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EXTENSION OF EIGENFUNCTION–EXPANSION SOLUTIONS OF A FOKKER–PLANCK EQUATION—II. SECOND ORDER SYSTEM

JAMES P. JOHNSON and RICHARD A. SCOTT
Department of Applied Mechanics and Engineering Science, The University of Michigan, Ann Arbor, MI, U.S.A.

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Abstract—In an earlier paper the authors presented results for eigenfunction–expansion solutions to the forward Fokker–Planck equation associated with a specific, non-linear, first-order system subject to white noise excitation. This work is concerned with eigenfunction–expansion solutions to the forward and backward Fokker–Planck equations associated with a specific, non-linear, second-order system subject to white noise excitation. Expansion terms through the fourth-order have been generated using a digital computer. Using this new information, inverted Domb–Sykes plots revealed a pattern in the coefficients for certain values of the parameters. Through this pattern, Dingle's theory of terminants was used to recast the series into a more favorable computational form.

1. INTRODUCTION

In his review article Caughey [1] outlined an eigenfunction expansion procedure for obtaining the response statistics of a weakly non-linear, second-order system undergoing white noise excitation. The present work builds on and extends Caughey's outline for a specific second-order, weakly non-linear system. This is achieved by means of tools developed elsewhere for the analysis and improvement of perturbation series (see Van Dyke [2]). Also, the theory of terminants developed by Dingle [3] is utilized.

The eigenfunction expansion procedure presents the response statistics as perturbation series. The spirit of this approach is to obtain a sufficient number of computer generated terms of the series that a pattern, if it exists, emerges for the series coefficients. This pattern is then used to recast the expansion into a more favorable computational form. Using this approach, new information is presented on the steady-state, mean square response of a specific second-order system to a white noise excitation. In an earlier paper [4], the authors gave results for a first-order system.

2. EIGENFUNCTION EXPANSION SOLUTIONS

The system considered is

$$\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + x + \varepsilon x^3 = n(t) \tag{1}$$

with initial conditions

$$\begin{aligned} x(0) &= x_0 \\ \dot{x}(0) &= \dot{x}_0 \end{aligned} \tag{2}$$

where $\dot{x} = dx/dt$.

Here ε is a small parameter and $n(t)$ is a white noise process with the properties:

(i) $n(t_i)$, $i = 1, 2, \ldots$ are mutually independent,

(ii) $n(t)$ has a Gaussian probability distribution with $E[n(t)] = 0$, $E[n(t)n(s)] = 2D\delta(t-s)$, E denoting expected value, δ being the delta function and D a constant which measures the white noise intensity.

The response x is modeled as a Markov process and the forward and backward Fokker–Planck equations associated with equation (1) are respectively

$$\frac{\partial p}{\partial t} = Lp \tag{3}$$

$$\frac{\partial p}{\partial t_0} = L^* p \tag{4}$$

where L and L^* represent the following spatial operators

$$L = -\dot{x}\frac{\partial}{\partial x} + \frac{\partial}{\partial \dot{x}}(\beta\dot{x} + x + \varepsilon x^3) + D\frac{\partial^2}{\partial \dot{x}^2} \tag{5}$$

$$L^* = \dot{x}_0\frac{\partial}{\partial x_0} - (\beta\dot{x}_0 + x_0 + \varepsilon x_0^3)\frac{\partial}{\partial \dot{x}_0} + D\frac{\partial^2}{\partial \dot{x}_0^2} \tag{6}$$

and the transition probability density $p(x, \dot{x}, t|x_0, \dot{x}_0)$ must satisfy

$$\lim_{t\to 0} p(x, \dot{x}, t|x_0, \dot{x}_0; \varepsilon) = \delta(x - x_0)\delta(\dot{x} - \dot{x}_0) \tag{7}$$

$$\int_{-\infty}^{\infty}\int p(x, \dot{x}, t|x_0, \dot{x}_0; \varepsilon)\mathrm{d}x\mathrm{d}\dot{x} = 1 \tag{8}$$

$$\lim_{t\to 0}\int_{-\infty}^{\infty}\int p(x, \dot{x}, t|x_0, \dot{x}_0; \varepsilon)\phi(x_0, \dot{x}_0)\mathrm{d}x_0\mathrm{d}\dot{x}_0 = \phi(x, \dot{x}) \tag{9}$$

for any continuous function $\phi(x_0, \dot{x}_0)$. The region of definition of $\{x, \dot{x}\}$ will be taken to be infinite and boundary conditions for the FPK equations are not necessary since an infinite time is required for any sample path of $\{x, \dot{x}\}$ to reach the boundary.
It can readily be shown that the steady-state solution of the FPK equation is

$$p_s(x, \dot{x}; \varepsilon) = \frac{\exp\left[-\frac{\beta}{2D}\left(\dot{x}^2 + x^2 + \varepsilon\frac{x^4}{2}\right)\right]}{\int_{-\infty}^{\infty}\int \exp\left[-\frac{\beta}{2D}\left(\dot{\eta}^2 + \eta^2 + \varepsilon\frac{\eta^4}{2}\right)\right]\mathrm{d}\eta\mathrm{d}\dot{\eta}}. \tag{10}$$

The non-steady state solution of the FPK equations is sought in the form

$$p(x, \dot{x}, t|x_0, \dot{x}_0; \varepsilon) = \sum_{i,j=1}^{\infty} c_{ij}u_{ij}(x, \dot{x}; \varepsilon)v_{ij}(x_0, \dot{x}_0; \varepsilon)T_{ij}(t; \varepsilon). \tag{11}$$

Substituting this expression into (3) and (4) gives

$$T_{ij}(t; \varepsilon) = \exp\left[-\lambda_{ij}(\varepsilon)t\right] \tag{12}$$

$$Lu_{ij}(x, \dot{x}; \varepsilon) + \lambda_{ij}(\varepsilon)u_{ij}(x, \dot{x}; \varepsilon) = 0 \tag{13}$$

$$L^* v_{ij}(x, \dot{x}; \varepsilon) + \lambda_{ij}(\varepsilon)v_{ij}(x, \dot{x}_0; \varepsilon) = 0. \tag{14}$$

Caughey [1] notes that (12) through (14) define an eigenvalue problem where u_{ij} and v_{ij} are the eigenfunctions of L and L^*, respectively, and λ_{ij} are the corresponding eigenvalues. Furthermore it can be shown that L and L^* are adjoint in that

$$\int_{-\infty}^{\infty}\int (u_{ij}Lv_{ij} - v_{ij}L^*u_{ij})\mathrm{d}x\mathrm{d}\dot{x} = 0$$

so that L and L^* have the same eigenvalues.

It is convenient to substitute for L and its eigenfunctions μ_{ij} the operator G and its eigenfunctions w_{ij}, where

$$Gw_{ij} = p_s^{-1}Lp_s w_{ij},\ w_{ij} = p_s^{-1}u_{ij}. \tag{15}$$

Then G and L^* will be adjoint operators with respect to the following inner product

$$(f, g) = \int_{-\infty}^{\infty}\int p_s(x, \dot{x}; \varepsilon) f(x, \dot{x}; \varepsilon) g(x, \dot{x}; \varepsilon) \mathrm{d}x\mathrm{d}\dot{x}. \tag{16}$$

By (5) and (15) we have

$$G = -\dot{x}\frac{\partial}{\partial x} - (\beta\dot{x} - x - \varepsilon x^3)\frac{\partial}{\partial \dot{x}} + D\frac{\partial^2}{\partial \dot{x}^2}. \tag{17}$$

Equations (6) and (17) show that L^* can be obtained from G by replacing $\dot{x}$ by $-\dot{x}$. Their eigenfunctions are then related by

$$w_{ij}(x, \dot{x}; \varepsilon) = v_{ij}(x, -\dot{x}; \varepsilon). \tag{18}$$

Assume the eigenfunctions form a complete bi-orthonormal set subject to the normalization conditions

$$\int_{-\infty}^{\infty}\int p_s(x, \dot{x}; \varepsilon) w_{ij}(x, \dot{x}; \varepsilon) v_{kl}(x, \dot{x}; \varepsilon) \mathrm{d}x\mathrm{d}\dot{x} = \delta_{ik}\delta_{jl} \tag{19a}$$

$$\int_{-\infty}^{\infty}\int p_s(x, \dot{x}; \varepsilon) v_{ij}(x, \dot{x}; \varepsilon) v_{ij}(x, \dot{x}; \varepsilon) \mathrm{d}x\,\mathrm{d}\dot{x} = 1 \tag{19b}$$

where δ_{kl} denotes the Kronecker delta.

