JOURNAL OF ALGEBRA 60, 96–125 (1979)

Finite Groups with Unbalancing 2-Components of $\{\hat{L}_3(4), He\}$-Type

ROBERT L. GRIESS, JR.* AND RONALD SOLOMON†

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109 and Department of Mathematics, Ohio State University, Columbus, Ohio 43210

Communicated by Walter Feit

Received September 18, 1977

1. INTRODUCTION

Considerable progress has been made in recent years towards the classification of finite simple groups of odd Chevalley type. These groups may be defined as simple groups G containing an involution t such that $C_G(t)/O(C_G(t))$ has a subnormal quasi-simple subgroup. This property is possessed by most Chevalley groups over fields of odd order but by no simple Chevalley groups over fields of characteristic 2, hence the name. It is also a property of alternating groups of degree at least 9 and eighteen of the known sporadic simple groups.

Particular attention in this area has focussed on the so-called B-Conjecture.

B-Conjecture: Let G be a finite group with $O(G) = \langle 1 \rangle$. Let t be an involution of G and L a perfect subnormal subgroup of $C_G(t)$ with $L/O(L)$ quasi-simple, then L is quasi-simple.

The B-Conjecture would follow as an easy corollary of the Unbalanced Group Conjecture (U-Conjecture).

U-Conjecture: Let G be a finite group with $F^*(G)$ quasi-simple. Suppose there is an involution t of G with $O(C_G(t)) \nsubseteq O(G)$. Then $F^*(G)/F(G)$ is isomorphic to one of the following:

(1) A Chevalley group of odd characteristic.

(2) An alternating group of odd degree.

(3) $L_3(4)$ or Held's simple group, He.

With the exception of the $L_2(q)$ component case, the cases (1) and (2) have been completely handled in [3], [5], [7], [8], [20], [29] and [30]. The $L_2(q)$ problem has been reduced in [19] to the solution of a small number of specific standard form problems in groups satisfying the U-Conjecture. This paper treats case (3).

Before we state our main results, we need to make some definitions. A perfect

* Partially supported by NSF Grant MCS76-07280.

† Partially supported by NSF Grant MCS75-08346.

0021-8693/79/090096-30$02.00/0

subnormal subgroup H of a group G is called a 2-*component* of G if $H/O(H)$ is quasi-simple. A 2-component, H, is a *component* if $O(H) \subseteq Z(H)$. $L(G)$ is the product of all 2-components of G. $\mathscr{L}(G)$ is the set of all 2-components of centralizers of involutions in G. For any elementary 2-subgroup A of G with $m(A) \geqslant 2$, we let $W_A = \langle \bigcap_{b \in B^\#} O(C_G(b)) \mid B \subseteq A, m(B) \geqslant 2 \rangle$. We call a group G *balanced* if $O(C_G(t)) \subseteq O(G)$ for all $t \in I(G)$. We call G 2-*balanced* if $W_A \subseteq O(G)$ for all elementary 4-subgroups A of G. We shall call $J \in \mathscr{L}(G)$ an *unbalancing* 2-*component* in G if there exists an *unbalancing triple* (a, x, J) with $\langle a, x \rangle$ a 4-subgroup of G, J a 2-component of $C_G(a)$ normalized by x and by $D = O(C_G(x)) \cap C_G(a)$ and with $[J, D] \nsubseteq O(J)$. We shall call G *locally* 2-*balanced* if for all unbalancing 2-components $J \in \mathscr{L}(G)$, all $a \in I(C_G(J))$ and all 4-subgroups E of $N_G(J) \cap C_G(a)$, we have

$$[J, W_E \cap C_G(a)] \subseteq O(C_G(a)).$$

We can now state our main results.

THEOREM 1.1. *Let G be a finite group. Suppose that*

(1) *The U-Conjecture holds in every proper section of G.*

(2) *G is locally 2-balanced.*

(3) *There exists $L \in \mathscr{L}(G)$ with $L/O(L) \simeq$ He.*

Let $\bar{G} = G/O(G)$ and let bars denote homomorphic images in $\bar{G}$. Then $\langle \bar{L}^{\bar{G}} \rangle = L_1 \times L_2 \times \cdots \times L_r$ for some $r \geqslant 1$, with $L_i \simeq$ He for all i, $1 \leqslant i \leqslant r$.

THEOREM 1.2. *Let G be a minimal counterexample to the U-Conjecture. Then G contains an unbalancing triple (b, y, K) with $K/O(K) \simeq L_2(q)$ for some odd $q \geqslant 4$ and with a Sylow 2-subgroup of $C_G(K/O(K))$ cyclic.*

We use the notation $\tilde{K} = K/Z^*(K)$, unless otherwise indicated. Besides this, our notation is standard. We collect the necessary assumed results and preliminary lemmas in Section 2. Theorem 1.1 is proved in Section 3. As will be clear from the discussion in Section 2, Theorem 1.2 is an immediate corollary of Theorem 1.1, Theorem 2.22 below and the following Theorem whose proof occupies Sections 4 and 5 of this paper.

THEOREM 1.3. *Let G be a minimal counterexample to the U-Conjecture. Suppose that G is locally 2-balanced. Then G does not contain any unbalancing triple (a, x, J) with $J/Z^*(J) \simeq L_3(4)$.*

Finally, we mention that the full solution of the $L_3(4)$ standard componen problem will follow from completion of the U-conjecture, which in turn depend on eliminating $L_2(q)$ as a maximal 2-component in a minimal counterexample See Theorem 2.13 and the comments following Lemma 2.23.

2. Preliminaries

We shall denote by $I(H)$ the set of all involutions of H. We first collect some needed properties of $L_3(4)$ and He.

Proposition 2.1. *Let $G \simeq SL(3,4)$ and let $\hat{G}$ be the covering group of G. Regard G as the group presented by the Steinberg generators $x_r(t)$ and relations*

$$\text{(A)}\quad x_r(t)\,x_r(u) = x_r(t+u) \qquad r \in \Sigma, \quad t, u \in K;$$

$$\text{(B)}\quad [x_r(t), x_s(u)] = \begin{cases} 1 & \text{if } r+s \notin \Sigma \\ x_{r+s}(tu) & \text{if } r+s \in \Sigma, \end{cases}$$

where $r, s \in \Sigma$, $r \neq \pm s$, $t, u \in K$. Here Σ is a root system of type A_2 and $K = \mathbf{F}_4$. Then $\hat{G}$ may be presented by generators $y_r(t)$, $r \in \Sigma$, $t \in K$, subject to the relations

$$\text{(A}'\text{)}\quad y_r(t)\,y_r(u) = y_r(t+u) \qquad r \in \Sigma, \quad t, u \in K;$$

$$\text{(B}'\text{)}\quad [y_r(t), y_s(u)] = \begin{cases} f_{r,s}(t,u) & \text{if } r+s \notin \Sigma \\ y_{r+s}(tu)\,f_{r,s}(t,u) & \text{if } r+s \in \Sigma, \end{cases}$$

where $r, s \in \Sigma$, $r \neq \pm s$, $t, u \in K$;

(C′) *all the $f_{r,s}(t,u)$ are central in $\hat{G}$.*

Also, $Z = O_2(Z(G))$ is isomorphic to $\mathbf{Z}_4 \times \mathbf{Z}_4$, the Schur multiplier of G. Let $\{a, b\}$ be a set of fundamental roots. The $f_{r,s}(t,u)$ satisfy the properties

(i) *if $tu \neq 0$ and r, s form a $60°$ angle, then $f_{r,s}(t,u)$ has order 2, $f_{r,s}(\ ,\ )$ is biadditive and $f_{r,s}(t,u) = f_{r,s}(\lambda^{-1}t, \lambda u)$ for all $\lambda \in K^\times$, $t, u \in K$.*

(ii) *if $tu \neq 0$ and r, s* form a $120°$ *angle, then $f_{r,s}(t,u)$ has order 4. Also*

$$f_{a,b}(t,u) = f_{a,b}(\lambda t, \lambda u) \text{ for all } \lambda \in K^\times,\ t, u \in K,$$
$$f_{a,b}(t,u)\,f_{a,b}(t',u)\,f_{a+b,a}(tu,t') = f_{a,b}(t+t',u), \text{ and}$$
$$f_{a,b}(t,u)\,f_{a,b}(t,u')\,f_{a+b,b}(tu,u') = f_{a,b}(t,u+u').$$

(iii) Aut$(G) \simeq \Sigma_3 \times \mathbf{Z}_2$ *acts on Z as follows: The $\mathbf{Z}_2$ direct factor is generated by the image of the unitary automorphism, which inverts Z. A Σ_3 direct factor acts faithfully on both $\Phi(Z)$ and $Z/\Phi(Z)$.*

(iv) *The preimage $\hat{X}_r$ in $\hat{G}$ of a root subgroup X_r of G is isomorphic to $\mathbf{Z}_4 \times \mathbf{Z}_4 \times \mathbf{Z}_2 \times \mathbf{Z}_2$.*

Proof. Let $X_r = \langle x_r(t) \mid t \in K\rangle$, $r \in \Sigma$, $U = \langle X_r \mid r = a, b, a+b\rangle$. Let ^ denote the preimage in $\hat{G}$ of a subset of G under the given map $\hat{G} \to G$. Since U has class 2, $\hat{U}$ has class at most 3, whence $\hat{U}'$ is abelian. Therefore $\hat{X}_r$ is abelian. Now let H be the standard Cartan subgroup of G. Then $X_r = [X_r, H]$ for all r, and so $Y_r = [\hat{X}_r, \hat{H}]$ complements $Z = O_2(Z(\hat{G}))$ in $\hat{X}_r$, since $\hat{X}_r$ is an abelian 2-group and $|H|$ is odd. Define $y_r(t) \in x_r(t)$ (regarded as a coset of Z in $\hat{G}$) by $y_r(t) = x_r(t) \cap Y_r$. At once, these $y_r(t)$ satisfy (A′). Define the various $f_{r,s}(t, u)$ by the relation (B′); then $f_{r,s}(t, u) \in Z$, so that (C′) holds.

