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Uniqueness Theorem for Abstract Hyperbolic Equations with Application to the Uniqueness of the Harmonic Coordinate System in General Relativity

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An analog of the radiation condition is found for an abstract hyperbolic equation. When this condition holds, the uniqueness theorem for $\partial_t(n\partial_t u) - \partial_r(\rho\partial_r u) + Au = f$ is valid. Here n, ρ, and A are some linear operators depending on t and r. For example, Eq. $\square\, u \equiv \partial_t(a_{oo}(t, x)\, \partial_t u) - \partial_i(a_{ij}\partial_j u) + q(t, x)\, u = f$ is of such a type. In what follows $\partial_t = \partial/\partial_t$, $\partial_i = \partial/\partial x_i$, $\partial_r = \partial/\partial_r$, $x = (x_1, ..., x_N)$, $1 \leqslant i, j \leqslant N$, where repeating indexes are summed. The results obtained can be applied to the scattering theory of hyperbolic equations. They will be used in the proof of Fock's conjecture of uniqueness (up to the Lorentz transformation) of the harmonic coordinate system. This conjecture has previously been shown to be valid only for $g^{\mu\nu} = -\delta^{\mu\nu}$, $1 \leqslant \nu, \mu \leqslant 3$, $g^{0\nu} = 0$, $1 \leqslant \nu \leqslant 3$, $g^{oo} = 1$, $g^{\mu\nu}$ being the fundamental tensor [1]. It follows from our results that the conjecture is valid for an arbitrary central-symmetric gravitation field which is Galilean at the infinity; in particular for the Schwarzschild tensor. If a_{oo}, a_{ij} do not depend on t, the uniqueness theorem follows from the uniqueness theorems for elliptic equations. In this case the radiation condition is less restrictive than in our more general case [2, I]. In [2, II] a uniqueness theorem for some special type of Eq. $\square\, u = 0$ with a_{oo} depending on t was announced. In Section 1 the main result (Theorem I and a corollary) is given, in Section 2 its proof is presented, in Section 3 an application to partial differential equations and to general relativity is considered.

1.

Let H, W_0, and W_1 be the Hilbert spaces, $W_0 \subset W_1 \subset H$, and W_0 and W_1 be dense in H. Consider an abstract hyperbolic equation in H:

$$Lw \equiv \partial_t\{n(t, r)\, \partial_t w\} - \partial_r\{\rho(t, r)\, \partial_r w\} + A(t, r)\, w = 0, \tag{1}$$

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where n, ρ, and A are self-adjoint operator functions in H, $(t, r) \in \Delta \equiv (-\infty, \infty) \times (0, \infty)$, satisfying the following assumptions:

(1) $n(t, r)$ and $\rho(t, r)$ are bounded self-adjoint operators, strongly continuously differentiable in r, t, for all t, r with $d_1 I \geqslant \rho \geqslant d_0 I$, $d_1 I \geqslant n \geqslant d_0 I$, I being the identity, and $|\partial_r n| + |\partial_r \rho| < N$, d_0, d_1, N being some positive constants;

(2) Let $A(t, r) = A_0(t, r) + Q(t, r)$, $A_0(t, r) \geqslant 0$ self-adjoint, and $D(A_0(t, r)) = W_0$, $A_0(t, r)$: $W_0 \to H$, $D(A)$ denotes the domain of A, and Q is a bounded self-adjoint operator strongly continuous in t, r such that $|Q| \leqslant N$. For all $w \in W_0$, $v \in W_1$ the equality $(A_0(t, r)\, w, v) = a(w, v)$ is valid where $a(w, v)$ is a sesquilinear symmetric nonnegative form defined on W_1 which is a function of t, r. The function $a(u, v)$ is continuously differentiable in r $\forall u$, $v \in W_1$, $\partial_r a(u, v) \equiv a'(u, v)$ is a symmetric form defined on W_1, $r\, |a'(u, u)| \leqslant Na(u, u)$.