When w_{ij} is substituted for u_{ij} the initial conditions and normalization conditions indicate that the non-steady-state solution can be written

$$p(x, \dot{x}, t | x_0, \dot{x}_0; \varepsilon) = p_s(x, \dot{x}; \varepsilon) \sum_{i,j=0}^{\infty} w_{ij}(x, \dot{x}; \varepsilon) \times v_{ij}(x_0, \dot{x}_0; \varepsilon) \exp\left[-\lambda_{ij}(\varepsilon)t\right] \tag{20}$$

where for the existence of $p_s(x, \dot{x}; \varepsilon)$ it is necessary that

$$\lambda_{00}(\varepsilon) = 0 \tag{21}$$

and

$$v_{00}(x, \dot{x}; \varepsilon) = w_{00}(x, \dot{x}; \varepsilon) = 1. \tag{22}$$

The spatial operators G and L^* are now written in the forms

$$G = G_0 + \varepsilon G_1 \tag{23}$$

$$L^* = L_0^* + \varepsilon L_1^* \tag{24}$$

where

$$G_0 = -\dot{x}\frac{\partial}{\partial x} - (\beta\dot{x} - x)\frac{\partial}{\partial \dot{x}} + D\frac{\partial^2}{\partial \dot{x}^2} \tag{25}$$

$$G_1 = x^3\frac{\partial}{\partial \dot{x}} \tag{26}$$

$$L_0^* = \dot{x}\frac{\partial}{\partial x} - (\beta\dot{x} + x)\frac{\partial}{\partial \dot{x}} + D\frac{\partial^2}{\partial \dot{x}^2} \tag{27}$$

$$L_1^* = -x^3\frac{\partial}{\partial \dot{x}}. \tag{28}$$

Then (13) and (14) can be written

$$(G_0 + \varepsilon G_1) w_{ij}(x, \dot{x}; \varepsilon) + \lambda_{ij}(\varepsilon) w_{ij}(x, \dot{x}; \varepsilon) = 0 \tag{29}$$

$$(L_0^* + \varepsilon L_1^*) v_{ij}(x, \dot{x}; \varepsilon) + \lambda_{ij}(\varepsilon) v_{ij}(x, \dot{x}; \varepsilon) = 0. \tag{30}$$

Now expand $w_{ij}(x, \dot{x}; \varepsilon)$, $v_{ij}(x, \dot{x}; \varepsilon)$ and $\lambda_{ij}(\varepsilon)$ in the forms

$$w_{ij}(x, \dot{x}; \varepsilon) = \sum_{k=0}^{\infty} w_{ijk}(x, \dot{x})\varepsilon^k \tag{31}$$

$$v_{ij}(x, \dot{x}; \varepsilon) = \sum_{k=0}^{\infty} v_{ijk}(x, \dot{x})\varepsilon^k \tag{32}$$

$$\lambda_{ij}(\varepsilon) = \sum_{k=0}^{\infty} \lambda_{ijk}\varepsilon^k \qquad i, j = 1, 2, \ldots . \tag{33}$$

Using (31) and (33) in (29) and equating the coefficients of like powers of ε to zero results in

$$G_0 w_{ijN} + G_1 w_{ij(N-1)} + \sum_{k,l=0}^{\infty} \delta(N-k-l)\lambda_{ijk} w_{ijl} = 0 \qquad N = 0, 1, 2, \ldots$$

where a negative subscript indicates a zero value of the entire subscripted quantity. By using the properties of the delta function and rearranging we obtain

$$G_0 w_{ij0} + \lambda_{ij0} w_{ij0} = 0 \tag{34a}$$

$$G_0 w_{ijN} + \lambda_{ij0} w_{ijN} = -G_1 w_{ij(N-1)} - \sum_{k=1}^{N} \lambda_{ijk} w_{ij(N-k)}. \tag{34b}$$

Likewise using (32) and (33) in (30) results in

$$L_0^* v_{ij0} + \lambda_{ij0} v_{ij0} = 0 \tag{35a}$$

$$L_0^* v_{ijN} + \lambda_{ij0} v_{ijN} = -L_1^* v_{ij(N-1)} - \sum_{k=1}^{N} \lambda_{ijk} v_{ij(N-k)}. \tag{35b}$$

Equations (34a) and (35a) are the equations for the eigensolutions of the system for $\varepsilon = 0$. Assume that the eigenvalues of (34a) and (35a) are discrete and distinct. Then from (2)

$$\int_{-\infty}^{\infty}\int p_s(x, \dot{x}) w_{ij0}(x, \dot{x}) v_{kl0}(x, \dot{x}) \mathrm{d}x\mathrm{d}\dot{x} = \delta_{ik}\delta_{jl} \tag{36}$$

$$\int_{-\infty}^{\infty}\int p_s(x, \dot{x}) v_{ij0}(x, \dot{x}) G_0 w_{kl0}(x, \dot{x}) \mathrm{d}x\mathrm{d}\dot{x} = -\lambda_{ij0}\delta_{ik}\delta_{jl} \tag{37}$$

$$\int_{-\infty}^{\infty}\int p_s(x, \dot{x}) w_{ij0}(x, \dot{x}) L_0^* v_{kl0}(x, \dot{x}) \mathrm{d}\dot{x} = -\lambda_{ij0}\delta_{ik}\delta_{jl}. \tag{38}$$

The eigensolutions of the system for $\varepsilon = 0$ are those corresponding to the two-dimensional Hermite equation. By using the change of variable

$$\zeta = x/\sqrt{D} \tag{40}$$

and by defining

$$\xi = \beta/2 \tag{41}$$

$$\nu = \sqrt{\xi^2 - 1} \tag{42}$$

$$\mu_1 = \xi + \nu \tag{43}$$

$$\mu_2 = \xi - \nu. \tag{44}$$

Atkinson [5] has demonstrated that the eigensolutions of (35a) are

$$\lambda_{ij0} = \mu_1 i + \mu_2 j \tag{45}$$

and

$$v_{ij0}(\zeta, \dot{\zeta}) = C_{ij} G_{ij}(\zeta_1, \zeta_2) \tag{46}$$

where

$$C_{ij}=\frac{(-1)^{i/2}}{(i!j!)^{1/2}}\left(\frac{\nu}{\xi}\right)^{1/2} \tag{47}$$

and G_{ij} is the Hermite polynomial

$$G_{ij}(\zeta_1, \zeta_2)=(-1)^{i+j}\exp(\phi/2)\frac{\partial^{i+j}}{\partial\eta_1^i\partial\eta_2^j}[\exp(-\phi/2)] \tag{48}$$

where

$$\phi=2\xi(\zeta^2+\dot{\zeta}^2) \tag{49}$$

and the coordinates (ζ_1, ζ_2) and $\eta_1, \eta_2)$ are related to $(\zeta, \dot{\zeta})$ via the transformations

$$\begin{Bmatrix}\zeta_1\\ \zeta_2\end{Bmatrix}=\sqrt{\frac{\xi}{\nu}}\begin{bmatrix}\mu_2^{1/2}\mu_1^{1/2}\\ \mu_1^{1/2}\mu_2^{1/2}\end{bmatrix}\begin{Bmatrix}\zeta\\ \dot{\zeta}\end{Bmatrix} \tag{50}$$

and

$$\begin{Bmatrix}\eta_1\\ \eta_2\end{Bmatrix}=\sqrt{\frac{\xi}{\nu}}\begin{bmatrix}-\mu_2^{1/2} & \mu_1^{1/2}\\ \mu_1^{1/2} & -\mu_2^{1/2}\end{bmatrix}\begin{Bmatrix}\zeta\\ \dot{\zeta}\end{Bmatrix}. \tag{51}$$