Note that if $y \in \hat{G}$ and $x \in G$, then y^x, defined to be $y^{x'}$, for some $x' \in x$, is independent of the choice of x'. If $g \in N = N_G(H)$ satisfies $x_r(t)^g = x_{r'}(t)$, $x_s(t)^g = x_{s'}(t)$, then $y_r(t)^g = y_{r'}(t)$ and $f_{r,s}(t, u) = f_{r',s'}(t, u)$. Thus, Z is generated by the $f_{r,s}(t, u)$ with $\{r, s\} \subseteq \{a, b, a + b\}$.

If r, s form a 60° angle, then $[y_r(t), y_s(u)] = f_{r,s}(t, u) \in Z$. Then, biadditivity of $f_{r,s}(\ ,\ )$ follows from the corresponding property of commutation in a class 2 2-group. Set $h = h_a(\lambda)$ for $\lambda \in K^\times$, $\lambda \neq 1$. Then

$$x_a(t)^h = x_a(\lambda^2 t),\ x_b(t)^h = x_b(\lambda^{-1}t) = x_b(\lambda^2 t),\ x_{a+b}(t)^h = x_{a+b}(\lambda t).$$

So, we have (i) and part of (ii). We obtain the last part of (ii) by applying the commutator identity $[xx', y] = [x, y]^{x'}[x', y]$ to $[y_a(t + t), y_b(u)]$ and then applying $[x, yy'] = [x, y'][x, y]^{y'}$ to $[y_a(t), y_b(u + u')]$. Now, as $f_{a,b}(t, u)^2 = f_{a+b,b}(tu, u')$, the order of $f_{a,b}(t, u)$ is 4, provided $f_{a+b,b}(tu, u')$ has order 2 for $tuu' \neq 0$. Also

$$f_{a,b}(t, u) = f_{a,b}(1, t^{-1}u) \text{ if } t \neq 0, \text{ and}$$
$$f_{a,b}(1, \alpha) f_{a,b}(1, \beta) \equiv f_{a,b}(1, \gamma)\ (\mathrm{mod} \langle f_{a,b}(1, \alpha)^2, f_{a,b}(1, \beta)^2 \rangle)$$

whenever $\{\alpha, \beta, \gamma\} = K^\times$. This implies that Z is a homomorphic image of $\mathbf{Z}_4 \times \mathbf{Z}_4$.

We shall not prove here that $Z \cong \mathbf{Z}_4 \times \mathbf{Z}_4$. This was first shown by Thompson in some unpublished notes. It can also be verified by exhibiting an extension of U with the above factor set, then showing that this factor set of U is stable with respect to G in the sense of Cartan-Eilenberg [6], Chapter XII, For examples of such an argument, see [16].

By Alperin's lemma (Assumed result (9) of [16]), Aut G does act on $\hat{G}$ so as to lift the natural action on $G = \hat{G}/Z$. Let $\langle \delta, \gamma, \varphi \rangle$ complement $\mathrm{Inn}(G)$ in Aut G, where γ, φ are the standard graph, field automorphisms, respectively, and δ is a diagonal outer automorphism of period 3, say $x_a(t) \mapsto x_a(wt)$, $x_b(t) \mapsto x_b(w^2 t)$, where $\langle w \rangle = K^\times$. Direct calculation shows that γ and φ invert δ. The way that the $y_r(t)$ were defined indicates that they are transformed by $\langle \delta, \gamma, \varphi \rangle$ as the corresponding $x_r(t)$ are. At this point, (iii) can be verified by direct calculation (i.e. apply members of $\langle \delta, \gamma, \varphi \rangle$ to the expressions in (B′)).

PROPOSITION 2.2. *Let $G = G'$ satisfy $O(G) = 1$ and $G/Z(G) \cong L_3(4)$. Let $\alpha \in \mathrm{Aut}\, G$ satisfy $\alpha^2 = 1$.*

(1) *If α induces the standard graph automorphism on $G/Z(G)$ then $C_G(\alpha) = Z_0 \times L_0$ where $Z_0 = C_{Z(G)}(\alpha)$ and $L_0 \cong A_5$. Z_0 is isomorphic to a subgroup of $\mathbf{Z}_4$.*

(2) *If α induces the standard field automorphism on $G/Z(G)$, then either $|Z(G)| = 16$ and $C_G(\alpha) \cong \mathbf{Z}_4 \circ SL(2, 7)$ or $|Z(G)| < 16$ and $C_G(\alpha) = Z_1 \times L_1$ where $Z_1 = C_{Z(G)}(\alpha)$ and $L_1 \cong L_2(7)$. Z_1 is isomorphic to a subgroup of $\mathbf{Z}_4$.*

(3) *If α induces a unitary automorphism on $G/Z(G)$, then α inverts $Z(G)$ and $|C_G(\alpha)| = 2^3 \cdot 3^2$. A Sylow 2-subgroup R of $C_G(\alpha)$ is isomorphic to*

Q_8	if $Z(G) = 1$;
$\mathbf{Z}_2 \times \mathbf{Z}_4$	if $Z(G)$ is cyclic but $Z(G) \neq 1$;
$\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$	otherwise.

Moreover, in all cases, $C_{Z(G)}(\alpha)$ is a direct factor of R.

Proof. In cases (1) and (2), $C_{G/Z(G)}(\alpha)$ is perfect. Hence, by the Three Subgroups Lemma, $C_{G/Z(G)}(\alpha) = C_G(\alpha)\, Z(G)/Z(G)$. We continue the notation of Proposition 2.1.

(1) In this case $C_{G/Z(G)}(\alpha) \cong A_5$ and we may assume that a Sylow 2-subgroup of $C_G(\alpha)\, Z(G)$ is $\langle Z(G), y_{a+b}(t) \mid t \in K\rangle$, an abelian group. Thus $C_G(\alpha)$ does not involve $SL(2, 5)$ and the result follows.

(2) In this case $C_{G/Z(G)}(\alpha) \cong L_2(7)$ and we may assume that a Sylow 2-subgroup of $C_G(\alpha)\, Z(G)$ is $\langle Z(G), y_a(1), y_b(1)\rangle$. If $Z(G) = Z$, then the commutator subgroup of this group is $\langle [y_a(1), y_b(1)]\rangle$, a cyclic group of order 4 intersecting Z in a group of order 2, $\langle f_{a,b}(1, 1)^2\rangle = \Omega_1(C_Z(\alpha))$. Thus if $Z(G) = Z$, then $C_G(\alpha) \cong \mathbf{Z}_4 \circ SL(2, 7)$. If $|Z(G)| \leqslant 8$, then $[y_a(1), y_b(1)]$ has order 2 in G, whence $C_{Z(G)}(\alpha)$ is a cyclic direct factor of $C_G(\alpha)$.

(3) Assume first that $Z = Z(G)$. We form a group $H = G\langle d\rangle$, where $|d| = 3$, d induces an outer diagonal automorphism on G/Z and α acts on H and centralizes d. By Proposition 2.1, d acts nontrivially on Z. Let bars denote images under $H \to H/Z$. Then $C_{\bar{H}}(\alpha) \cong PGU(3, 2)$. Let C be the preimage in H of $C_{\bar{H}}(\alpha)$ and let $T \in \mathrm{Syl}_2(C)$. Then $[T, \alpha] \subseteq Z$ and since $\alpha d = d\alpha$, $[T, \alpha]$ is either 1, $\Phi(Z)$ or Z.

We argue that $|T'| = 2$ and $T/T' \cong \mathbf{Z}_8 \times \mathbf{Z}_8$. We note first that $T'Z \subseteq Z(T)$. Thus, as $T/T'Z \cong \mathbf{Z}_2 \times \mathbf{Z}_2$, $|T'| = 2$ and $T' \cap Z = \langle 1\rangle$. As $\langle d\rangle$ acts on T/T', T/T' is isomorphic either to $\mathbf{Z}_8 \times \mathbf{Z}_8$ or to $\mathbf{Z}_4 \times \mathbf{Z}_4 \times \mathbf{Z}_2 \times \mathbf{Z}_2$. We may choose α so that $y_a(w)\, y_b(w^{-1}) \in T$. As $y_a(w)\, y_b(w^{-1})$ has order 8, $T/T' \cong \mathbf{Z}_8 \times \mathbf{Z}_8$. Now as α inverts $ZT'/T' = \mho^1(T/T')$, $C_T(\alpha) = T'\Omega_1(Z)$.