(3) $\exists R_0 \in (0, \infty)$, such that $2ra(u, u) + r^2 a'(u, u) \leqslant 0$ for $u \in W_1$, $r \geqslant R_0$; $a(u, v) = r^{-2}(Bu, Bv) + a_1(u, v)$, u, $v \in W_1$. Here $B \geqslant 0$ is a self-adjoint operator defined on W_1, $a_1(u, v)$ is a symmetric sesquilinear form in H defined on W_1; $|a_1(u, u)| \leqslant Nr^{-2-\delta} |Bu|^2$, $\delta > 0$; $\forall \eta$, $\eta > 0$ the operator $(\eta + A_0{}^2)^{-1}$ is compact; $\partial_r \rho \geqslant r\, |Q|\, I$, $r\, |\partial_r \rho| + r\, |\partial_r \rho| + |I - \rho| + |I - n| \leqslant Nr^{-\delta}$.

We denote by $C_0{}^2$ the space of functions $f(t, r) \in W_0$ with continuous first and second derivatives in r, t, $\partial_t f$, $\partial_r f \in W_1$, $\partial^2_{tt} f$, $\partial^2_{rr} f$, $\partial^2_{rt} f \in H$.

THEOREM I. *Let assumptions* (1)–(3) *hold, and let* $w \in C_0{}^2$ *be a solution to Eq.* (1) *in* Δ, *such that*

$$\int_{-\infty}^{\infty} \{|\partial_t w|^2 + |\partial_r w|^2 + a(w, w)\}\, dt \leqslant N < \infty \tag{α}$$

and

$$\liminf_{r \to \infty} \int_{-\infty}^{\infty} \{|\partial_t w|^2 + |\partial_r w|^2\}\, dt = 0. \tag{A}$$

Then $w = 0$ *in* Δ.

COROLLARY I. *The conclusion of Theorem* I *remains valid if* (a) $k^2\, |\tilde{w}(k, r)|^2 \leqslant \varphi(k) \in L_1(0, \infty)$, *where* $\tilde{w}$ *is the Fourier transform of* w, *or* (b) *if* $\int_{-\infty}^{\infty} |\partial^2_{tr} w|^2\, dt \leqslant N$ *holds and instead of* (A) *one of the following conditions*

$$\liminf_{r \to \infty} \int_{-\infty}^{\infty} \{|\partial_t w + \partial_r w|^2 + |w|^2\}\, dt = 0, \tag{A'}$$

$$\liminf_{r \to \infty} \int_{-\infty}^{\infty} \{|\partial_t w - \partial_r w|^2 + |w|^2\}\, dt = 0 \tag{A''}$$

is valid.

2.

Proof. Denote by $\|u\|^2$, $\langle u, v\rangle$, $b(u, v)$, $b_1(u, v)$, $b'(u, v)$ the magnitudes $\int_{-\infty}^{\infty} |u|^2\, dt$, $\int_{-\infty}^{\infty} (u, v)\, dt$, $\int_{-\infty}^{\infty} a(u, v)\, dt$, $\int_{-\infty}^{\infty} a_1(u, v)\, dt$, and $\int_{-\infty}^{\infty} a'(u, v)\, dt$, respectively. For any v, $v \in C_0{}^2$ it is true that

$$\operatorname{Re} \int_{R_0}^{R} \varphi(r) \langle Lw, v\rangle\, dr$$

$$= \int_{R_0}^{R} [\varphi(r) \{\langle \rho\partial_r w, \partial_r v\rangle + b(w, v) - (n\partial_t w, \partial_t v) + \langle Qw, v\rangle\} \tag{2}$$

$$+ \partial_r\varphi\langle \rho\partial_r w, v\rangle]\, dr - \varphi(r) \operatorname{Re}\langle \rho\partial_r w, v\rangle|_{R_0}^{R} = 0;$$

$$\operatorname{Re} \int_{R_0}^{R} \varphi(r) \langle Lw, \partial_r w\rangle\, dr$$

$$= \int_{R_0}^{R} [0.5\partial_r\varphi\{\| \rho^{1/2}\partial_r w \|^2 + \| n^{1/2}\partial_t w \|^2\} + (\varphi(r) r^{-1} - 0.5\partial_r\varphi(r)) \cdot b(w, w)$$