The second eigenfunction, that for (34a), can be expressed in terms of the other of the Hermite polynomials by using (18), (50) and (51). It is

$$w_{ij0}(\zeta, \dot{\zeta})=(-1)^i C_{ij}H_{ij}(\zeta_1, \zeta_2) \tag{52}$$

where

$$H_{ij}(\zeta_1, \zeta_2=(-1)^{i+j}\exp(\phi/2)\frac{\partial^{i+j}}{\partial\zeta_1^i\partial\zeta_2^j}[\exp(-\phi/2)]. \tag{53}$$

The eigenfunctions for the $\varepsilon=0$ system form a complete, bi-orthonormal set of functions. When $\varepsilon\neq 0$, but small, one may think of the process as perturbing the $\varepsilon=0$ eigensolutions. Within this conceptual framework the following expansions are set forth (see Courant and Hilbert [6]):

$$w_{ijk}(x, \dot{x})=\sum_{l,m=0}^{\infty} a_{ijklm}w_{lm0}(x, \dot{x}) \tag{54}$$

and

$$v_{ijk}(x, \dot{x})=\sum_{l,m=0}^{\infty} b_{ijklm}v_{lm0}(x, \dot{x}) \tag{55}$$

Two requirements on a_{ijklm} and b_{ijklm} present themselves immediately. First, (54) and (55) lead to

$$a_{ij0lm}=b_{ij0lm}=\delta_{il}\delta_{jm}. \tag{56}$$

Second, the existence of $p_s(x, \dot{x}; \varepsilon)$ requires $w_{00}\equiv v_{00}\equiv 1$. By using (22), (31), (32), (54), (55) and (56) we have

$$a_{00klm}=b_{00klm}=0 \qquad k\geqslant 1. \tag{57}$$

A recursion relation for the coefficient a_{ijklm} can be developed by using the expansion (54) in (34b). If each term in the resulting equation is multiplied by $v_{rs0}p_s$, integrated over the entire space and summation is interchanged with integration then (36) and (37) can be used to give

$$-\sum_{l,m=0}^{\infty} a_{ijNlm}\lambda_{rs0}\delta_{rl}\delta_{sm}+\sum_{l,m=0}^{\infty} a_{ijNlm}\lambda_{ij0}\delta_{rl}\delta_{sm}$$

$$=-\sum_{l,m=0}^{\infty} a_{ij(N-1)lm}(v_{rs0}, G_1 w_{lm0})-\sum_{l,m=0}^{\infty}\sum_{k=1}^{N} a_{ij(N-k)lm}\lambda_{ijk}\delta_{rl}\delta_{lm} \tag{58}$$

where $(-, -)$ is the inner product defined as

$$(u, v)=\int\int_{-\infty}^{\infty} p_s(x, \dot{x})u(x, \dot{x})v(x, \dot{x})\mathrm{d}x\mathrm{d}\dot{x}. \tag{59}$$

By using properties of the delta function, (58) can be written as

$$a_{ijNrs}(\lambda_{ij0}-\lambda_{rs0})=-\sum_{l,m=0}^{\infty} a_{ij(N-1)lm}(v_{rs0}, G_1 w_{lm0})-\sum_{k=1}^{N} a_{ij(N-k)lm}\lambda_{ijk}. \tag{60}$$

Performing similar operations on (35b) gives

$$b_{ijNrs}(\lambda_{ij0}-\lambda_{rs0})=-\sum_{l,m=0}^{\infty} b_{ij(N-1)lm}(w_{rs0}, L_1^* v_{lm0})-\sum_{k=1}^{N} b_{ij(N-k)rs}\lambda_{ijk}. \tag{61}$$

By requiring $i\neq r$ and $j\neq s$ in (60) and (61) and then allowing $i=r$ and $j=s$ in these equations, the following recursion relations result:

$$a_{ijNrs}=\frac{1}{\lambda_{rs0}-\lambda_{ij0}}\left[\sum_{l,m=0}^{\infty} a_{ij(N-1)lm}(v_{rs0}, G_1 w_{lm0})+\sum_{k=1}^{N} a_{ij(N-k)rs}\lambda_{ijk}\right], \quad i\neq r, j\neq s \qquad N=1, 2,\ldots \tag{62}$$

$$b_{ijNrs}=\frac{1}{\lambda_{rs0}-\lambda_{ij0}}\left[\sum_{l,m=0}^{\infty} b_{ij(N-1)lm}(w_{rs0}, L_1^* v_{lm0})+\sum_{k=1}^{N} b_{ij(N-k)rs}\lambda_{ijk}\right], \quad i\neq r, j\neq s \qquad N=1, 2, \ldots \tag{63}$$

$$\begin{aligned}\lambda_{ijN}=&-\tfrac{1}{2}\sum_{l,m=0}^{\infty}\left[a_{ij(N-1)lm}(v_{ij0}, G_1 w_{lm0})+b_{ij(N-1)lm}(w_{ij0}, L_1^* v_{lm0})\right]\\&-\tfrac{1}{2}\sum_{k=0}^{N-1}\left[a_{ij(N-k)ij}+b_{ij(N-k)ij}\right]\lambda_{ijk} \qquad N=1, 2, \ldots.\end{aligned} \tag{64}$$

The recursion relations for a_{ijNij} and b_{ijNij} must now be derived. Begin by applying (40) and (41) to the normalization conditions (19) and to the steady state density (10) to get

$$D\int\int_{-\infty}^{\infty} p_s(\zeta, \dot{\zeta}; \varepsilon)w_{ij}(\zeta, \dot{\zeta}; \varepsilon)v_{kl}(\zeta, \dot{\zeta}; \varepsilon)\mathrm{d}\zeta\mathrm{d}\dot{\zeta}=\delta_{ik}\delta_{jl} \tag{65}$$

$$D\int\int_{-\infty}^{\infty} p_s(\zeta, \dot{\zeta}; \varepsilon)v_{ij}(\zeta, \dot{\zeta}; \varepsilon)v_{ij}(\zeta, \dot{\zeta}; \varepsilon)\mathrm{d}\zeta\mathrm{d}\dot{\zeta}=1 \tag{66}$$

$$p_s(\zeta, \dot{\zeta}; \varepsilon)=\frac{\exp\left[-\xi(\dot{\zeta}^2+\zeta^2+\varepsilon D\zeta^4/2)\right]}{D\int\int_{-\infty}^{\infty}\exp\left[-\xi(\dot{\eta}^2+\eta^2+\varepsilon D\eta^4/2)\right]\mathrm{d}\eta\mathrm{d}\dot{\eta}}. \tag{67}$$

$\int_{-\infty}^{\infty}\exp\left[-ax^2\right]\mathrm{d}x=\sqrt{(\pi/a)}$ to get

$$p_s(\zeta, \dot{\zeta}; \varepsilon)=\frac{\exp\left[-\xi(\dot{\zeta}^2+\zeta^2)\right]\Sigma_{n=0}^{\infty}(-\xi D\zeta^4/2)^n(\varepsilon^n/n!)}{D\sqrt{(\pi/\xi)}\Sigma_{m=0}^{\infty}(-\xi D/2)^m(\varepsilon^m/m!)\int_{-\infty}^{\infty}\exp\left[-\xi\eta^2\right]\eta^{4m}\mathrm{d}\eta}. \tag{68}$$

Evaluate the integral in the denominator and perform the indicated division of series to get

$$p_s(\zeta, \dot{\zeta}; \varepsilon)=p_s(\zeta, \dot{\zeta})\sum_{n,m=0}^{\infty}\left(\frac{-D}{2}\right)^{n+m}\frac{\xi^{n-m}}{n!}B_m\zeta^{4n}\varepsilon^{n+m} \tag{69}$$

where B_m is defined by

$$B_0 = 1$$

$$B_m = -\sum_{i=1}^{m} A_i B_{m-i} \qquad m \geqslant 1 \tag{70}$$

and

$$A_i = \begin{cases} 1 & i = 0 \\ 1 \cdot 3 \cdot 5 \cdots (4i-1)/4^i i! & i \neq 0 \end{cases} \tag{71}$$

and

$$p_s(\zeta, \dot{\zeta}) = p_s(\zeta, \dot{\zeta}; 0).$$

Consider (65). Use (31), (32), (54), (55), (59) and (69) in (65) to get

$$D \sum_{n,m=0}^{\infty} \left(-\frac{D}{2}\right)^{n+m} \frac{\xi^{n-m}}{n!} B_m \sum_{p,r,s,q,t,u=0}^{\infty} a_{ijprs} b_{klqtu} (\zeta^{4n} w_{rs0}, v_{tu0}) \varepsilon^{n+m+p+q} = \delta_{ik}\delta_{jl}. \tag{72}$$