At this point we obtain all the conclusions in one stroke. First, we put H aside and go back to G. Let G^* be any group satisfying the hypotheses of our proposition. Choose an epimorphism $*$: $G \to G^*$. Let $R \in \mathrm{Syl}_2(C_{G^*}(\alpha))$. We may arrange for $R \subseteq T^*$. Then $R \supseteq (T^*)' \cong Z_2$ and the way α acts on T/T' implies that the image of R in $(T/T')^*$ is the four-group of fixed points of α. If Z^* has rank 2, then clearly $R = (T^*)' \times Z^* \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$. If Z^* is cyclic but

$Z^* \neq \langle 1 \rangle$, then $|R \cap Z^*| = 2$. As $T^*/Z^* \cong Q_8$ and $R \supset (T^*)' \times (R \cap Z^*)$, $R \cong \mathbf{Z}_2 \times \mathbf{Z}_4$. If $Z^* = \langle 1 \rangle$, the result is clear.

PROPOSITION 2.3. *Let $G = G'$ satisfy $O(G) = 1$ and $G/Z(G) \cong L_3(4)$. Let $\alpha \in \mathrm{Aut}\, G$ induce an inner automorphism of order 2 on G. Then $R = C_G(\alpha)$ is a 2-group containing $Z = Z(G)$ and if S is the unique Sylow 2-subgroup of G containing R, then $[S, \alpha] = \Phi(Z)$. Thus $\Phi(Z)\alpha \subseteq \alpha^S$.*

(1) *If Z is elementary, then $R = S$. $Z(S)$ is the inverse image in S of $Z(S/Z)$. S has exactly two elementary subgroups of* rank $m(Z) + 4$. *Each is normal in S. Thus $SCN_4(S) \neq \varnothing$ and if $Z \neq \langle 1 \rangle$, $SCN_5(S) \neq \varnothing$.*

(2) *If $\mathbf{Z} \cong \mathbf{Z}_4 \times \mathbf{Z}_4$, then $R/Z \cong \mathbf{Z}_2 \times Q_8$. $Z_2(S) = Z \times Y_{a+b} \cong \mathbf{Z}_4 \times \mathbf{Z}_4 \times \mathbf{Z}_2 \times \mathbf{Z}_2$. Thus $SCN_4(S) \neq \varnothing$.*

(3) *If $\mathbf{Z} \cong \mathbf{Z}_4$ or $\mathbf{Z}_2 \times \mathbf{Z}_4$, R is a maximal subgroup of S. $Z_2(S) = Z \times Y_{a+b}$. Thus if $Z \cong \mathbf{Z}_2 \times \mathbf{Z}_4$, $SCN_4(S) \neq \varnothing$.*

Proof. First we consider the case $Z \cong \mathbf{Z}_4 \times \mathbf{Z}_4$. As before, we form $H = G\langle d \rangle$ with d chosen so that the actions of d and α on G commute. The structure of Aut $\bar{G}$ implies that $C_{\bar{G}}(d) \cong A_4$ and since $\langle d \rangle$ is fixed point free on Z, $C_G(d) \cong A_4$. Choose $y \in C_G(d)^{\#}$ so that $g^y = g^\alpha$ for all $g \in G$. In the notation of Proposition 2.1, we may assume that $y = y_{a+b}(1)$. Since $\Phi(Z) = [S, y]$, we get that $S/R \cong \mathbf{Z}_2 \times \mathbf{Z}_2$. Also if $x \in S$ and $x^2 \equiv y \pmod{Z}$, then $x \in R$. Thus $y_a(t) \cdot y_b(t^{-1}) \in R$ for all $t \in K^\times$. At once, this gives $\bar{R} \cong \mathbf{Z}_2 \times Q_8$. In Proposition 2.1, it was shown that $Z_2(S) = Z \times Y_{a+b}$.

Now let G^* be any group satisfying our hypotheses and let $*: G \to G^*$ be an epimorphism. Since $[S, y] = \Phi(Z)$, we obviously get $R = S$ if Z^* is elementary and that R is maximal in S if $Z^* \cong \mathbf{Z}_4$ or $\mathbf{Z}_4 \times \mathbf{Z}_2$. Again, the statements about $Z(S)$ and $Z_2(S)$ are clear. From Proposition 2.1, we know that if Z^* is elementary, then no element of $(Z^*)^{\#}$ is a square in S^*. Thus the inverse image of every elementary subgroup of S^*/Z^* is elementary.

COROLLARY 2.4. *Let G be a finite group with $Z(G)$ an elementary 2-group $Z(G) \subseteq G'$ and $G/Z(G)$ isomorphic to a subgroup of* Aut $L_3(4)$ *containing* Inn $L_3(4)$. *Let $S \in \mathrm{Syl}_2(G)$, $S_0 = S \cap G'$.*

(1) *$Z(S_0)$ is elementary of* rank $2 + m(Z(G))$.

(2) *If $S_1 \subseteq S_0$ of index 2, then $Z(S_0) \subseteq S_1$.*

(3) *If $y \in S$ with $|S : C_S(y)| \leqslant 2$, then $y \in Z(S_0)$.*

(4) *If E is a normal elementary subgroup of $S/Z(G)$ and $m(E) \geqslant 3$, then $Z(S_0)/Z(G) \subseteq E$.*

Proof. (1) is clear from 2.3. As $Z(S_0) \subseteq S_0'$, $Z(S_0) \subseteq S_1$ for any $S_1 \subseteq S_0$ of index 2. If $y \in S$ with $|S : C_S(y)| \leqslant 2$, then y centralizes $Z(S_0)$ by (2).

Also y normalizes the two elementary subgroups of S of maximum rank. Thus $y \in S_0$. If $y \notin Z(S_0)$, then y is conjugate to an element of $Z(G)\, zy$ for each $z \in Z(S_0)$. This proves (3) and also (4), once we observe that E must be in $S_0/Z(G)$.

COROLLARY 2.5. *Let G be a finite group with $Z(G) \cong \mathbf{Z}_4$, $Z(G) \subseteq G'$ and $G/Z(G) \cong L_3(4)$. Let $S_0 \in \mathrm{Syl}_2(G)$. Let $B = \Omega_1(Z_2(S_0))$. Then*

(1) *B is elementary of order* 8.

(2) $C_{S_0}(B)/Z(G) \cong \mathbf{Z}_4 \times \mathbf{Z}_4$.

(3) *S_0 induces on B the stability group of the chain $B \supseteq B \cap Z(G) \supseteq \langle 1 \rangle$.*

Proof. (1) is immediate from the fact that $\hat{X}_{a+b}$ is abelian. Let $r \in N_G(S_0)$ of order 3. Then r is fixed point free on $S_0/Z(G)$. Now S_0 stabilizes the chain $B \supseteq B \cap Z(G) \supseteq \langle 1 \rangle$. Thus $| S_0 : C_{S_0}(B)| \leqslant 4$. On the other hand, by Proposition 2.3 (3), $| S_0 : C_{S_0}(b)| = 2$ for $b \in B - Z(G)$. As r is fixed point free on $S_0/C_{S_0}(B)$, $| S_0 : C_{S_0}(B)| = 4$ and (3) follows. Finally r is fixed point free on $C_{S_0}(B)/Z(G)$, a group of order 16. Thus $C_{S_0}(B)/Z(G)$ is isomorphic either to E_{16} or to $\mathbf{Z}_4 \times \mathbf{Z}_4$. In the former case,

$$C_{S_0}(B)/Z(G) = \langle x_{a+b}(t), x_r(t) \mid t \in K \rangle$$

for some $c \in \{a, b\}$. But $\{f_{a+b,c}(1, \lambda) \mid \lambda \in K^{\times}\}$ is the full set of elements of order 2 in the Schur multiplier of $G/Z(G)$. Thus the inverse image of $\langle x_{a+b}(t), x_c(t) \mid t \in K \rangle$ in G does not have B in its center. Hence $C_{S_0}(B)/Z(G) \cong \mathbf{Z}_4 \times \mathbf{Z}_4$.

PROPOSITION 2.6. *Let $H = \mathrm{He}$, $H_0 = \mathrm{Aut}\, H$.*

(1) *The Schur multiplier of H is trivial.*

(2) $| H_0 : H | = 2$.

(3) *H has two classes of involutions with representatives z, u.*

(a) *Let $C = C_H(z)$, $Q = O_2(C)$. Then $Q \cong 2_+^{1+6}$ and $C/Q \cong GL(3, 2)$. $C = C'$ and $Z(C) = \langle z \rangle$.*

(b) *$C_H(u)$ contains a perfect subgroup, K, of index 2 with $Z(K) \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ and $K/Z(K) \cong L_3(4)$.*

(4) *H_0 has one class of involutions not in H. If $x \in I(H_0 - H)$, then $C_H(x)$ contains a perfect subgroup, J, of index 2 with*

$$Z(J) \cong \mathbf{Z}_3 \textit{ and } J/Z(J) \cong A_7 .\ C_H(x)/Z(J) \cong S_7 .$$

(5) *Let z, C and Q be as in* (3a). *Let Q_1, Q_2 be the two normal subgroups of C isomorphic to E_{16}. Let $t \in I(H_0 - H)$ normalize C. Then t interchanges Q_1 and Q_2. Consequently, $C_{G/Q}(t) \cong S_3$.*

Proof. Property (3) may be found in [21]. Property (1) is proved in [17]. The existence of an extension H_0 of H which satisfies $|H_0 : H| = 2$ and Property (4) was established by G. Higman and J. H. McKay [23].