$$- 0.5r^{-2}\varphi(r) \{r^2 b'(w, w) + 2rb(w, w)\} + \varphi(r) \operatorname{Re}\langle Qw, \partial_r w\rangle$$

$$+ 0.5\varphi(r) \{-\langle \partial_r\rho\partial_r w, \partial_r w\rangle + \langle \partial_r n\partial_r w, \partial_t w\rangle\}]\, dr - 0.5\varphi(r)$$

$$\cdot \{\| \rho^{1/2}\partial_r w \|^2 + \| n^{1/2}\partial_t w \|^2 - b(w, w)\}|_{R_0}^{R} = 0. \tag{3}$$

Let $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots, \lambda_n \to \infty$ be the eigenvalues of B^2, p_n the corresponding orthogonal projectors, $B^2 u = \lambda_n u$, $u \in p_n H$, $Q_n = \sum_{i=0}^{n} p_i$, $P_n = I - Q_n$. Choosing in (3) $\varphi(r) = 1 - \epsilon^{-1} r^{-\epsilon}$, $\epsilon = 0{,}5 \min(1, \delta)$ and making use of assumptions (3) and (A) we get (as in the proof of Theorem 1 in [3]) the inequality

$$\begin{aligned} &\int_R^{\infty} \left\{\frac{r^{-1-\epsilon}}{3} \{\| \rho^{1/2}\partial_r w \|^2 + \| n^{1/2}\partial_t w \|^2\} + \frac{\| Bw \|^2}{r^3}\right\} dr \\ &\quad + \tfrac{1}{2}\varphi(R) \{\| \rho^{1/2}\partial_r P_0 w \|^2 + \| \rho^{1/2}\partial_r w \|^2 + \| n^{1/2}\partial_t Q_0 w \|^2\} \\ &\leqslant N_1 \left(\int_R^{\infty} \{r^{-3-\delta} \| w \|^2 + \| Bw \|^2 r^{-3-\epsilon}\}\, dr + R^{-\delta} \| \partial_t w \|^2\right) \\ &\quad + \frac{\varphi(R)}{2} \{b(w, w) + \| \rho^{1/2}\partial_r P_0 w \|^2 - \| n^{1/2}\partial_t P_0 w \|^2\}. \end{aligned} \tag{4}$$

We note that from assumption (3) it follows that $b(w, w) = b(P_0 w, P_0 w)$. Integrating both parts of ineqiality (4) over (R, ∞) and then once again, making use of (2) with $v = p_0 w$ we get as in [3], for all sufficiently large R the inequality

$$\int_R^\infty 0.5(r-R)^2 r^{-3} \| Bw \|^2 \, dr + \int_R^\infty 0.5 \cdot (r-R) \{\| \rho^{1/2} \partial_r w \|^2 + \| n^{1/2} \partial_t Q_0 w \|^2\} \, dr$$
$$\leqslant N_1 \int_R^\infty r^{-1-\epsilon} [\| w \|^2 + \| Bw \|^2 + r^2 (\| \partial_t w \|^2 + \| \partial_r w \|^2)] \, dr$$
$$+ 0.25 \cdot P_0 w(t, R)\|^2$$
$$\leqslant N_2 \int_R^\infty r^{-1-\epsilon} [\| w \|^2 + \| Bw \|^2 + r^2 \| \partial_r w \|^2] \, dr + 0.25 \cdot \| P_0 w(t, R) \|^2$$
$$+ N_3 (R^{1-\epsilon} \| \partial_r w \|^2 + R^{-1-\epsilon} \| w \|^2). \tag{5}$$