Let $i = k$ and $j = l$. Group according to powers of ε and set the coefficient of ε^0 equal to unity and the coefficients of all higher powers of ε equal to zero. It can be shown by (56) and (59) that the coefficient of ε^0 equals unity. Consider coefficients of ε^N, $N \geqslant 1$. Equation (72) becomes

$$D \sum_{n,m,p,q=0}^{N} \delta(N-n-m-p-q) \left(-\frac{D}{2}\right)^{n+m} \frac{\xi^{n-m}}{n!} B_m \times \sum_{r,s,t,u=0}^{\infty} a_{ijprs} b_{ijqtu} (\zeta^{4n} w_{rs0}, v_{tu0}) = 0 \qquad N \geqslant 1. \tag{73}$$

Performing similar operations on (66) results in

$$D \sum_{n,m,p,q=0}^{N} \delta(N-n-m-p-q) \left(-\frac{D}{2}\right)^{n+m} \frac{\xi^{n-m}}{n!} B_m \times \sum_{r,s,t,u=0}^{\infty} b_{ijprs} b_{ijqtu} (\zeta^{4n} v_{rs0}, v_{tu0}) = 0 \qquad N \geqslant 1. \tag{74}$$

By systematically isolating a_{ijNij} and b_{ijNij} in (73) and (74) and using (56), (59) and (70), the definition of the delta function, and orthogonality properties of Hermite polynomials the recursion relations for a_{ijNij} and b_{ijNij} can be obtained. The recursion relations are quite lengthy due to the multiple series in (73) and (74). For convenience they are contained in the Appendix.

The recursion relations for the eigenvalues and eigenfunctions can be used once the inner product expressions have been evaluated. Before doing this an expression for the mean square response of the system will be developed. The response is assumed to be a stationary Markov process. Its autocorrelation function is

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} \int\int\int x_{t+\tau} x_t p(x_{t+\tau}, \dot{x}_{t+\tau}, t+\tau; x_t, \dot{x}_t, t) \mathrm{d}x_t \mathrm{d}\dot{x}_t \mathrm{d}x_{t+\tau} \mathrm{d}\dot{x}_{t+\tau}. \tag{75}$$

Introduce (20) and (40) into (75) and use the Markov property of the response to get

$$R_{xx}(\tau) = D^3 \int_{-\infty}^{\infty} \int\int\int \zeta_{t+\tau} \zeta_t [p_s(\zeta_{t+\tau}, \dot{\zeta}_{t+\tau}; \varepsilon) \times \sum_{i,j=0}^{\infty} w_{ij}(\zeta_{t+\tau}, \dot{\zeta}_{t+\tau}; \varepsilon) v_{ij}(\zeta_t, \dot{\zeta}_t; \varepsilon) \exp(-\lambda_{ij}(\varepsilon)\tau)] \times [p_s(\zeta_t, \dot{\zeta}_t; \varepsilon) \sum_{k,l=0}^{\infty} w_{kl}(\zeta_t, \dot{\zeta}_t; \varepsilon) v_{kl}(\zeta_0, \dot{\zeta}_0; \varepsilon) \exp(-\lambda_{kl}(\varepsilon)t)] \mathrm{d}\zeta_t \mathrm{d}\dot{\zeta}_t \mathrm{d}\zeta_{t+\tau} \mathrm{d}\dot{\zeta}_{t+\tau}. \tag{76}$$

Since the response is stationary $R_{xx}(\tau)=\lim_{t\to\infty} R_{xx}(\tau)$, a fact that affects only the second $[-]$ term in (76) which reduces to $p_s(\zeta_t, \zeta_t; \varepsilon)$ by using (22) and (45). The autocorrelation can now be written as

$$R_{xx}(\tau)=D^3\sum_{i,j=0}^{\infty}\alpha_{ij}(\varepsilon)\beta_{ij}(\varepsilon)\exp\left[-\lambda_{ij}(\varepsilon)\tau\right] \tag{77}$$

where

$$\alpha_{ij}(\varepsilon)=\int\int_{-\infty}^{\infty} p_s(\zeta_t, \dot{\zeta}_t; \varepsilon)\zeta_t w_{ij}(\zeta_t, \dot{\zeta}_t; \varepsilon)\mathrm{d}\zeta_t\mathrm{d}\dot{\zeta}_t \tag{78}$$

and

$$\beta_{ij}(\varepsilon)=\int\int_{-\infty}^{\infty} p_s(\zeta_t, \dot{\zeta}_t; \varepsilon)\zeta_t v_{ij}(\zeta_t, \dot{\zeta}_t; \varepsilon)\mathrm{d}\zeta_t\mathrm{d}\dot{\zeta}_t. \tag{79}$$

Therefore the mean square response is

$$R_{xx}(0)=D^3\sum_{i,j=0}^{\infty}\alpha_{ij}(\varepsilon)\beta_{ij}(\varepsilon). \tag{80}$$

Consider α_{ij} and β_{ij}. From (46), (47), (48), (52) and (53) it can be shown that $v_{000}=w_{000}=1$. Using this along with (31), (32), (54), (55), (59), (69), (78) and (79) become

$$\alpha_{ij}(\varepsilon)=\sum_{n,m=0}^{\infty}\left(-\frac{D}{2}\right)^{n+m}\frac{\xi^{n-m}}{n!}B_m\sum_{k,r,s=0}^{\infty}a_{ijkrs}(\zeta^{4n+1}w_{rs0}, v_{000})\varepsilon^{n+m+k} \tag{81}$$

$$\beta_{ij}(\varepsilon)=\sum_{n,m=0}^{\infty}\left(-\frac{D}{2}\right)^{n+m}\frac{\xi^{n-m}}{n!}B_m\sum_{k,r,s=0}^{\infty}b_{ijkrs}(\zeta^{4n+1}w_{000}, v_{rs0})\varepsilon^{n+m+k}. \tag{82}$$

By using (81) and (82) $R_{xx}(0)$ can be written in the convenient form

$$R_{xx}(0)=D^3\sum_{N,i,j=0}^{\infty}A_{Nij}\varepsilon^N \tag{83}$$

where A_{Nij} are defined by

$$A_{Nij}=\sum_{n,m,p,q,k,l=0}^{N}\delta(N-n-m-p-q-k-l)\left(-\frac{D}{2}\right)^{n+m+p+q}$$

$$\times\frac{\xi^{n+p-m-q}}{n!p!}B_mB_q\sum_{r,s,t,u=0}^{\infty}a_{ijkrs}b_{ijltu}(\zeta^{4n+1}w_{rs0}, v_{000})(\zeta^{4p+1}w_{000}, v_{tu0}). \tag{84}$$

The recursion relations involve inner products of the forms

$$(\zeta^n w_{ij0}, v_{kl0}),\ (\zeta^n v_{ij0}, v_{kl0}),\ (v_{ij0}, G_1 w_{kl0}) \quad \text{and} \quad (w_{ij0}, L_1^* v_{kl0}).$$

It will be demonstrated that the inner products involving G_1 and L_1^* reduce to the general form $(\zeta^n w_{ij0}, v_{kl0})$. Equation (59) defines the inner product. By using (26), (40) and then (46) and (52) with (59) we get

$$(v_{ij0}, G_1 w_{kl0})=D^2(-1)^k C_{ij}C_{kl}\int\int_{-\infty}^{\infty} p_s(\zeta, \dot{\zeta})G_{ij}(\zeta_1, \zeta_2)\zeta^3\frac{\partial}{\partial\dot{\zeta}}H_{kl}(\zeta_1, \zeta_2)\mathrm{d}\zeta\mathrm{d}\dot{\zeta}. \tag{85}$$

Equations (28), (40) and then (46) and (52) with (59) give

$$(w_{ij0}, L_1^* v_{kl0})=-D^2(-1)^i C_{ij}C_{kl}\int\int_{-\infty}^{\infty} p_s(\zeta, \dot{\zeta})H_{ij}(\zeta_1, \zeta_2)\zeta^3\frac{\partial}{\partial\dot{\zeta}}G_{kl}(\zeta_1, \zeta_2)\mathrm{d}\zeta\mathrm{d}\dot{\zeta} \tag{86}$$

where C_{ij} are given by (47) and ζ, $\dot{\zeta}$ and ζ_1, ζ_2 are related by (50).