(2) Suppose false. Then by the above there is a group G with $|G : H| = 2p$ for some prime p and there exists $x \in I(G)$ with $C_G(x) \simeq 3 \cdot S_7$. Let z, u and C be as in (3). Let $V = O_2(C_H(u))$, $N = N_G(V)$. The structure of $\mathrm{Aut}\, L_3(4)$ implies that $\langle g \rangle = C_N(O^2(N))$ has order p and meets H trivially. We may assume that $z \in O^2(N)$. Then g centralizes a maximal subgroup of a Sylow 2-subgroup of C. If $[C, g] \subseteq Q$, it follows that g centralizes C, whence g centralizes H, a contradiction. Thus $[C, g] \nsubseteq Q$ and g effects a nontrivial automorphism on $C/Q \simeq GL(3, 2)$. But then $|Q : C_Q(g)| \geqslant 4$, a contradiction.

(5) Suppose false. Then t leaves invariant the representations of C/Q on $Q_i/\langle z \rangle$, $i = 1, 2$. Thus t induces an inner automorphism of C/Q. We may choose $s \in Ct$ so that $[C, s] \subseteq Q$. Since $Q_1/\langle z \rangle$ and $Q_2/\langle z \rangle$ are nonisomorphic modules for C/Q, we get $[Q, s] \subseteq \langle z \rangle$. By the Three Subgroups Lemma, $[S, s] = 1$. Thus $C_{C\langle s \rangle}(Q) = \langle z, s \rangle$ is normal in $C\langle s \rangle$, whence $[C, s] \subseteq \langle z \rangle$. By the Three Subgroups Lemma, $[C, s] = 1$. As $s \in H_0 - H$, we get $|s| = 4$ by (4). But then Qs contains an involution s' such that $|C_H(s')|_2 = 2^7$, against (4). This proves (5).

We now catalogue the properties of $\mathscr{L}(G)$ which we shall need.

THEOREM 2.7 (L-Balance [15, Sections 3 and 4].) *Let G be a finite group, A and B 2-subgroups of G with $[A, B] \subseteq A \cap B$, N a subgroup of G normalized by A and J a 2-component of $N_G(A)$ with $[J, B] \subseteq O(J)$. Then*

(1) $L(C_N(A)) \subseteq L(N)$

(2) $L(C_J(B)) = L(C_K(A))$ *where* $K = \langle L^A \rangle$ *for some 2-component* L *of* $N_G(B)$.

DEFINITION 2.8. Let A, B, J, L be as in Theorem 2.7 with $|A| = |B| = 2$, $N = G$. We say that L *corresponds to* J and write $J \rightarrow L$. We extend $\rightarrow$ to a transitive relation $\twoheadrightarrow$ on $\mathscr{L}(G)$. We call $J \in \mathscr{L}(G)$ *maximal* if and only if $J \twoheadrightarrow M$ implies $J/Z^*(J) \simeq M/Z^*(M)$. We let $\mathscr{L}^*(G)$ be the set of maximal elements of $\mathscr{L}(G)$.

THEOREM 2.9 (Maximal 2-Component Theorem): *Let $J \in \mathscr{L}^*(G)$ with $m(J) > 1$ and $J/O(J) \not\simeq \mathrm{Sp}(4, q)$, q odd. Let $t \in I(C_G(J))$. Then $\langle J^g \mid J^g \subseteq C_G(t) \rangle \subseteq O(C_G(t))J$. Moreover if $m(J) > 2$, then $[J, J^g] \nsubseteq O(J) \cap O(J^g)$ for all $g \in G$.*

Proof. This is immediate from [28, Theorem 1.2].

DEFINITION. Let $J \in \mathscr{L}(G)$ with $O(J) \subseteq Z(J)$. Let $K = C_G(J)$. J is *standard* in G if $|K \cap K^g|$ is odd for all $g \in G - N_G(K)$ and $[J, J^g] \neq \langle 1 \rangle$ for all $g \in G$.

LEMMA 2.10. *Let $J \in \mathscr{L}(G)$ with $O(J) \subseteq Z(J)$. Suppose that $J \trianglelefteq C_G(t)$ for all $t \in I(C_G(J))$ and $[J, J^g] \neq \langle 1 \rangle$ for all $g \in G$. Then J is standard in G.*

Proof. Let $K = C_G(J)$ and suppose that $g \in G$ with $|K \cap K^g|$ even. Let $t \in I(K \cap K^g)$. Then $J \trianglelefteq C_G(t)$ and, as $t \in K^g$, $J^g \trianglelefteq C_G(t)$. Suppose that $J \neq J^g$. Then $[J, J^g] \subseteq J \cap J^g \subseteq Z(J)$. Hence, by the 3-subgroups lemma, $[J, J^g] = \langle 1 \rangle$, contrary to hypothesis. Thus $g \in N_G(J) \subseteq N_G(K)$ and J is standard in G.

COROLLARY 2.11. *Let $J \in \mathscr{L}^*(G)$ with $m(J) \geqslant 3$. Suppose that $O(J) \subseteq Z(J)$ and $O(\langle J^{L(C_G(t))} \rangle) \subseteq Z(J)$ for all $t \in I(C_G(J))$. Then J is standard in G.*

Proof. Clearly, by Theorem 2.9, it will suffice to prove that $\langle J^{C_G(t)} \rangle = J$ for all $t \in I(C_G(J))$. By Theorem 2.9,

$$\langle J^{C_G(t)} \rangle \subseteq (O(C_G(t))J)^{(\infty)} = \langle J^{L(C_G(t))} \rangle.$$

By hypothesis, $\langle J^{L(C_G(t))} \rangle = J$, and we are done.

THEOREM 2.12 (Aschbacher-Seitz [4]). *Let G be a finite group with $F^*(G)$ simple having a standard component $L \simeq$ He. Then $m_2(C_G(L)) = 1$.*

THEOREM 2.13. (1) (Nah [25]), Aschbacher-Seitz [31]: *Let G be a simple group having a standard subgroup L with $\bar{L} \simeq L_3(4)$. Then G is isomorphic to* He *or to Suzuki's sporadic simple group or to O'Nan's simple group.*

(2) (Seitz [27]): *Let G be a finite group with $F^*(G)$ simple having a standard subgroup L with $\bar{L} \simeq L_3(4)$, $|Z(L)|$ odd and $|C_G(L)|_2 = 2$. Then one of the following holds:*

(a) $F^*(G) \simeq L_3(16)$.

(b) *G contains a subgroup H with $L(H) = L_1(L_1)^t$ with $L_1/O(L_1) \simeq L_3(4)$, $t \in I(G)$, $L = L(C_{L(H)}(t))$ and $|G : H|$ odd. Also, the B-Conjecture does not hold in G.*

COROLLARY 2.14. *Let G be a finite group in which the B-Conjecture holds. Suppose that $O(G) = \langle 1 \rangle$ and $L \in \mathscr{L}(G)$ with $\bar{L} \simeq L_3(4)$. Suppose that a Sylow 2-subgroup of $Z(L)$ has 2-rank 2. Then one of the following holds:*

(1) $L \trianglelefteq G$.

(2) *$\langle L^G \rangle = KK^t$ for some $K \lhd L(G)$, $t \in I(G)$ with $L = C_{KK^t}(t)'$.*

(3) *There exists a 2-subgroup, R, of G and $M \lhd\lhd C_G(R)$ with $L \subseteq M$ and $M \simeq$* He.

Proof. By Theorem 2.7, $L \subseteq L(G)$. We assume that G is a minimal counterexample. Then $K = \langle L^{L(G)} \rangle$ is quasi-simple and $G = \langle K, t \rangle$ for some $t \in I(G)$ with $L \lhd\lhd C_G(t)$. If $t \in Z(K)$, then $L = K \trianglelefteq G$, a contradiction. Thus, by the

3-subgraphs lemma, if $\bar{G} = G/Z(K)$, then $\bar{L} \in \mathscr{L}(\bar{G})$. By [7, Corollary 1.4], there exists a chain

$$\bar{L} = L_0, L_1, \ldots, L_{n-1}, L_n = \bar{K}$$

satisfying:

(1) If $L_i = L_j$, then $i = j$.

(2) L_i is a component of $C_{\bar{G}}(T_i)$ for some 2-subgroup T_i of $\bar{G}$.

(3) For $i \geqslant 1$, $T_i \subseteq S_{i-1} \in \mathrm{Syl}_2(C_{\bar{G}}(L_{i-1}))$ and L_{i-1} is a component of $C_{\bar{G}}(\langle T_i, s_i \rangle)$ for some $s_i \in N_{S_{i-1}}(T_i) - T_i$ with $(s_i)^2 \in T_i$.

(4) $L_i \subseteq \langle (L_{i-1}) L(C_{\bar{G}}(T_i)) \rangle$.

(5) For each i, $1 \leqslant i \leqslant n$, one of the following holds:

(a) $L_i = \langle (L_{i-1})^{L(C_{\bar{G}}(T_i))} \rangle$ and $L_{i-1} C_{\bar{G}}(L_i)/C_{\bar{G}}(L_i)$ is standard in some subgroup of $N_{\bar{G}}(L_i)/C_{\bar{G}}(L_i)$ containing $L_i C_{\bar{G}}(L_i)/C_{\bar{G}}(L_i)$.