Multiplying (5) by $2\alpha R^{2\alpha-1}$, integrating over (R, ∞), making use of the equality $\| Bw \|^2 = \| BP_0 w \|^2 \geqslant \lambda_1 \| P_0 w \|^2$, $\lambda_1 > 0$, and assuming that $\alpha(\alpha+1)(2\alpha+1) < \lambda_1$, $0 < 2\alpha < \epsilon$, we get $\int_R^\infty \{r^{2\alpha-1} \| P_0 w \|^2 + r^{2\alpha+1} \| \partial_r w \|^2\} \, dr < \infty$. From here and (2) it follows that $\int_R^\infty r^{2\alpha+1} \| \partial_t w \|^2 \, dr < \infty$. If $\liminf_{r\to\infty} \| Q_0 w \|^2 = 0$, then from the inequality $\| w(t, R) \|^2 \leqslant \int_R^\infty r^{m+1} \| \partial_r w \|^2 \, dr + \int_R^\infty r^{-m-1} \| w \|^2 \, dr$, $m > 0$, which holds if $\liminf_{r\to\infty} \| w \|^2 = 0$, we get $\int_R^\infty r^{k-1} \| w \|^2 \, dr < \infty$, $0 < k < \alpha$. Let us assume that $\liminf_{r\to\infty} \| Q_0 w \|^2 = c_0 > 0$. For any sequence $R_j \to \infty$ and for $R_i > R_j$ we have

$$\| w(t, R_i) - w(t, R_j) \|^2 = \int_{R_j}^{R_i} \langle \partial_r w(t, r), w(t, R_i) - w(t, R_j) \rangle \, dr$$
$$\leqslant \frac{\| w(t, R_i) - w(t, R_j) \|}{(2\alpha)^{1/2} (R_j)^\alpha} \left(\int_{R_j}^\infty r^{2\alpha+1} \| \partial_r w \|^2 \, dr \right)^{1/2}. \tag{6}$$

Thus the sequence $w(t, R_j)$ tends to a limit in $L^2[(-\infty, \infty), H]$, $\lim_{r\to\infty} Q_0 w(t, r) = w_\infty(t) \in L^2[(-\infty, \infty); H]$, $Q_0 w = w_\infty + w_1$, $\| w_1(t, r) \|^2 \to 0$, $r \to \infty$. From (6) it follows that $\| w_1 \|^2 \leqslant N r^{-2\alpha}$. As $\liminf_{r\to\infty} \| \partial_t w \|^2 = 0$, so for some sequence $R_j \to \infty$ we have $\| \partial_t w(t, R_j) \|^2 \to 0$, $j \to \infty$. Denote ny $\tilde{w}_0(k, r)$ the Fourier transformation of $Q_0 w$. Then for sufficiently large N we have

$$\lim_{j\to\infty} \int_{-\infty}^\infty | \partial_t Q_0 w(t, R_j) |^2 \, dt = \lim_{j\to\infty} \int_{-\infty}^\infty k^2 \, | Q_0 \tilde{w}_0(k, R_j) |^2 \, dk$$
$$\geqslant \lim_{j\to\infty} \int_{-N}^N k^2 \, | Q_0 \tilde{w}_0(k, R_j) |^2 \, dk$$
$$> \tfrac{1}{2} \int_{-N}^N k^2 \, | \tilde{w}_\infty(k) |^2 \, dk > 0,$$

if N is large enough, as $\int_{-\infty}^\infty | w_\infty(k) |^2 \, dk = C_0 > 0$. Thus $w_\infty(t) = 0$, $\int_R^\infty r^{k-1} \| w \|^2 \, dr < \infty$, for $k < \alpha$. Let $m = \sup\{k, \int_R^\infty r^{2k-1} \| w \|^2 \, dr < \infty\}$, then $m = \infty$. To show this suppose $0{,}5\alpha \leqslant m < \infty$, and set $w_m = r^m w$. From the earlier inequalities it follows that (a) $\int_R^\infty r^{2k-1}[\| w \|^2 + r^2\{b(w, w) + \| \partial_t w \|^2 + \| \partial_r w \|^2\}] \, dr < \infty$ for $0 < k < m$, (b) $L_m w_m \equiv L w_m + 2mr^{-1}\partial_r w_m + mr^{-1}\partial_r \rho w_m - m(m+1) r^{-2} \rho w_m = 0$, c) $\exists n \geqslant 0$ such that $\lambda_n \leqslant m(m+1) < \lambda_{n+1}$. As in the case $m = 0$, $w_m = w$ we obtain the inequality