Using (50) the partial derivatives in (85) and (86) are

$$\frac{\partial}{\partial\xi} H_{kl}(\zeta_1, \zeta_2)=\left(\frac{\xi}{\nu}\right)^{1/2}\left(\mu_1^{1/2}\frac{\partial}{\partial\zeta_1} H_{kl}(\zeta_1, \zeta_2)+\mu_2^{1/2}\frac{\partial}{\partial\zeta_2} H_{kl}(\zeta_1, \zeta_2)\right) \tag{87}$$

and

$$\frac{\partial}{\partial\xi} G_{kl}(\zeta_1, \zeta_2)=\left(\frac{\xi}{\nu}\right)^{1/2}\left(\mu_1^{1/2}\frac{\partial}{\partial\zeta_1} G_{kl}(\zeta_1, \zeta_2)+\mu_2^{1/2}\frac{\partial}{\partial\zeta_2} G_{kl}(\zeta_1, \zeta_2)\right). \tag{88}$$

From Appell [7] we have

$$\frac{\partial}{\partial\zeta_1} G_{kl}(\zeta_1, \zeta_2)=kG_{(k-1)l}(\zeta_1, \zeta_2) \tag{89}$$

$$\frac{\partial}{\partial\zeta_2} G_{kl}(\zeta_1, \zeta_2)=lG_{k(l-1)}(\zeta_1, \zeta_2) \tag{90}$$

$$\frac{\partial}{\partial\zeta_1} H_{kl}(\zeta_1, \zeta_2)=\frac{\xi}{\nu}kH_{(k-1)l}(\zeta_1, \zeta_2)-\frac{1}{\nu}lH_{k(l-1)}(\zeta_1, \zeta_2) \tag{91}$$

$$\frac{\partial}{\partial\zeta_2} H_{kl}(\zeta_1, \zeta_2)=-\frac{1}{\nu}kH_{(k-1)l}(\zeta_1, \zeta_2)+\frac{\xi}{\nu}lH_{k(l-1)}(\zeta_1, \zeta_2). \tag{92}$$

Finally, using (87) through (92) in (85) and (86) and then (46), (52) and (59) we get

$$(v_{ij0}, G_1 w_{kl0})=D^2\frac{C_{kl}}{\nu}\left(\frac{\xi}{\nu}\right)^{1/2}\left[\frac{-k}{C_{(k-1)l}}(\mu_1^{1/2}\xi-\mu_2^{1/2})(\zeta^3 v_{ij0}, w_{(k-1)l0})\right.$$
$$\left.+\frac{l}{C_{k(l-1)}}(\mu_2^{1/2}\xi-\mu_1^{1/2})(\zeta^3 v_{ij0}, w_{k(l-1)0})\right] \tag{93}$$

and

$$(w_{ij0}, L_1^* v_{kl0})=-D^3 C_{kl}\left(\frac{\xi}{\nu}\right)^{1/2}\left[\frac{k}{C_{(k-1)l}}\mu_1^{1/2}(\zeta^3 w_{ij0}, v_{(k-1)l0})+\frac{l}{C_{k(l-1)}}\mu_2^{1/2}(\zeta^3 w_{ij0}, v_{k(l-1)0})\right]. \tag{94}$$

Thus inner product evaluation reduces to developing expressions for $(\zeta^n w_{ij0}, v_{kl0})$ and $(\zeta^n v_{ij0}, v_{kl0})$. The development for $(\zeta^n w_{ij0}, v_{kl0})$ will be outlined.

Set $\varepsilon=0$ in (67) to get

$$p_s(\zeta, \dot\zeta)=\frac{\xi}{\pi D}\exp\left[-\xi(\zeta^2+\dot\zeta^2)\right]. \tag{95}$$

Equations (46), (52), (59) and (95) combine to give

$$(\zeta^n w_{ij0}, v_{kl0})=(-1)^i C_{ij}C_{kl}\frac{\xi}{\pi D}\int\int_{-\infty}^{\infty}\exp-\xi(\zeta^2+\dot\zeta^2)\zeta^n H_{ij}(\zeta_1, \zeta_2)G_{kl}(\zeta_1, \zeta_2)\mathrm{d}\zeta\mathrm{d}\dot\zeta. \tag{96}$$

From Appell [7] we have the relationships

$$G_{kl}(\zeta_1, \zeta_2)=(\xi/\nu)^{(k+l)/2}\sum_{r=0}^{\min(k,l)}\frac{(-1)^r(-k, r)(-l, r)}{\xi^r r!}H_{k-r}\left(\sqrt{\frac{\nu}{\xi}}\zeta_1\right)H_{l-r}\left(\sqrt{\frac{\nu}{\xi}}\zeta_2\right) \tag{97}$$

and

$$H_{ij}(\zeta_1, \zeta_2)=(\xi/\nu)^{(i+j)/2}\sum_{s=0}^{\min(i,j)}\frac{(-i, s)(-j, s)}{\xi^s s!}H_{i-s}\left(\sqrt{\frac{\nu}{\xi}}\eta_1\right)H_{j-s}\left(\sqrt{\frac{\nu}{\xi}}\eta_2\right) \tag{98}$$

where $H_n(\cdot)$ is the nth order Hermite polynomial in one variable, ζ_1, ζ_2 and η_1, η_2 are defined

by (50) and (51), respectively, and (n, m) is Pochhammer's symbol, defined by

$$(n, m)=\frac{\Gamma(n+m)}{\Gamma(n)}=n(n+1)(n+2)\ldots(n+m-1) \tag{99}$$

$(n, 0)=1$ and $(1, m)=m!$, $\Gamma(\cdot)$ being the gamma function.

We also have Appell's general expression for the Hermite polynomial

$$H_n(z)=\sum_{m=0}^{m\leq n/2}\frac{(-1)^m(-n, 2m)}{2^m m!}z^{n-2m}. \tag{100}$$

Define the function $E(n)$ as

$$E(n)=\int_{-\infty}^{\infty}\exp(-\xi x^2)x^n dx=\begin{cases}0, & n \text{ odd}\\ \sqrt{(\pi/\xi)}, & n=0\\ \dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2^{n/2}\xi^{(n+1)/2}}\sqrt{\pi}, & n \text{ even}\end{cases} \tag{101}$$

Use (50) and (51) in (96) to make the integral uniform with respect to variables of integration; then use (97), (98), (100) and (101) and the binomial theorem to give

$$(\zeta^n w_{ij0}, v_{kl0})=(-1)^i C_{ij}C_{kl}\frac{\xi}{\pi D}\left(\frac{\xi}{v}\right)^{(i+j+k+l)/2}\sum_{s=0}^{\min(i,j)}\sum_{r=0}^{\min(k,l)}$$

$$\times\frac{(-1)^r(-i, s)(-j, s)(-k, r)(-l, r)}{\xi^{r+s}r!s!}\sum_{p=0}^{p\leq(i-s)/2}\sum_{q=0}^{q\leq(j-s)/2}\sum_{t=0}^{t\leq(k-r)/2}\sum_{u=0}^{u\leq(l-r)/2}$$

$$\times\frac{(-i+s, 2p)(-j+s, 2q)(-k+r, 2t)(-l+r, 2u)(i-s-2p)!(j-s-2q)!(k-r-2t)!}{(-2)^{p+q+t+u}p!q!t!u!}$$

$$\times(l-r-2u)!\sum_{\alpha=0}^{i-s-2p}\sum_{\beta=0}^{j-s-2q}\sum_{\sigma=0}^{k-r-2t}\sum_{\rho=0}^{l-r-2u}$$

$$\times\frac{(-1)^{i-s-2p-\alpha+\beta}\mu_1^{\frac{\alpha+j-s-2q-\beta+\sigma+l-r-2u-\rho}{2}}\mu_2^{\frac{i-s-2p-\alpha+\beta+k-r-2t-\sigma+\rho}{2}}}{\alpha!\beta!\sigma!\rho!(i-s-2p-\alpha)!(j-s-2q-\beta)!(k-r-2t-\sigma)!(l-r-2\mu-\rho)!}$$

$$\times E(\alpha+\beta+\sigma+\rho)E(n+i+j+k+l-2(s+r+p+q+t+u)-\alpha-\beta-\sigma-\rho). \tag{102}$$