(b) $\langle (L_{i-1})^{L(C_{\bar{G}}(T_i))} \rangle = L_i (L_i)^{s_i}$ with $L_i \neq (L_i)^{s_i}$ and $L_i/Z(L_i) \simeq L_{i-1}/Z(L_{i-1})$

If $\bar{K} \simeq \bar{L}$, then $L = K \leqslant G$, a contradiction. Thus there is some i, $0 \leqslant i \leqslant n - 1$, with $L_i/Z(L_i) \simeq L_3(4)$ and $L_i C_{\bar{G}}(L_{i+1})/C_{\bar{G}}(L_{i+1})$ standard in some subgroup of $N_{\bar{G}}(L_{i+1})/C_{\bar{G}}(L_{i+1})$. It follows from Theorem 2.13 that $L_{i+1}/Z(L_{i+1}) \simeq L_3(16)$, He, Suzuki's sporadic simple group or O'Nan's simple group. For $j = 0, 1, \ldots, n$, let M_j be the commutator subgroup of the inverse image of L_j in G and let R_j be the inverse image of T_j in G. Then by conditions (2)–(5) and the choice of i, $M_i/Z_i \simeq L$ for some $Z_i \subseteq Z(M_i)$. In particular $M_i/Z(M_i) \simeq L_3(4)$ and $Z(M_i)$ has 2-rank 2. Thus by Theorem 2.13, $M_{i+1} \simeq$ He and conclusion (3) of the corollary holds with $M = M_{i+1}$, $R = R_{i+1}$.

THEOREM 2.15 (Gilman-Solomon [9]). *Let G be a minimal counterexample to the U-Conjecture. Suppose that (a, x, J) is an unbalancing triple in G. Then*

(1) *If $\tilde{J} \simeq L_3(4)$, then either $J \in \mathscr{L}^*(G)$ or $J \twoheadrightarrow L \in \mathscr{L}^*(G)$ with $L/O(L) \simeq$* He.

(2) *If $\tilde{J} \simeq A_7$. Then $J \twoheadrightarrow L \in \mathscr{L}(G)$ with $L/O(L) \simeq$* He. *Moreover, for all $a \in I(C_G(J))$ and all 4-subgroups, E, of $N_G(J) \cap C_G(a)$, we have*

$$[J, W_E \cap C_G(a)] \subseteq O(C_G(a)).$$

(3) *If $\tilde{J} \simeq L_2(q)$, then either $J \in \mathscr{L}^*(G)$ or $q \in \{5, 7\}$ and $J \twoheadrightarrow L \in \mathscr{L}^*(G)$ with $L/Z^*(L) \simeq L_3(4)$ and (b, y, L) an unbalancing triple in G for some (b, y).*

PROPOSITION 2.16. *Let G be a minimal counterexample to either Theorem 1.1 or Theorem 1.2. Then one of the following holds:*

(1) *G is simple.*

(2) *$G = G'\langle t \rangle$ for some $t \in I(G)$. G' is simple and $L \trianglelefteq C_G(t)$ with $L/O(L) \simeq$* He *or $L_3(4)$. Also $L \in \mathscr{L}^*(G)$.*

Proof. Clearly $O(G) = \langle 1 \rangle$. Let $t \in I(G)$, L a 2-component of $C_G(t)$ with either $\tilde{L} \cong \mathrm{He}$ or $\tilde{L} \cong L_3(4)$, $|Z^*(L)|$ even and (t, x, L) an unbalancing triple for some $x \in C_G(t)$. By L-Balance, $L \subseteq L(G)$. Set $M = \langle L^{L(G)} \rangle$. Assume that M is not quasi-simple. Then G is a counterexample to Theorem 1.1. Since L is subnormal in $C_G(t)$ and projects nontrivially into each component of $M/Z(M)$, it follows firstly, that t normalizes each component of M, and secondly that M is quasisimple, since $LZ(M)/Z(M) \trianglelefteq C_M(t)\, Z(M)/Z(M)$. Thus M is quasi-simple and, clearly, if G is a minimal counterexample to Theorem 1.1, then $G = \langle M, t \rangle$. Thus in all cases, $M = L(G)$ is quasi-simple and $Z(G) \subseteq M$. Let bars denote homomorphic images in $\bar{G} = G/Z(G)$. If G is a minimal counterexample to Theorem 1.1 and $Z(G) \neq \langle 1 \rangle$, then $\bar{G}$ is balanced and $\bar{L} \in \mathscr{L}(\bar{G})$ with $\bar{L}/O(\bar{L}) \cong \mathrm{He}$. Then $M \cong \mathrm{He}$, a contradiction. Thus we may assume G is a minimal counterexample to Theorem 1.2 and there exists an unbalancing triple (a, x, J) in G satisfying the hypotheses of Theorem 2.15. Moreover, as G is not a counterexample to Theorem 1.1, $J \in \mathscr{L}^*(G)$. Now $J \subseteq M$. As $J \neq M$ and $J \in \mathscr{L}^*(G)$, $Z(M) = \langle 1 \rangle$. Thus M is simple and if $Z^*(J) = O(J)$, then (2) holds with $L = J$.

Hence we may assume that $|Z^*(J)|$ is even and $G \neq M$. Then M is balanced. Thus $O(J) = \langle 1 \rangle = O(\langle L^{L(C_G(t))} \rangle)$ for all $t \in I(C_G(J))$. Hence J is standard in the simple group M and Aut M is unbalanced.

As G is a counterexample to the U-Conjecture, $M \ncong \mathrm{He}$. Thus by Theorem 2.13, $|Z(J)|$ is odd and G contains a subgroup H with $L(H) = J_1 J_2$, $J_i/O(J_i) \cong L_3(4)$, $J = L(C_{L(H)}(t))$ and $|G : H|$ odd. Let $S \in \mathrm{Syl}_2(H \cap M)$. Then $|S| \leqslant 2^{16}$ because $G \neq M$. Let $S_i = S \cap J_i$, $Z_i = Z(S_i)$. Let $z_1 \in I(Z_1)$, $L = \langle L(C_{J_2}(z_1))^{L(C_G(z_1))} \rangle$. As M is a balanced group, Theorem 2.13 implies that L is isomorphic to $L_3(4)$, $L_3(16)$ or Suz or $L = L_1 L_2$ with $L_i/Z(L_i) \cong L_3(4)$. Now S_1 normalizes L. Assume that $L \ncong L_3(4)$. Then $|L| \geqslant 2^{12}$, in which case $|S| \leqslant 2^{16}$ implies that $|C_{S_1}(L)| \leqslant 2^4$. As S_1 centralizes $J_2 \cap L$ and $|S_1 : C_{S_1}(L)| \geqslant 4$, $L \cong \mathrm{Suz}$. But then $|L| = 2^{13}$ and $|C_{S_1}(L)| = 2^6$, a contradiction. Thus $L = L(C_{J_2}(z_1)) \cong L_3(4)$. It follows that $L \trianglelefteq C_M(t_1)$ for all $t_i \in I(S_1)$. Thus L is standard in the balanced group M. By Theorem 2.13, M is isomorphic to $L_3(16)$ or Suz. But then Aut M is a balanced group, contrary to hypothesis.

This proves that $G = M$, a simple group.

Corollary 2.17. *Let G be a minimal counterexample to Theorem* 1.2. *Suppose Theorem* 1.1 *is valid for G. Then G is simple and if (a, x, J) is an unbalancing triple in G with $\tilde{J} \cong L_3(4)$, then $J \in \mathscr{L}^*(G)$ and $G = \langle G', a \rangle$.*

Proof. This is immediate from Theorem 2.15 and Proposition 2.16.

We now collect some results on transfer and fusion.

Lemma 2.18 (Up-and-Down Fusion). *Let G be a finite group, $M \subseteq G$, $S \in \mathrm{Syl}_2(M)$, $a, b \in S$ with a an M-extremal G-conjugate of b in S.*

(i) *Suppose* $S \in \mathrm{Syl}_2(C_G(a))$ *and* $a^G \cap S = \{a\} \cup b^S$. *Then for some* $b' \in b^S$, *there exists* $g \in N_G(C_S(b'))$ *with* $(b')^g = a$.

(ii) *Suppose that* $S \in \mathrm{Syl}_2(M)$ *with* $m_2(S) \geqslant 3$, *and that, whenever* E, E^g *are two eights-groups in* M, $g \in M$. *Then* $S \in \mathrm{Syl}_2(G)$ *and if* $m_2(C_S(b)) \geqslant 3$, *then* $b \in a^M$.

(iii) *Suppose* $S \in \mathrm{Syl}_2(G)$ *and* $J \trianglelefteq M$. *Let* $R = S \cap J$ *and* $Z = \Omega_1(R \cap Z^*(J)) \subseteq C_R(b)$. *Suppose that if* E *and* E^g *are two elementary eights-subgroups of* R *containing* Z, *then* $g \in M$. *Then, if* $m_2(C_R(b)) \geqslant 3$, $b \in a^M$.

Proof. Suppose that $S \notin \mathrm{Syl}_2(G)$ and let $S < T \in \mathrm{Syl}_2(G)$. Let $r \in N_T(S) - S$ with $r^2 \in S$. In (i), set $b' = a^r$. Then $S = C_S(b')$ and $(b')^r = a$. Thus (i) holds in this case. Suppose we are in case (ii). Then if E is an eights-group in S, so is E^r. Thus $\langle S, r\rangle$ is a 2-subgroup of M, a contradiction. Thus we may assume in all cases that $S \in \mathrm{Syl}_2(G)$.