$$\int_R^\infty [0.5(r - R)^2 \{\| BP_n w_m \|^2 r^{-3} - m(m+1) r^{-3} \| P_n w_m \|^2\} + R(r-R)\{m(m+1) \| Q_n w_m \|^2 - \| BQ_n w_m \|^2\} r^{-3}] \, dr + \int_R^\infty 0.5(r-R)\{\| \rho^{1/2}\partial_r w_m \|^2 + \| n^{1/2}\partial_t Q_n w_m \|^2\} \, dr$$

$$\leqslant N_3 \int_R^\infty r^{-1-\epsilon}(\| w_m \|^2 + \| Bw_m \|^2 + r^2 \| \partial_b w_m \|^2) \, dr + \int_R^\infty \frac{mR}{r^2} \| P_n w_m \|^2 \, dr + 0.25 \| P_n w_m \|^2 + N_5\{R^{1-\epsilon} \| \partial_r w_m \|^2 + R^{-1-\epsilon} \| w_m \|^2\}. \tag{7}$$

Using the inequality $\| BP_n w_m \|^2 - m(m+1) \| P_n w_m \|^2 \geqslant (\lambda_{n+1} - m(m+1)) \| P_n w_m \|^2$, $\lambda_{n+1} > m(m+1)$, multiplying (7) by $2\alpha R^{2\alpha - 1}$ and integrating over (R, ∞), and assuming that $0 < 2\alpha < \epsilon$, $\alpha(\alpha+1)(2\alpha + 4m + 1) < \lambda_{n+1} - m(m+1)$, we obtain

$$\int_R^\infty \{r^{2\alpha-1} \| P_n w_m \|^2 + r^{2\alpha+1}(\| \partial_r w_m \|^2 + \| n^{1/2}\partial_t Q_n w_m \|^2) + (m(m+1) - \lambda_n) r^{2\alpha-1} \| P_n w_m \|^2 + r^{2\alpha-1}(m(m+1) - \lambda_{n-1}) \| Q_{n-1} w_m \|^2\} \, dr < \infty.$$

If $m(m+1) > \lambda_n$, it follows that $\int_R^\infty r^{2\alpha-1} \| w_m \|^2 \, dr < \infty$ and that the assumption $m < \infty$ leads to a contradiction. If $m(m+1) = \lambda_n$, then as in the case $m = 0$, $n = 0$ we obtain $\int_R^\infty r^{k-1} \| w_m \|^2 \, dr < \infty$, $0 < k < \alpha$. Since this is also a contradiction it follows that $m = \infty$.

Taking m sufficiently large and using the equality $\int_R^\infty r^2 \langle L_m w_m, \partial_r w_m \rangle \, dr = 0$ we obtain

$$\int_R^\infty \Big[r\{\| \rho^{1/2}\partial_r w_m \|^2 (2m+1) + \| n^{1/2}\partial_t w_m \|^2\}(1 - N_1 r^{-\delta}) - \{r^2 b'(w_m, w_m) + 2rb(w_m, w_m)\} + \left(\frac{(m+1)m}{4} - \frac{N}{2}\right) \langle \partial_r \rho w_m, w_m \rangle \Big] \, dr + 0.5\{R^2(\| \rho^{1/2}\partial_r w_m \|^2 + \| n^{1/2}\partial_t w_m \|^2) + R^{2m}(m(m+1) \| w \|^2 - R^2 b(w, w))|_{r=R}\} \leqslant 0. \tag{8}$$

When $m(m+1) > 2N$ and R is large enough, the integral on the left side of (8) is positive. If for a sufficiently large R $\| w(t, R) \|^2 > 0$, then $m^2 \| w(t, R) \|^2 > R^2 b(w(t, R), w(t, R))$ if m is large enough. From here and (8) it follows that