A similar development for $(\zeta^n v_{ij0}, v_{kl0})$ results in

$$(\zeta^n v_{ij0}, v_{kl0})=C_{ij}C_{kl}\frac{\xi}{\pi D}\left(\frac{\xi}{v}\right)^{(i+j+k+l)/2}\sum_{r=0}^{\min(i,j)}\sum_{s=0}^{\min(k,l)}$$

$$\times\frac{(-1)^{r+s}(-i, r)(-j, r)(-k, s)(-l, s)}{\xi^{r+s}r!s!}\sum_{p=0}^{p\leq(i-r)/2}\sum_{q=0}^{q\leq(j-r)/2}\sum_{t=0}^{t\leq(k-s)/2}\sum_{u=0}^{u\leq(l-s)/2}$$

$$\times\frac{(-i+r, 2p)(-j+r, 2q)(-k+s, 2t)(-l+s, 2u)(i-r-2p)!(j-r-2q)!(k-s-2t)!(l-s-2u)!}{(-2)^{p+q+t+u}p!q!t!u!}$$

$$\times\sum_{\alpha=0}^{i-r-2p}\sum_{\beta=0}^{j-r-2q}\sum_{\sigma=0}^{k-j-2t}\sum_{\rho=0}^{l-s-2u}$$

$$\times\mu_1^{\frac{\alpha-\beta+\sigma-\rho+j+l-r-s-2q-2u}{2}}\mu_2^{\frac{-\alpha+\beta-\sigma+\rho+i+k-r-s-2p-2t}{2}}$$

$$\times E(\alpha+\beta+\sigma+\rho)E(n+i+j+k+l-2(r+s+p+q+t+u)-\alpha-\beta-\sigma-\rho). \tag{103}$$

By means of (102) and (103), the recursion relations for the eigenvalues and eigenfunctions and the perturbation expansion for the mean square response can be calculated to desired order using a digital computer.

3. COMPUTER EXTENSION OF PERTURBATION SOLUTIONS

The recursion relations to obtain the eigenvalues and eigenfunction expansion coefficients and the formulas for the perturbation expansion of the mean square response were programmed on the Ford Motor Company Honeywell 6000 computer. The programming involved considerable effort, and the authors would be pleased to supply interested readers with details on program listings and organization.

Because of the large number of multiple power series in these expansions the computation time and cost of very high order perturbation expansions were prohibitively large. Fourth-order expansions in the perturbation quantity ε were carried out for white noise excitations of intensities $D=0.001$ and 5.0. The value of damping factor ξ was taken to be 2.0. This new information on the coefficients of the perturbation expansion for the mean square response is given in Table 1.

Table 1. Mean square response perturbation coefficients for two values of the intensity D and for the damping factor $\xi=2.0$

	$D=0.001$	$D=5.0$
a_0	0.2499999E-03	0.1249999E01
a_1	−0.9418733E-04	0.3242193E01
a_2	−0.1697558E-03	0.4785780E03
a_3	−0.7050253E-03	0.2856254E04
a_4	−0.4432641E-02	0.7554378E06

These results, valuable in themselves, can be made considerably more useful by means of the theory of terminants advanced by Dingle [3], and recently used by Buchanan [8] in a study on improvement of series representations. A basic aim of Dingle's work is the analysis and improvement of divergent asymptotic series, and inspection of Table 1 indicates that the series at hand are divergent. Each series is a single sign series for all terms of order greater than or equal to unity, although of opposite signs.

Further insight can be gained by constructing Domb–Sykes plots, that is, plots of a_{n-1}/a_n vs $1/n$, a feature impossible before this work since sufficient data was not available. Figures 1 and 2 show these plots. Figure 1 is a typical result for the Domb–Sykes plot of a divergent

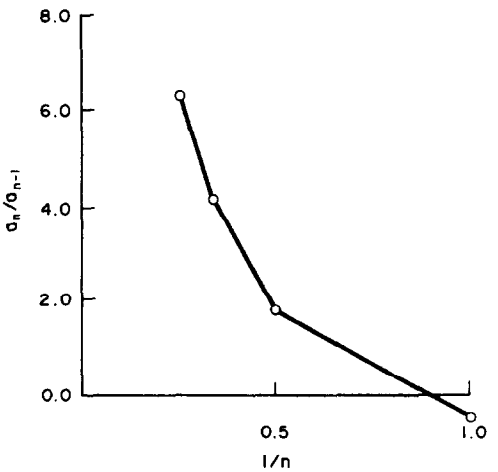

Fig. 1. Domb–Sykes plot for mean square response coefficients, $\xi=2.0$, $D=0.001$.

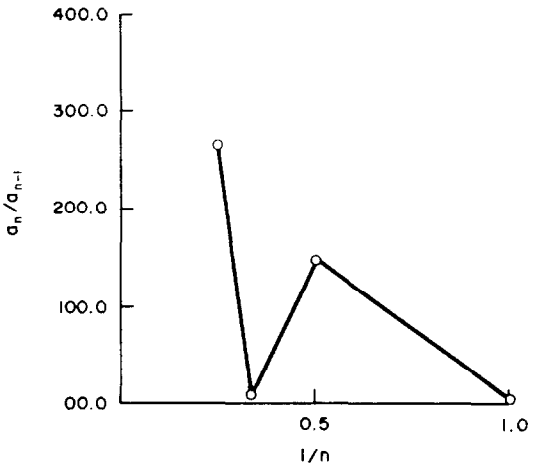

Fig. 2. Domb–Sykes plot for mean square response coefficients, $\xi = 2.0$, $D = 5.0$.

asymptotic series. Figure 2 is judged to be inconclusive toward establishing the analytical structure of the expansion for $D = 5.0$. It appears that, while all the a_n coefficients for $n \geq 1$ have changed from a negative to a positive value as D increased from 0.001 to 5.0, they have not all increased at the same rate. This accounts for the large 'dip' in Fig. 2. Higher order perturbation coefficients will be necessary before a stable relationship is established between coefficient magnitudes for this $D = 5.0$ case. This will not be pursued further in this work.

An inverted Domb–Sykes plot, that is, a plot of a_n/a_{n-1}, for the coefficients of the $D = 0.001$ expansion is shown in Fig. 3. This plot is typical for that of a divergent asymptotic series in that a linear relationship between coefficient ratios is becoming established as n increases. Here this pattern is developing for $n \geq 3$. The information available from a fourth-order expansion is obviously limited and conclusions must have some degree of qualification. Nevertheless, it is known that a small number of terms can supply a close approximation to a function represented by an asymptotic expansion, and the viewpoint is taken here that results drawn from the fourth-order expansion will provide a close approximation to the mean square response of the system.

The dashed line on Fig. 3 is the limiting slope towards which the inverted Domb–Sykes plot is progressing as it approaches the origin. Graphically we obtain $a_{n-1}/a_n = 0.6(1/n)$, behavior which is reproduced by taking $a_n = c(1/0.6)^n n!$ where c is a constant and has been calculated from the knowledge of the fourth-order coefficient to be $c = -0.2393626\text{E-}04$. The general form of the coefficients for the $D = 0.001$ expansion is now taken to be

$$a_n = -0.2393626\text{E-}04 \times \left(\frac{1}{0.6}\right)^n n! \tag{104}$$

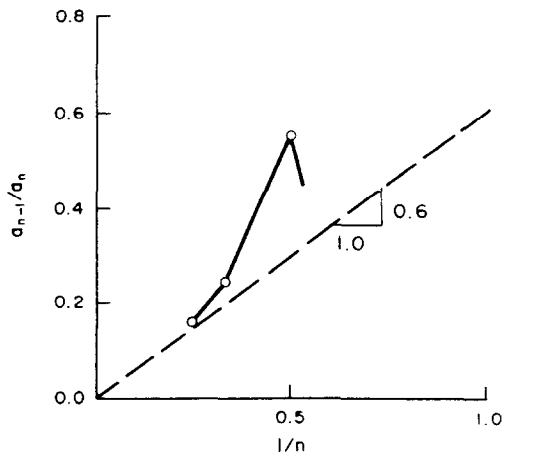

Fig. 3. Inverted Domb–Sykes plot for mean square response coefficients, $\xi = 2.0$, $D = 0.001$.