By Up-and-Down Fusion [1], there exist chains of elements of S,

$$b = b_0, b_1, b_2, \ldots, b_n = a,$$

subgroups, $H_1, H_2, \ldots, H_n$ of S and elements g_i of $N_G(H_i)$, $i = 1, 2, \ldots, n$, with $b_{i-1}^{g_i} = b_i$ and $C_S(b_{i-1}) \subseteq H_i$. Choose such a chain of minimal length.

(i) Since $a^G \cap S = \{a\} \cup b^S$, $b_{n-1} \in b^S$. Set $b' = b_{n-1}$, $g = g_n$. Since $H_{n-1} \subseteq S$ and $a \in Z(S)$, we get $b' \in Z(H_{n-1})$. As $C_S(b') \subseteq H_{n-1}$, $C_S(b') = H_{n-1}$.

(iii) Suppose $m_2(H_i \cap J) \geqslant 3$ and $Z \subseteq H_i$. If $g \in N_G(H_i \cap J)$, then g permutes the elementary subgroups of $H_i \cap J$ of any given rank. By our hypothesis, $g \in M$. As $C_S(b) \subseteq H_1$, we get

$$m_2(H_1 \cap J) \geqslant m_2(C_S(b) \cap J) \geqslant m_2(C_R(b)) \geqslant 3.$$

Also $Z \subseteq H_1$. Thus $g_1 \in M$. Now $C_S(b)^{g_1} \subseteq C_S(b_1)$. So, $C_R(b)^{g_1} \subseteq C_{R^{g_1}}(b_1) \cap S$. As $R^{g_1} \cap S \subseteq J \cap S = R$, since $J \trianglelefteq M$, we get $C_R(b)^{g_1} \subseteq C_R(b_1)$. Thus $Z \subseteq C_R(b_1)$ and $m_2(C_R(b_1)) \geqslant 3$. Hence $g_2 \in M$, and we may repeat the argument to get $g_i \in M$ for all i. So, $b \in a^M$.

Part (ii) may be proved in the same way as part (iii).

LEMMA 2.19 (Goldschmidt [11]). *Let* G *be a finite group,* $S \in \mathrm{Syl}_2(G)$, $a \in I(S)$. *Then either* a *has an extremal* G-*conjugate in every maximal subgroup of* S *or* $a \notin O^2(G)$.

THEOREM 2.20. *Let* G *be a finite group with* $F^*(G)$ *simple.*

(1) (Goldschmidt [12]): *Suppose that* A *is a strongly closed* 2-*subgroup of* G. *Then* $C_G(A)$ *is solvable.*

(2) (Holt [24]): *Suppose that G acts transitively on a set X and* $t \in I(G)$ *with* $|G : C_G(t)|$ *odd, fixing exactly one point of X. Then* $L(C_G(t)) = \langle 1 \rangle$.

We now discuss the U-Conjecture and the context of this paper. Major work of Gorenstein-Harada [13], Gorenstein-Walter [15], Aschbacher [2] and others proves the following result.

THEOREM 2.21. *Let G be a finite unbalanced group with* $F^*(G)$ *quasi-simple. Suppose that there is no unbalancing* $J \in \mathscr{L}(G)$. *Then* $F^*(G)/Z(F^*(G))$ *is isomorphic to* A_7, $L_3(4)$, *or* $L_2(q)$ *for some* odd q.

Proof. By the above cited results. G has a sectional 2-rank at most 4, whence G is known by [13]. If $F^*(G)/Z(F^*(G))$ is a simple Chevalley group of odd characteristic, then by Proposition A of [5], either there is an unbalancing $J \in \mathscr{L}(G)$ or $F^*(G)/Z(F^*(G)) \simeq L_2(q)$ for some odd q. The remaining unbalanced groups of sectional 2-rank at most 4 have $F^*(G)/Z(F^*(G)$ isomorphic to A_7, A_9, A_{11} or $L_3(4)$. If $F^*(G)/Z(F^*(G))$ is isomorphic to A_9 or A_{11}, then G has an unbalancing $J \in \mathscr{L}(G)$ with $J/Z(J) \simeq A_7$. Thus the result holds.

Thus if G is a minimal counterexample to the U-conjecture, then G contains an unbalancing $J \in \mathscr{L}(G)$. Moreover, $N_G(J)$ is an unbalanced group, whence by minimal choice of G, $J/Z^*(J)$ is isomorphic to one of the groups listed in the conclusion of the U-Conjecture. The next result eliminates most of these cases.

THEOREM 2.22 (Aschbacher [3], Thompson, Burgoyne [5], Solomon [29, 30], Foote [8], Gorenstein-Harada [13], Harris [18], Harris-Solomon [20], Gilman-Solomon [9]). *Let G be a minimal counterexample to the U-Conjecture. Let J be an unbalancing 2-component in G. Then one of the following holds*:

(1) $J/O(J) \simeq L_2(q)$ *for some odd* $q \geqslant 49$ *and a Sylow 2-subgroup of* $C_G(J/O(J))$ *is cyclic.*

(2) $J/O(J) \simeq L_2(q)$ *for some* $q \in \{5, 7\}$ *and* $J \twoheadrightarrow L \in \mathscr{L}^*(G)$ *with* $\tilde{L} \cong L_3(4)$ *and* (b, y, L) *an unbalancing triple in G for some* (b, y).

(3) $\tilde{J} \cong L_3(4)$.

(4) $\tilde{J} \cong$ He *and* $J \in \mathscr{L}^*(G)$.

(5) $J/O(J) \cong A_7$ *and* $J \twoheadrightarrow L \in \mathscr{L}^*(G)$ *with* $\tilde{L} \cong$ He. *Moreover for all* $a \in I(C_G(J))$ *and all 4-subgroups E of* $N_G(J) \cap C_G(a)$,

$$[J, W_E \cap C_G(a)] \subseteq O(C_G(a)).$$

Proof. The results enumerated above eliminate all possibilities for J except those listed above and the following:

(6) $J/Z^*(J)$ is a Chevalley group over $GF(7)$ and G contains another unbalancing triple (b, y, K) with $K/Z^*(K) \cong L_3(4)$, $b \in Z(K)$, $\langle b, y\rangle \subseteq S \in$

$\mathrm{Syl}_2(N_G(K))$, $Z = S \cap Z^*(K) \simeq \mathbf{Z}_4 \times \mathbf{Z}_4$ and $v \in C_S(\langle b, y\rangle)$ induces a field automorphism on $\tilde{K}$. We shall eliminate this configuration and thereby prove Theorem 2.22.

First we argue that $b^G \cap C_S(\tilde{K}) \subseteq Z$. Suppose on the contrary that $b^g \in C_S(\tilde{K}) - Z$. By Theorem 2.15, both K^g and $\langle L(C_K(b^g))^{L(C_G(b^g))}\rangle = K^*$ are maximal in G. As $b^g \in Z(K^g)$ but $b^g \notin Z(K^*)$, K^g and K^* are distinct 2-components of $C_G(b^g)$. This violates Theorem 2.9.

As $C_Z(v)$ is cyclic, $\Omega_1(Z(S)) \cap Z = \langle b\rangle$. As $Z(S) \subseteq C_S(\tilde{K})$, we have shown that $b^G \cap Z(S) = \{b\}$. Thus $S \in \mathrm{Syl}_2(G)$. Suppose that v^g is an extremal G-conjugate of v in S with $v^g \notin v^{C_G(b)}$ and $C_S(v)^g \subseteq S$. Then $b^g \in S - C_S(\tilde{K})$. Let $\langle z_1\rangle = C_Z(v)$, $b_0 = b^g$, $z_0 = z_1{}^g$, $T = C_S(v)^g$ and let bars denote images in $\bar{S} = S/C_S(\tilde{K})$. Then $\bar{b}_0 \neq 1$ implies that $|\bar{z}_0| = 4$ and $|\bar{T}| \geqslant 2^5$. Let $\bar{S}_0 = \bar{S} \cap K$. If r is an involution of $\bar{T}$ not in $\bar{S}_0$, then $|\bar{T}| \geqslant 2^5$ implies that r induces a field or unitary automorphism on $\bar{K}$. Then $\langle \bar{b}_0\rangle \subseteq Z(\bar{T})$ implies $\bar{T} \cap \bar{S}_0 = \langle \bar{b}_0\rangle$ and so $|\bar{T}| \leqslant 2^4$, a contradiction. Therefore $\Omega_1(\bar{T}) \subseteq \bar{S}_0$, whence $C_{\bar{S}_0}(v) \simeq Z_4 \circ Q_8$ implies that $|\bar{T} \cap \bar{S}_0| \geqslant 2^5$. Again, $\langle \bar{b}_0\rangle \subseteq Z(\bar{T} \cap \bar{S}_0)$ and Corollary 2.4(3) are in conflict. Thus, by Lemma 2.19, $v \notin O^2(G)$. But G is simple by Proposition 2.15, and we are done.

Finally we discuss the 2-balanced functor and its relevance to this paper.

LEMMA 2.23. *Let $H \subseteq G \subseteq \mathrm{Aut}\, H$ with H isomorphic to $L_3(4)$,* He *or $L_2(q)$ for some odd q. If G is not 2-balanced, then $H \simeq L_2(q)$ for some* odd $q \geqslant 27$.