$\| w(t, R)\|^2 = 0$, i.e., $\| w(t, r)\| \equiv 0$ for sufficiently large r. Using the method given in [3, 4] we show now that $w(t, r) \equiv 0$, $r \geqslant 0$. Set $r_1 = \inf\{R: w(t, r) = 0, r \geqslant R\}$. If $r_1 > 0$, then it follows from the continuity of $w(t, r)$ that there exists $r_2 > 0$, such that $\| w(t, r)\| > 0$, $r_2 < r < r_1$. Denote by $c = 3N$, $b = \min\{0.5; c^{-1}(\exp(cr_1) - \exp(cr_2)); (8c^4)^{-1}\}$. Let $r_0 > r_1$ be chosen so that $c^{-1}(\exp(cr_0) - \exp(cr_1)) < 0.5b$. If $s \equiv c^{-1}(\exp(cr_0) - \exp(cr))$, then $r(s) = c^{-1}\ln(\exp(cr_0) - cs)$, $dr/ds = -\exp(-cr)$, $s(r_0) = 0$, $s(r_1) < 0.5b$, $(0, b) \subset (s(r_0), s(r_2))$.

Now assume that $\psi(s) \in C^2(-b, b)$, $\psi(s) = 1$ if $|s| \leqslant 0.5b$, $\psi(s) = 0$ if $s \geqslant 0.8b$, $|d\psi/ds| \leqslant N_1$, $|d^2\psi/ds^2| \leqslant N_1$ and consider the function $u(s) = \psi(s)\, w(r(s))$. This function is the solution to the equation $\partial_t(\tilde{n}\partial_t u) - \partial_s(\tilde{\rho}\partial_s u) + \tilde{A}u = f(t, s)$, $f(t, s) \equiv 2\rho(t, r(s))\, d\psi/ds \cdot \partial_r w(t, r(s)) + \exp(cr(s))\ [(d\psi/ds)(c\rho + \partial\rho/\partial r) - (d^2\psi/ds^2)\,\rho]\, w(t, r(s))$, $\tilde{n}(t, s) = n(t, r(s))\exp(-cr(s))$, $\tilde{A}(t, s) = A(t, r(s))\exp\{-cr(s)\}$, $\tilde{\rho}(t, s) = \rho(t, r(s))\exp\{cr(s)\}$, $\tilde{b}(\cdot, \cdot) = \exp\{-cr(s)\}\, b(\cdot, \cdot)$, $\tilde{b}'(\cdot, \cdot) = \exp\{-2cr(s)\}\,(cb(\cdot, \cdot) - b'(\cdot, \cdot))$, $\partial\tilde{n}/\partial s = \exp(-2cr)\,(cn - \partial_r n)$, $\partial\tilde{\rho}/\partial s = -(c\rho + \partial_r\rho)$.

Assumptions (1) and (2) and the choice of c imply that $\tilde{b}^1(\cdot, \cdot) \geqslant 2N\exp\{-2cr(s)\}\, b(\cdot, \cdot)$, $\partial\tilde{\rho}/\partial s \leqslant -2N\rho$. Setting $v_m(t, s) = s^{-m/2}u(t, s)$ and taking into account that $u(s) = 0$, $0 = s(r_0) \leqslant s < s(r_1)$, we obtain $v_m(0) = 0$, $\forall m$. The function $v_m(t, s)$ is the solution of the equation

$$\partial_t(\tilde{n}\partial_t v_m) = \partial_s(\tilde{\rho}\partial_s v_m) + \tilde{A}v_m - \left(\frac{m}{s}\right)\tilde{\rho}\partial_s v_m - \frac{m(m-2)}{4s^2}\tilde{\rho}v_m - \frac{m}{2s}\cdot\frac{\partial\tilde{\rho}}{\partial s}v_m$$
$$= s^{-m/2}\cdot f(s) = g_m(s).$$

Multiplying this equation by $ms^2\partial_s v_m(t, s)$ in H and integrating over $(0, b)$ we obtain

$$\int_0^b \operatorname{Re}\Big[ms\{\|\tilde{\rho}^{1/2}\cdot\partial_s v_m\|^2 + \|\tilde{n}^{1/2}\partial_t v_m\|^2\} - m\{s\tilde{b}(v_m, v_m) + \tfrac{1}{2}s^2\tilde{b}'(v_m, v_m)\}$$
$$+ \frac{m^2(m-2)}{8}\langle\partial_s\tilde{\rho}v_m, v_m\rangle - \frac{ms}{2}\langle\partial_s\tilde{\rho}v_m, \partial_s v_m\rangle$$
$$+ \tfrac{1}{2}ms^2\{\langle\partial_s\tilde{n}\partial_t v_m, \partial_t v_m\rangle - \langle\partial_s\tilde{\rho}\partial_s v_m, \partial_s v_m\rangle - 2\langle g_m, \partial_s v_m\rangle\}$$
$$- m^2 s\,\|\tilde{\rho}^{1/2}\partial_s v_m\|^2\Big]\, ds$$
$$= 0. \tag{9}$$