With this information the theory of terminants can now be employed. (104) allows writing the series, (83), as

$$R_{xx}(0)=\sum_{n=0}^{4} a_n\varepsilon^n-(0.2393626\text{E-}04)\sum_{n=5}^{\infty} n!\left(\frac{-\varepsilon}{0.6}\right) \tag{105}$$

where a_0 through a_4 are listed in Table 1.

Using Dingle's terminant for a single-sign asymptotic series, (105) can be written in the much more useful form

$$R_{xx}(0)=\sum_{n=0}^{4} a_n\varepsilon^n-(0.2393626\text{E-}04)(5!)\left(\frac{\varepsilon}{0.6}\right)^5\bar{\Lambda}_5\left(\frac{-\varepsilon}{0.6}\right) \tag{106}$$

where the terminant $\bar{\Lambda}$, a tabulated function, is given by

$$\bar{\Lambda}_m(-x)=\frac{1}{m!}P\int_0^{\infty}\frac{\zeta^m e^{-\zeta}}{1-\zeta/x}\,d\zeta \tag{107}$$

P denoting principal value.

Dingle has also developed an absolutely convergent expansion for the $\bar{\Lambda}$ terminant. Using this, an absolutely convergent representation for the mean square response can be found and is given by

$$R_{xx}(0)=\sum_{n=0}^{4} a_n\varepsilon^n-(0.2393696\text{E-}04)(5!)\left(\frac{\varepsilon}{0.6}\right)^5\left[-\frac{(\varepsilon/0.6)}{5}\left\{1+\frac{(\varepsilon/0.6)}{4}+\ldots+\frac{(\varepsilon/0.6)^4}{4!}\right\}\right.$$
$$\left.-\frac{(\varepsilon/0.6)}{5!}\sum_{t=0}^{\infty}\frac{(-\varepsilon/0.6)^t(\psi(t)-\ln(\varepsilon/0.6))}{t!}\right] \tag{108}$$

where

$$\psi(t)=\frac{1}{\Gamma(t+1)}\frac{d}{dt}\Gamma(t+1) \tag{109}$$

Γ denoting the gamma function.

For purposes of comparison, calculations based on (i) linear system, (ii) first-order perturbation expansion, (iii) second-order expansion and (iv) terminant expansion were carried out for $D=0.001$ and for values of the non-linear parameter ε ranging between 0.01 and 0.3. Results are shown in Table 2.

Table 2. A comparison of mean square response according to the linear system, first-order expansion, second-order expansion and terminant expansion for intensity $D=0.001$, damping factor $\xi=2.0$ and various values of the non-linear parameter ε

	Mean square response			
ε	Linear system	1st order expansion	2nd order expansion	Terminant expansion
0.3	0.249999E-03	0.2217437E-03	0.2064657E-03	0.1618777E-03
0.1	0.249999E-03	0.2405812E-03	0.2388836E-03	0.2377460E-03
0.06	0.249999E-03	0.2443487E-03	0.2437375E-03	0.2435284E-03
0.03	0.249999E-03	0.2471743E-03	0.2470215E-03	0.2469989E-03
0.01	0.249999E-03	0.2490580E-03	0.2490411E-03	0.2490403E-03

Several observations can be made. Note that for $\varepsilon\leqslant 0.03$ the linear portion of the system is dominant and that the first-order expansion essentially gives the response of the non-linear system. The second-order and terminant expansions adjust the accuracy for third or higher order significant figures. As the non-linearity of the system increases a deviation from linear response should occur, a result borne out by the data of Table 2 and the contribution of the terminant approach toward the accuracy of the mean square response calculation increases. This is illustrated by $\varepsilon\geqslant 0.1$ in Table 2. Finally, when using a terminant approach values of ε must be used to insure the system is weakly non-linear since this is the mathe-

matical basis of the perturbation solution. Large values of ε can be expected to give erroneous results for perturbation expansions of any order as well as for a terminant expansion. The value of $\varepsilon = 0.3$ in Table 2 is very likely the upper limit of the non-linear parameter for that system.

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APPENDIX
Recursion relations

This appendix presents the recursion relations for the a_{ijNij} and b_{ijNij} coefficients for the eigenfunctions of the weakly non-linear, second order system described by (1).

For convenience of notation the following definitions are made:

$$F_{nm} = \frac{(-D)^{n+m+1}\xi^{n-m}B_m}{2^{n+m}n!}$$

$$Y(n, i, j, p, q, r, s, t, u) = b_{ijprs}b_{ijqtu}(\zeta^{4n}v_{rs0}(\zeta, \dot{\zeta}), v_{tu0}(\zeta, \dot{\zeta}))$$

$$Z(n, i, j, p, q, r, s, t, u) = a_{ijprs}b_{ijqtu}(\zeta^{4n}w_{rs0}(\zeta, \dot{\zeta}), v_{tu0}(\zeta, \dot{\zeta}))$$

where $(\cdot, \cdot)$ is the inner product (16), and D, ξ, B_m, $v_{rs0}(\zeta, \dot{\zeta})$ and $w_{rs0}(\zeta, \dot{\zeta})$ are defined as before.

The recursion relation for b_{ijNij} is

$$\begin{aligned}
b_{ijNij} = -\tfrac{1}{2}\sum_{n,m=0}^{N}\sum_{p,q=0}^{N-1}\delta(N-n-m-p-q)F_{nm}\Bigg\{&\sum_{r=0}^{i-1}\sum_{s=0}^{j-1}[Y(n,i,j,p,q,r,s,i,j)+Y(n,i,j,p,q,i,j,r,j)]\\
&+\sum_{r=0}^{i-1}[Y(n,i,j,p,q,r,j,i,j)+Y(n,i,j,p,q,i,j,r,j)]\\
&+\sum_{r=0}^{i-1}\sum_{s=j+1}^{\infty}[Y(n,i,j,p,q,r,s,i,j)+Y(n,i,j,p,q,i,j,r,s)]\\
&+\sum_{s=0}^{j-1}[Y(n,i,j,p,q,i,s,i,j)+Y(n,i,j,p,q,i,j,i,s)]+Y(n,i,j,p,q,i,j,i,j)]\\
&+\sum_{s=j+1}^{\infty}[Y(n,i,j,p,q,i,s,i,j)+Y(n,i,j,p,q,i,j,i,s)]\\
&+\sum_{r=i+1}^{\infty}\sum_{s=0}^{j-1}[Y(n,i,j,p,q,r,s,i,j)+Y(n,i,j,p,q,i,j,r,s)]\\
&+\sum_{r=i+1}^{\infty}[Y(n,i,j,p,q,r,j,i,j)+Y(n,i,j,p,q,i,j,r,j)]\\
&+\sum_{r=i+1}^{\infty}\sum_{s=j+1}^{\infty}[Y(n,i,j,p,q,r,s,i,j)+Y(n,i,j,p,q,i,j,r,s)]\Bigg\}-\tfrac{1}{2}\sum_{n,m,p,q=0}^{N}\delta(N-n-m-p-q)F_{nm}\\
\times\Bigg\{&\sum_{r,t=0}^{i-1}\sum_{s,u=0}^{j-1}Y(n,i,j,p,q,r,s,t,u)+\sum_{r,t=0}^{i-1}\sum_{s=0}^{j-1}[Y(n,i,j,p,q,r,s,t,j)+Y(n,i,j,p,q,r,j,t,s)]\\
&+\sum_{r,t=0}^{i-1}\sum_{s=0}^{j-1}\sum_{u=j+1}^{\infty}[Y(n,i,j,p,q,r,s,t,u)+Y(n,i,j,p,q,r,u,t,s)]\\
&+\sum_{r,t=0}^{i-1}Y(n,i,j,p,q,r,j,t,j)+\sum_{r,t=0}^{i-1}\sum_{s=j+1}^{\infty}[Y(n,i,j,p,q,r,j,t,s)+Y(n,i,j,p,q,r,s,t,j)]
\end{aligned}$$