Proof. If H is isomorphic to $L_3(4)$ or He and $t \in I(G)$ with $O(C_G(t)) \neq \langle 1\rangle$, then t is in a unique non-trivial coset of H in G. Thus if A is a 4-subgroup of G with $W_A \neq \langle 1\rangle$, then $H \simeq L_2(q)$. Also every involution of A induces an inner or diagonal automorphism on H. If $q < 27$, then $H = O^{2'}(\mathrm{Aut}\, H)$. Thus we may assume that $G = HA$. But then $C_G(a)$ is dihedral for each $a \in A^\#$. Hence $W_A = \langle 1\rangle$ in this case as well.

The work of Gilman-Solomon [9] shows that if G is a minimal counterexample to the U-Conjecture with unbalancing triple (a, x, J), $J/O(J) \simeq L_2(q)$, $q \geqslant 9$, then G has a maximal 2-component K with $K/O(K) \simeq L_2(q_1)$ for some $q_1 \geqslant 9$. Foote has then shown in [8] that $m_2(C_G(\tilde{K})) = 1$. The independent classification problem of handling the case $m_2(C_G(\tilde{K})) = 1$ has been undertaken by M. Harris. As this problem is unaffected by our considerations, it is convenient for us to ignore such unbalancing components. Then Theorem 2.22 and Lemma 2.23 justify our hypothesis of local 2-balance. The significance of this hypothesis for us rests in the following theorem and it consequences.

THEOREM 2.24 (Signalizer Functor Theorem of Gorenstein-Goldschmidt [10]). *Let A be an elementary 2-subgroup of G with $m(A) \geqslant 4$. Then $|W_A|$ is odd.*

DEFINITION 2.25. Let A, B be abelian 2-groups in G and $k \geqslant 2$ an integer. We say that A and B are *k-connected* if there exists a sequence $A = A_1, A_2, \ldots,$

$A_n = B$ such that each A_i is an abelian 2-group of rank at least k and $[A_i, A_{i+1}] = 1$ for $i = 1, 2, \ldots, n-1$. If S is a 2-subgroup of G with $m(S) \geqslant 3$, we note that S is 3-*connected* if A and B are 3-connected for any two eights-subgroups A and B of S.

COROLLARY 2.26 (Gorenstein-Walter [15]). *Let A be an elementary 2-subgroup of G with $m(A) \geqslant 4$. Let E and E_1 be abelian 2-subgroups of G which are 3-connected to A. Let $M = N_G(W_A)$. Then*

(1) $W_E = W_{E_1} = W_A$

(2) *If $E_1 = E^g$, then $g \in M$.*

(3) *If $S \in \mathrm{Syl}_2(M)$ is 3-connected, then $S \in \mathrm{Syl}_2(G)$ and if $a, b \in S$ with a an extremal G-conjugate of b and $m(C_S(b)) \geqslant 3$, then $b \in a^M$.*

Proof. (1) and (3) are in [15]. (3) follows from 2 and Lemma 2.18(ii). We remark that even if a Sylow 2-subgroup of M is not 3-connected, we can use Lemma 2.18 (iii) to obtain information about G-fusion of involutions in M.

We conclude this section with some useful lemmas about balance and connectivity.

LEMMA 2.27. *Let G be a finite group and let S and T be non-trivial commuting elementary abelian 2-subgroups of G. Let $D = O(C_G(S)) \cap C_G(T)$. Let T be a 2-component of $C_G(T)$. Then*

(1) $[O_{2',2}(C_G(T)), D] \subseteq O(C_G(T))$; $[O_2(C_G(T)), D] = 1$.

(2) *D normalizes L.*

(3) *If $[L, D] \nsubseteq O(L)$, then $S_L = S \cap N_G(L) \neq 1$ and $L = O(L)[L, D] = O(L)[L, S_L]$.*

(4) *If $D \nsubseteq O(C_G(T))$, then $[M, D] \nsubseteq O(M)$ for some 2-component, M, of $C_G(T)$.*

Proof. See [28, Lemma 2.5].

LEMMA 2.28. *Let G be a finite group with $F^*(G)$ simple, all of whose proper sections satisfy the U-Conjecture. Let L be a 2-component of $C_G(t)$ for some $t \in I(G)$. Let E be an elementary 2-subgroup of $C_G(\tilde{L}) \cap C_G(t)$ with $m(E) \geqslant 2$ and $W_E = \langle 1 \rangle$. Suppose that either $E \subseteq C_G(L)$ or $m(E) \geqslant 3$. Then $O(L) \subseteq Z(L)$.*

Proof. Let $X = O(L)$. For a subset F of E, let $L_F = L(C_L(F))$, $X_F = C_X(F)$, $Y_F = [X_F, L_F]$. Then $X_E \subseteq X_F$ and $L_E \subseteq L_F$, whence

$$Y_E \subseteq Y_F \subseteq M_e = \langle (L_e)^{L(C_G(e))} \rangle$$

for all $e \in F^{\#}$. If M_e is a product of two 2-components of $C_G(e)$, then clearly $Y_e \subseteq O(M_e)$. If M_e is a single 2-component, then as L_e is a 2-component of

$C_{M_e}(t)$ and $M_e < G$, $Y_e \subseteq O(M_e)$ again. Thus in all cases, $Y_F \subseteq \bigcap_{e \in F^\#} O(C_G(e))$. Hence if $m(F) \geqslant 2$, then $Y_F = 1$. Now if $E \subseteq C_G(L)$, then $1 = Y_E = [X_E, L_E] = [X, L]$. So $X \subseteq Z(L)$, as desired. Suppose that $m(E) \geqslant 3$. Then $[X_F, L_E] \subseteq Y_F = 1$, for all $F \subseteq E$ with $m(F) \geqslant 2$. Now

$$X = \langle X_F \mid F \subseteq E, m(F) \geqslant 2 \rangle.$$

Thus $[X, L_E] = 1$. Then $C_L(X) \trianglelefteq L$ and $L = XC_L(X)$. Thus $C_L(X) \supseteq O^{2'}(L) = L$, as desired.

LEMMA 2.29. *If S is a 2-group with $SCN_5(S) \neq \varnothing$, then S is 3-connected.*

Proof. Let E be a normal elementary subgroup of S of rank at least 5. It is enough to show that if A is an eights-subgroup of S, then A is 3-connected to E. Let $1 \subset A_0 \subset A_1 \subset A$ be a strictly increasing chain of subgroups of A. Let $E_i = C_E(A_i)$, $E_2 = C_E(A)$. Then $E_2 \neq \langle 1 \rangle$, $m(E_1) \geqslant 2$ and $m(E_0) \geqslant 3$. Then A, $\langle A_1, E_2 \rangle$, $\langle A_0, E_1 \rangle$, E_0, E is a chain which 3-connects A to E.

3. THE HELD GROUP CASE

Throughout this section G will be a minimal counterexample to Theorem 1.1. We fix $L \in \mathscr{L}^*(G)$ with $\tilde{L} \simeq \mathrm{He}$. Let $S \in \mathrm{Syl}_2(N_G(L))$, $P = S \cap C_G(\tilde{L})$ and $t \in I(Z(S) \cap C_G(L))$. As a consequence of Theorem 2.9, L is a 2-component of $C_G(t)$. By Proposition 2.16, $G = \langle G', t \rangle$ with G' simple.

LEMMA 3.1. *The Sylow 2-subgroups of* He *and* Aut He *are 3-connected.*

Proof. Let $H = \mathrm{He}$ and $H_0 = \mathrm{Aut\,He}$. We may assume $H_0 \supseteq H$. So $|H_0 : H| = 2$ by Proposition 2.6(2). Take $T \in \mathrm{Syl}_2(H)$ and let $T \subset T_0 \in \mathrm{Syl}_2(H_0)$. Let $\langle z \rangle = Z(T) \simeq \mathbf{Z}_2$, $C = C_H(z)$, $Q = O_2(C) \simeq 2_+^{1+6}$.

First, we take $A \simeq E_8$, $A \subseteq T_0$, $A \nsubseteq T$ and show that A is 3-connected to some $E \subseteq T$, $E \simeq E_8$. Let $E_8 \simeq B \subseteq Q$. We may assume $z \in A \cap B$. Choose $a \in A - S$. By Proposition 2.6(5), $C_{C/Q}(a) \simeq S_3$ and $Q_1{}^a = Q_2$ where Q_1 and Q_2 are the normal subgroups of C isomorphic to E_{16}. Thus $BB^a \simeq 2_+^{1+4}$ and $D = C_{BB^a}(a) \simeq E_8$. Now take $a_1 \in (A \cap T) - \langle z \rangle$. Then $|C_D(a_1)| \geqslant 4$. Thus A is 3-connected to $E = \langle A \cap T, C_D\langle a_1 \rangle \rangle$ which is in T, as required.