From here and the equality

$$\int_0^b \operatorname{Re}\Big[s^k\|\tilde{\rho}^{1/2}\partial_s v_m\|^2 + s^k\tilde{b}(v_m, v_m) - s^k\|\tilde{n}^{1/2}\partial_t v_m\|^2 - s^{k-1}(m+k)\langle\rho\partial_s v_m, v_m\rangle$$
$$- \frac{m(m-2)}{4}s^{k-2}\|\tilde{\rho}^{1/2}v_m\|^2 - \frac{1}{2}ms^{k-1}\langle\partial_s\tilde{\rho}v_m, v_m\rangle - s^k\langle g_m, v_m\rangle\Big]$$
$$= 0 \tag{10}$$

for m sufficiently large it follows that

$$\int_0^b \left\{ m(m-d_3)\, s \,\| \partial_s v_m \|^2 + m s^2 \tilde{b}(v_m, v_m) + \frac{m^2(m-d_3)}{s} \| \tilde{\rho}^{1/2} v_m \|^2 \right\} ds \leqslant N_1 \int_0^b s^3 \| g_m \|^2 \, ds, \qquad m > d_3,$$

where d_3 does not depend on m. From this inequality it follows that

$$\int_0^b s^{-m} \{ \| u \|^2 + \| \partial_s u \|^2 + b(u, u) \} \, ds \leqslant b \int_0^b s^{-m} \| f \|^2 \, ds.$$

As $f(t, s) = 0$, $0 \leqslant s \leqslant 0.5b$,

$$\| f \|^2 \leqslant b^{-1} N_1 \sup \{ \| \partial_r w \|^2 + \| w \|^2 \}, \qquad r(b) \leqslant r \leqslant r(0.5b)$$

we finally establish the inequality

$$\begin{aligned} \int_0^{b/2} \{ \| u \|^2 + \| \partial_r u \|^2 + b(u, u) \} \, ds &\leqslant \frac{b^m}{2^m} \int_0^{b/2} s^{-m} \{ \| u \|^2 + \| \partial_s u \|^2 + b(u, u) \} \, ds \\ &\leqslant \frac{b^{m+1}}{2^m} \int_{b/2}^b s^{-m} \| f \|^2 \, ds \\ &\leqslant \frac{N_1}{m} \sup \{ \| w \|^2 + \| \partial_r w \|^2 \}, \qquad r(b) \leqslant r \leqslant r(b/2). \end{aligned}$$

As m is arbitrary, $u = 0$ if $0 \leqslant s \leqslant 0.5b$, i.e., $w(r) = 0$, $r \in [r(0.5b), r_1]$. This contradiction proves that $w(t, r) \equiv 0$, $r > 0$.

The assumptions of Corollary 1 imply assumption A of Theorem 1 since from the equality $\liminf_{r \to \infty} \| w \|^2 = 0$ and either inequality (a) of Corollary 1 or inequality (b) of this corollary if follows that $\liminf_{r \to \infty} \| \partial_t w \|^2 = 0$ or $\liminf_{r \to \infty} \operatorname{Re} \langle \partial_t w, \partial_r w \rangle = 0$.

COROLLARY 2. *If in assumption* (3) *instead of the inequality* $r \mid Q \mid I \leqslant N \partial_r \rho$, *the inequality* $r \mid Q \mid \leqslant N r^{-1-\delta}$ *is valid, then for any solution* $w \in C_0^2$ *to Eq.* (1) *the inequality* $\int_R^\infty r^k \{ \| w \|^2 + \| \partial_r w \|^2 + \| \partial_t w \|^2 + b(w, w) \} \, dr < \infty$ *for* $0 < k < \infty$ *holds.*

3.

