated point: we have

\[
p \geq |\{v\}| + \deg v + |\{u_4; v\}| + |\{u_1; v\}| + |\{u_2; u_1\}| + |\{u_3; u_2; u_1\}| \geq 12.
\]

If link v has a line, then let \( u_1 \) and \( u_2 \) be the points on the line and let \( u_3 \) and \( u_4 \) be the other two points of link v. Without loss of generality we can suppose that only v is adjacent to both \( u_1 \) and \( u_3 \). For if there is another point adjacent to both \( u_1 \) and \( u_3 \), and another point adjacent to both \( u_2 \) and \( u_4 \) then \( P_{1,2} \) fails. So by symmetry we can suppose that no other point is adjacent to both \( u_1 \) and \( u_3 \). Also no point but v is adjacent to both \( u_3 \) and \( u_4 \). Then

\[
p \geq 1 + 4 + |\{u_1; v\}| + |\{u_2; u_1\}| + |\{u_3; u_2\}| + |\{u_4; u_1, u_2\}| \geq 12.
\]

And as \( P \) is the unique \((3,5)\)-cage, Lemma 3 can be used to show that no 10-point graph other than \( P \) is in \( P_{1,2} \). \( \square \)

**Corollary 1a.** The smallest graph in \( P_{2,1} \) is \( \bar{P} \), the complement of the Petersen graph.

\[
\text{Fig. 2. A 12-point graph in } P_{1,2}.
\]
Theorem 2. Robertson's graph, $R$, the $(4, 5)$-cage, is the smallest graph in $P_{1,3}$.

Proof. In Fig. 3 we show $R$, Robertson's graph, which has 19 points. We know $R \in P_{1,3}$ by Lemma 8. We will show that any other graph in $P_{1,3}$ has at least 20 points or has 19 points and more lines than $R$. (In fact the second case can be eliminated, but to do so is unnecessary here.)

From [6] we know that $R$ is smaller than any other 4-regular graph of girth 5. By Lemma 7, it is also smaller than any graph $G$ with girth 5 and $\Delta(G) > 4$, and is smaller than any graph $G$ with $\delta(G) \geq 4$ and $g \geq 6$. So we proceed to tackle the case $g < 5$.

We show that if $G \in P_{1,3}$ and has a point $u$ of degree 5, then $G$ is larger than $R$. Let

$$A = \{v : v \in V(G) \text{ and } d(u, v) = 2\},$$

i.e., $A$ consists of those points not in the closed neighborhood of $u$ which are adjacent to a point of link $u$. And we let

$$B = \{v : v \in A \text{ and } v \text{ is adjacent to at least 2 points of link } u\}.$$

Observe that link $u$ can contain at most one line and that no point of $B$ is adjacent to more than two points of link $u$, lest $P_{1,3}$ be violated. Thus there are two possibilities: in Case 1, some point $v_1$ of link $u$ is adjacent to all points of $B$ and is a point on every line in link $u$, whilst in Case 2, there are at least two isolated points in link $u$, $v_4$ and $v_5$, with neither adjacent to a point of $B$.

To prove that these two cases exhaust the possibilities, suppose neither is true. We say two points of link $u$ are matched if they are adjacent or if a point of $B$ is adjacent to both. Since Case 2 fails to hold, at least four points of link $u$ are matched. Without loss of generality, let $v_1$ be matched to $v_2$, and since Case 1 does not hold, suppose $v_1$ is not matched with $v_3$ and that $v_2$ is not matched with $v_4$. Thus since we do not have Case 2, either $v_1$ is matched with $v_4$, and $v_2$ with $v_3$, or else $v_1$ is matched with $v_2$, and $v_3$ with $v_4$. Either situation implies $G \notin P_{1,3}$. Thus either Case 1 or Case 2 holds as claimed.

In both cases, the points of link $u$ are denoted $v_1$ to $v_5$. In Case 1,

$$p \geq |u| + \deg u + \sum_{i=2}^{3} |(v_i; v_1)| + |(v_1; u)| \geq 1 + 5 + 4 \cdot 3 + 3 = 21.$$
Note the repeated use of Lemma 4 for bounding purposes.