$$+\sum_{r,t=0}^{i-1}\sum_{s,u=j+1}^{\infty} Y(n,i,j,p,q,r,s,t,u)+\sum_{r=0}^{i-1}\sum_{s,u=0}^{j-1}[Y(n,i,j,p,q,r,s,i,u)+Y(n,i,j,p,q,i,s,r,u)]$$

$$+\sum_{r=0}^{i-1}\sum_{s=0}^{j-1}[Y(n,i,j,p,q,i,s,r,j)+Y(n,i,j,p,q,r,j,i,s)]$$

$$+\sum_{r=0}^{i-1}\sum_{s=0}^{j-1}\sum_{u=j+1}^{\infty}[Y(n,i,j,p,q,r,s,i,u)+Y(n,i,j,p,q,i,s,r,u)+Y(n,i,j,p,q,r,u,i,s)+Y(n,i,j,p,q,i,u,r,s)]$$

$$+\sum_{r=0}^{i-1}\sum_{s=j+1}^{\infty}[Y(n,i,j,p,q,r,j,i,s)+Y(n,i,j,p,q,i,s,r,j)]$$

$$+\sum_{r=0}^{i-1}\sum_{s,u=j+1}^{\infty}[Y(n,i,j,p,q,r,s,i,u)+Y(n,i,j,p,q,i,s,r,u)]$$

$$+\sum_{r=0}^{i-1}\sum_{s,u=0}^{j-1}\sum_{t=i+1}^{\infty}[Y(n,i,j,p,q,r,s,t,u)+Y(n,i,j,p,q,t,s,r,u)]$$

$$+\sum_{r=0}^{i-1}\sum_{s=0}^{j-1}\sum_{t=i+1}^{\infty}[Y(n,i,j,p,q,r,s,t,j)+Y(n,i,j,p,q,t,s,r,j)$$
$$+Y(n,i,j,p,q,r,j,t,s)+Y(n,i,j,p,q,t,j,r,s)]$$

$$+\sum_{r=0}^{i-1}\sum_{s=0}^{j-1}\sum_{t=i+1}^{\infty}\sum_{u=j+1}^{\infty}[Y(n,i,j,p,q,r,s,t,u)+Y(n,i,j,p,q,t,s,r,u)$$
$$+Y(n,i,j,p,q,r,u,t,s)+Y(n,i,j,p,q,t,u,r,s)]$$

$$+\sum_{r=0}^{i-1}\sum_{t=i+1}^{\infty}[Y(n,i,j,p,q,r,j,t,j)+Y(n,i,j,p,q,t,j,r,j)]$$

$$+\sum_{r=0}^{i-1}\sum_{s=j+1}^{\infty}\sum_{t=i+1}^{\infty}[Y(n,i,j,p,q,r,j,t,s)+Y(n,i,j,p,q,t,j,r,s)$$
$$+Y(n,i,j,p,q,r,s,t,j)+Y(n,i,j,p,q,t,s,r,j)]$$

$$+\sum_{r=0}^{i-1}\sum_{t=i+1}^{\infty}\sum_{s,u=j+1}^{\infty}[Y(n,i,j,p,q,r,s,t,u)+Y(n,i,j,p,q,t,s,r,u)]$$

$$+\sum_{s,u=0}^{j-1}Y(n,i,j,p,q,i,s,i,u)+\sum_{s=0}^{j-1}\sum_{u=j+1}^{\infty}[Y(n,i,j,p,q,i,s,i,u)+Y(n,i,j,p,q,i,u,i,s)]$$

$$+\sum_{s,u=j+1}^{\infty}Y(n,i,j,p,q,i,s,i,u)+\sum_{r=i+1}^{\infty}\sum_{s,u=0}^{j-1}[Y(n,i,j,p,q,i,s,r,u)+Y(n,i,j,p,q,r,s,i,u)]$$

$$+\sum_{r=i+1}^{\infty}\sum_{s=0}^{j-1}[Y(n,i,j,p,q,i,s,r,j)+Y(n,i,j,p,q,r,j,i,s)]$$

$$+\sum_{r=i+1}^{\infty}\sum_{s=0}^{j-1}\sum_{u=j+1}^{\infty}[Y(n,i,j,p,q,i,s,r,u)+Y(n,i,j,p,q,r,s,i,u)$$
$$+Y(n,i,j,p,q,i,u,r,s)+Y(n,i,j,p,q,r,u,i,s)]$$

$$+\sum_{r=i+1}^{\infty}\sum_{s=j+1}^{\infty}[Y(n,i,j,p,q,r,j,i,s)+Y(n,i,j,p,q,i,s,r,j)]$$

$$+\sum_{r=i+1}^{\infty}\sum_{s,u=j+1}^{\infty}[Y(n,i,j,p,q,i,s,r,u)+Y(n,i,j,p,q,r,s,i,u)]$$

$$+\sum_{r,t=i+1}^{\infty}\sum_{s,u=0}^{j-1}Y(n,i,j,p,q,r,s,t,u)+\sum_{r,t=i+1}^{\infty}\sum_{s=0}^{j-1}[Y(n,i,j,p,q,r,s,t,j)+Y(n,i,j,p,q,r,j,t,s)]$$

$$+\sum_{r,t=i+1}^{\infty}\sum_{s=0}^{j-1}\sum_{u=j+1}^{\infty}[Y(n,i,j,p,q,r,s,t,u)+Y(n,i,j,p,q,r,u,t,s)]$$

$$+\sum_{r,t=i+1}^{\infty}Y(n,i,j,p,q,r,j,t,j)+\sum_{r,t=i+1}^{\infty}\sum_{s=j+1}^{\infty}[Y(n,i,j,p,q,r,j,t,s)+Y(n,i,j,p,q,r,s,t,j)]$$

$$+\sum_{r,t=i+1}^{\infty}\sum_{s,u=j+1}^{\infty}Y(n,i,j,p,q,r,s,t,u)\Bigg\}.$$

The recursion relation for a_{ijNij} is

$a_{ijNij} = -b_{ijNij} + 2\{$The right hand side of the recursion relation for b_{ijNij} with $Y(n,i,j,p,q,r,s,t,u)$ replaced by $Z(n,i,j,p,q,r,s,t,u)\}$.

Resume:

Dans un article precedent (1), les auteurs ont presente des resultats pour les solutions en developpement de fonctions propres a l'equation directe de Fokker - Planck associee a un systeme particulier non lineaire du premier ordre soumis a une excitation avec un bruit blanc. Ce travail concerne les solutions en developpement de fonctions propres des equations de Fokker - Planck directe et inverse associees a un systeme particulier non lineaire du second ordre soumis a une excitation avec un bruit blanc. Les termes du developpement jusqu'au quatrieme ordre ont ete generes en utilisant un ordinateur. Avec cette nouvelle information, des graphiques inverses de Domb - Sykes revelent un modele dans les coefficients pour certaines valeurs des parametres. On a alors utilise avec ce modele la theorie de Dingle pour refondre les series sous une forme plus favorable au calcul.

Zusammenfassung:

Eine fruhere Arbeit der Verfasser (1) befasste sich mit Losungen mit Eigenfunktionsentwicklungen fur die vorwarts wirkende Gleichung nach Fokker und Planck, die ein bestimtes, nichtlineares Systems erster Ordnung unter Erregung durch weisses Rauschen beschreibt. Die vorliegende Arbeit befasst sich mit Losungen mit Eigenfunktionsentwicklungen fur die vorwarts und ruckwarts wirkende Gleichung nach Fokker und Planck, due ein bestimmtes, nichtlineares, durch weisses Rauschen erregtes System zweiter Ordnung beschreibt. Unter Verwendung eines Dgitalrechners wurden Entwicklungsglieder bis zur vierten Ordnung bestimmt. Mit Hilfe dieser neuen Information zeigt sich in umgekehrten Domb-Sykes Diagrammen ein Muster fur die Koeffizienten fur bestimmte Werte der Parameter. Dindles Terminantentheorie wurde benutzt um mit Hilfe dieses Musters die Reihen in eine fur die Berechnung besser geeignete Form zu bringen.