Consider the equation

$$\square u \equiv \partial_t (a_{oo}(t, x) \, \partial_t u) - \partial_i (a_{ij}(t, x) \, \partial_j u) + q(t, x) \, u = 0. \tag{11}$$

THEOREM 2. *Suppose the following conditions hold:*

(1) $a_{ij}(t, x) = p_0(t, x)\,\delta_{ij} - r^{-2}a(t, x)\,x_i x_j$, δ_{ij} *being the Kronecker delta.* $r^2 = x_i x_i$, $p_0(t, x)$, $a(t, x)$, $a_{oo}(t, x)$, $q(t, x) \in C^2(\mathbb{R}^1 \times (\mathbb{R}^N - S_{r_0}))$, $S(r_0) = \{x: |x| \leqslant r_0, x \in \mathbb{R}^N\}$;

(2) $a_{oo} > 0$, $p_0(t, x) - a(t, x) > 0$, $r > r_0$.

(3) $r\,|q| \leqslant N_0 \partial_r(p_0 - a)$, $\partial_r p_0 \leqslant 0$, $|a| + |a_{oo} - (1/c)| + |p_0 - a - c| + r\partial_r(p_0 - a) + r\,|\partial_r a_{oo}| \leqslant N_0 r^{-\delta}$, $\delta > 0$, $r > R_0 > r_0$, $0 < c =$ const.

Let $u(t, x)$ *be a solution to Eq.* (11), $w \equiv r^{(N-1)/2}u$, $r \geqslant r_0$. *Then* $u \equiv 0$ *provided that conditions* (α) *and* (A) *from Theorem* 1 *are valid for* w. *For the function* w *the assertions of Corollaries* 1 *and* 2 *hold.*

Proof. Equation (11) for the function w can be written in spherical coordinates:

$$\partial_t(a_{oo}(t, x)\,\partial_t w) - \partial_r((p_0 - a)\,\partial w) - \frac{1}{q_j \sin^{N-1-j}\theta_j}\frac{\partial}{\partial\theta_j}\left(\sin^{N-1-j}\theta_j \frac{P_0}{r^2}\frac{\partial w}{\partial\theta_j}\right)$$
$$+ qw + (p_0 - a)\frac{(N-1)(N-3)}{4r^2} + (\partial_r(p_0 - a))\frac{(N-1)}{2r}w$$
$$= 0.$$

$$q_j = (\sin\theta_1 \cdots \sin\theta_{j-1})^2, \qquad j \geqslant 2, \quad q_1 = 1. \tag{12}$$

It is easy to verify that for Eq. (12) the assumptions from Theorem 1 and Corollaries 1 and 2 are valid, $H = L^2(S^{N-1})$, S^{N-1} being the unit sphere in $\mathbb{R}^N$.

From [1] it follows, that for central-symmetric gravitational fields the theorem of the uniqueness of the harmonic coordinate system can be reduced to the uniqueness theorem for Eq. (11) for $N = 3$, $q(t, x) = 0$. In particular for the Schwarzschild field $a = cr^{-2}\alpha^2$, $p = c$, $a_{oo} = c^{-1}(1 + \alpha r^{-1})(I - \alpha r^{-1})^{-1}$, $S_{r_0} = S_\alpha$ [1, p. 288].

REFERENCES

1. V. A. FOCK, "Theory of Space Time and Gravitation," Fizmatgiz, Moscow, 1961 (in Russian) [Engl. transl. by McMillan, New York, 1964].
2. P. A. MISHNAEVSKY AND A. G. RAMM, Uniqueness theorem of use in general relativity I, II, *Notices Amer. Math. Soc.* **24** (1977), A-388, A-431.
3. P. A. MISHNAEVSKY, Spectral theory for the Sturm–Lionville equation with an operator coefficient, *Izv. Acad. Sci. USSR* **40** (1976), 152–189 [Engl. transl., *Soviet Math. Izv.* **10** (1976), 145–180].
4. W. JÄGER, Das asymptotische Verhalten von Lösungen eines Typs von Differentialgleichungen, *Math. Z.* **112** (1969), 26–36.