Case 2 gives
\[ p \geq 1 + 5 + |(v_4; u)| + |(v_5; u)| + |(v_2; v_1)| + |(v_3; v_1, v_2)| \geq 20, \]
where we label the \( v_i \) so that \( v_2 \Delta v_3 \).

If there is a point \( u \) in \( G \) of degree \( d \geq 6 \) then either \( G_u \) is the Petersen graph \( P \), or \( p \geq 19 \) and \( G \) has more lines than \( R \). This follows from Theorem 1 and Lemmas 2 and 5. So suppose \( G_u = P \). The argument in the preceding paragraphs allows us to conclude that \( G \) has no point of degree 5. So we have \( G_u = P \) and \( \deg u = 6 \) or 7. Let \( w_1, \ldots, w_{10} \) denote the points of \( G_u \). Since \( \delta(G) \geq 4 \), if \( G \) is to be smaller than \( R \) then every point of \( G_u \) is adjacent to a point of link \( u \). And since for any \( v_i \in N(u) \), \( (v_i; u) \geq 2 \), we know that every point of link \( u \) is adjacent to at least two points of \( G_u \). So the fact that \( \text{diam } P = 2 \) implies that every point of \( G \) is on a 3- or 4-cycle. Since \( G \) has no points of degree 5, we infer from Lemma 3 that every point of \( G \) has degree 6 or 7. So every point of \( G_u \) is adjacent to at least three points of link \( u \). Thus there are at least 30 lines between \( G_u \) and link \( u \), so that two of the link \( u \) points, which we can call \( v_1 \) and \( v_2 \), are adjacent to a total of at least seven points of \( G_u \), or else \( P_{1,3} \) is violated. But given any seven points of \( P \), there is a point of \( P \) adjacent to three of the seven; let \( w_1 \) be such a point. Then \( (w_1; u, v_1, v_2) \) is empty, which means that \( G \neq P_{1,3} \). This contradiction shows that we cannot have \( G_u = P \) when \( p < 19 \), and eliminates all candidates for graphs in \( P_{1,3} \) smaller than \( R \). Hence \( R \) is the smallest graph in \( P_{1,3} \). \( \square \)

In order to obtain a general proof that an \((n + 1,5)\)-cage is a smallest graph in \( P_{1,n} \), the following conjecture would be useful.

**Conjecture.** If \( G \in P_{1,n} \) and has girth \( g < 5 \), then \( p \geq n^2 + 3n + 2 \).

We have been unable to devise a proof of this conjecture for all \( n \). We have, however, proved it for \( n \leq 6 \) by considering each of these values separately. The techniques used are very similar to those used in Theorems 1 and 2. Note that Theorem 1 establishes the conjecture for \( n = 2 \); verification for \( n = 1 \) is easy.

Of course proving that the smallest graph in \( P_{1,n} \) is a cage would probably be very difficult if no \((n + 1,5)\)-cage is known. However, the conjecture is motivated by the following assertion which is verified by Table 1.

### Table 1. The smallest 5-cages

<table>
<thead>
<tr>
<th>Degree</th>
<th>( p )</th>
<th>Discovered by</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>10</td>
<td>Petersen [3]</td>
</tr>
<tr>
<td>4</td>
<td>19</td>
<td>Robertson [6]</td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td>Wegner [7]</td>
</tr>
<tr>
<td>6</td>
<td>40</td>
<td>O'Keefe and Wong [5]</td>
</tr>
<tr>
<td>7</td>
<td>50</td>
<td>Hoffman and Singleton [4]</td>
</tr>
</tbody>
</table>
Observation. For each of the known \((n + 1, 5)\)-cages, \(p \leq n^2 + 3n + 2\).

So the conjecture might be useful for checking whether \((n + 1, 5)\)-cages which will be discovered in the future are also smallest graphs in \(P_{1,n}\).

The bound in the conjecture is sharp, at least for \(n = 1\), in that \(\tilde{C}_6 \in P_{1,1}\).

We remark that the irregularity of the numbers \(p\) in Table 1 is startling.

Unsolved Problems. What are the smallest graphs in \(P_{1,n}\) when \(n \geq 7\) and more generally what are the answers for \(P_{m,n}\) with \(m, n \geq 2\)?

References