Now let $A, B \simeq E_8$, $A \subseteq T$ and $B \subseteq Q$, $B \trianglelefteq S$. We may assume $A \cap B \supseteq \langle z \rangle$. Let Q_1, Q_2 be as above with $B \subseteq Q_1$. Suppose $|A \cap Q| \geqslant 4$. For $a \in (A \cap Q) - \langle z \rangle$, $|C_{Q_1}(a)| \geqslant 8$. Thus, $|C_{Q_1}(A)| \geqslant 4$ and A is 3-connected to B. Now assume $A \cap Q = \langle z \rangle$. Let $a \in A - \langle z \rangle$. Then $D = \langle [Q_1, a], [Q_2, a], z \rangle \simeq E_8$. Thus $|C_D(A)| \geqslant 4$ and A is 3-connected to $\langle C_D(A), a \rangle$, which by our previous argument is 3-connected to B. Thus A is 3-connected to B in all cases.

Corollary 3.2. *$W_A = \langle 1\rangle$ for every 4-subgroup A of $N_G(L)$.*

Proof. It follows trivially from Lemma 3.1 and Corollary 2.26 that $W_A = W_B$ for every 8-subgroup A of S. Then if $N = N_G(W_A)$, we have $N_G(B) \subseteq N$ for every 8-subgroup B of S. By Proposition 2.1(iv) and the fact that $C_S(L) \neq 1$, $L = \langle N_L(B) \mid B \subseteq S,\ B \text{ an 8-group}\rangle \subseteq N$. If $N = G$, then as $W_B = \langle 1\rangle$ for every 4-subgroup B_0 of B, the Corollary follows from Sylow's Theorem.

We shall derive a contradiction from the assumption that $N < G$. As G is a minimal counterexample to Theorem 1.1, induction and Theorem 2.9 yield

$$L^* = \langle L^N\rangle = O(L^*)L.$$

Thus $N = O(N)\, N_G(L)$ and $S \in \mathrm{Syl}_2(N)$. Now $m_2(C_{\bar{L}}(s)) \geqslant 3$ for all $s \in I(\mathrm{Aut}\,\bar{L})$. Thus $m_2(C_N(s)) \geqslant 3$ for all $s \in N$ with $s^2 \in P$. Thus, by Corollary 2.26(iii), $S \in \mathrm{Syl}_2(G)$ and if s, $s^g \in S$ with $s^2 \in P$ and $(s^g)^2 \in P$, then $g \in N$. Thus P is strongly closed in G, but $C_G(P)^{(\infty)} \supseteq L(C_L(P)) \neq \langle 1\rangle$, violating Theorem 2.20(1).

Lemma 3.3. *L is standard in G.*

Proof. By Corollary 2.11, it suffices to prove that $O(L) = \langle 1\rangle$ and that $O(\langle L^{C_G(t)}\rangle) = \langle 1\rangle$ for all $t \in I(C_G(L))$.

Let U be a 4-subgroup of $S \cap L$ with $\bar{L}_U = L(C_L(U))/Z^*(L(C_L(U)) \simeq L_3(4)$. Let $u \in U^{\#}$ and let $K = \langle L(C_L(u))^{L(C_G(u))}\rangle$. Hypothesis 1 of Theorem 1.1 permits us to apply Corollary 2.14 in K to conclude that $K/Z^*(K)$ is isomorphic either to $L_3(4)$ or to $L_3(4) \times L_3(4)$. In particular, $U \subseteq Z^*(L(C_G(u)))$. Let K_1 be a 2-component of K. If $K_1 = K$, then $U \times P \subseteq C_S(\bar{K}_1)$. If $K = K_1K_1{}^t$, then let $Z_1 \in \mathrm{Syl}_2(Z^*(K) \cap N_K(S \cap C_K(t)))$. Then $m_2(Z_1) = 4$ and Z_1 is 3-connected to $S \cap C_K(t)$. Thus by Lemmas 3.2 and 2.28, $O(K) \subseteq Z(K)$ and $U \subseteq Z(E(C_G(u))) \subseteq O_2(C_G(u))$. Thus we have $[U, C_{O(L)}(u)] \subseteq O_2(C_G(u)) \cap O(L) = \langle 1\rangle$. Then $C_{O(L)}(u) = \langle C_{O(L)}(u_1) \mid u_1 \in U^{\#}\rangle = O(L)$. Hence $O(L) \subseteq Z(L) = \langle 1\rangle$.

As the same argument applies to $\langle L^{C_G(t)}\rangle$ for any $t \in I(C_G(L))$, the proof is complete.

Corollary 3.4. $m_2(P) = 1$.

Proof. This is immediate from Theorem 2.12 and Lemma 3.3.

We now fix a 4-subgroup U of $S \cap L$ with $L(C_L(U))/U \simeq L_3(4)$ and we fix $u \in U^{\#}$. Let $t \in I(P)$ and $K = L(C_L(U))$.

Lemma 3.5. *$K \trianglelefteq C_G(u)$ and K is the unique normal subgroup in $C_G(u)$ of its isomorphism type.*

Proof. We consider first $K^* = \langle K^{L(C_G(u))}\rangle$. By Lemma 2.28, $O(K^*) \subseteq Z(K^*)$. We argue that $K^* = K$. So, assume $K^* \neq K$. Again, by Hypothesis 1 of Theorem 1.1, we may apply Corollary 2.14 to K^* and conclude that $K^* = K_1K_1{}^t$ with $K_1/Z(K_1) \simeq K/Z(K)$ and $m_2(Z(K^*)) = 4$. Also $Z(K^*)$ is 3-connected to every eights group in $S \cap K$. Thus if $v \in I(Z(K^*))$, then by Lemma 2.28, $L(C_G(v)) = E(C_G(v))$ and if $K_1 \leqslant E(C_G(v))$, then $\langle K_1^{E(C_G(v))}\rangle = L_1 \simeq \mathrm{He}$. Now the proofs of Lemmas 3.2 and 3.3 apply to prove that L_1 is standard in G. But $m_2(C_G(L_1)) > 1$, violating Theorem 2.12. Thus $K^* \trianglelefteq E(C_G(v))$ for each $v \in I(Z(K^*))$. As $E(C_G(U)) = K_1E(C_G(K_1))$, we have for $w \in I(C_G(K_1))$ that $m_2(C_{Z(E(C_G(U)))}(w)) \geqslant 3$. We conclude first, using Corollary 2.14, that $\langle K_1^{L(C_G(w))}\rangle = O(\langle K_1^{L(C_G(w))}\rangle)K_1$ and, second, using Lemma 2.28, that $O(\langle K_1^{L(C_G(w))}\rangle) \subseteq Z(\langle K_1^{L(C_G(w))}\rangle)$. Thus $K_1 \trianglelefteq L(C_G(w))$ for all $w \in I(C_G(K_1))$. Hence, $K_1 \in \mathscr{L}^*(G)$. But $K_1K_2 = K_2K_1$, violating Theorem 2.9. Thus $K^* = K$. Now $U \times P \subseteq C_G(K)$. Again, using Lemma 2.28 and Corollary 2.14, we deduce that $K \leqslant L(C_G(u))$.

Now $U \times P \in \mathrm{Syl}_2(C_G(\langle t\rangle \times K))$. Thus $K \trianglelefteq C_G(u)$. The last statement follows for the same reasons.

LEMMA 3.6. *If $v \in I(L)$ and $K_1 \trianglelefteq C_G(v)$ with $K_1 \simeq K$, then $v \in u^L$.*

Proof. Suppose that $v \in I(L)$, $v \notin u^L$ and $K_1 \trianglelefteq C_G(v)$ with $K_1 \simeq K$. Let $H = C_L(v)$. Then $H = H^{(\infty)} \subseteq K_1C_G(K_1)$. If $H_0 \trianglelefteq H$ and $H_0 \neq H$, then $H_0 \subseteq O_2(H)$. As $\langle u^L \cap H\rangle \not\subseteq O_2(H)$, $\langle u^L \cap H\rangle = H$. As $\{u^L\} \cap C_G(K_1) = \varnothing$, $|C_H(K_1)| = 2$ and $H/C_H(K_1)$ is embedded in K, which is impossible.

COROLLARY 3.7. $C_G(u) \subseteq N_G(L)$.

Proof. By Lemma 3.6, $u^G \cap L = u^L$. For $u_1 \in u^L$, let $K(u_1)$ be the normal subgroup of $C_G(u_1)$ isomorphic to K. Let $\mathscr{V} = \{V \subseteq K \mid V \simeq \mathbf{Z}_2 \times \mathbf{Z}_2,\ V \cap Z(K) = \langle u\rangle\}$. By the known fusion in L, $|u^L \cap V| = 2$ for all $V \in \mathscr{V}$. Take $V \in \mathscr{V}$ and let $u_1 \in u^L \cap (V - \langle u\rangle)$. Then $K(u_1) \subseteq L$ but $K(u_1) \not\subseteq N_L(K)$. Thus $\langle K(w) \mid w \in \langle u, u_1\rangle^{\#}\rangle = \langle K(u), K(u_1)\rangle = L$ for all $\langle u_1, u\rangle \in \mathscr{V}$ with $u_1 \in u^L$. As $K \trianglelefteq C_G(u)$, $C_G(u)$ permutes the members of $\mathscr{V}$. Hence $C_G(u)$ normalizes L.

We can now complete the proof of Theorem 1.1. By Glauberman's Z^*-Theorem, there exist $t_1 \in (t^G \cap N_G(L)) - \{t\}$. By the properties of Aut He, $t_1 \in C_G(u_1)$ for some $u_1 \in u^L$. $L_1 = E(C_G(t_1))$. The structure of $C_G(u)$ implies that $C_{L_1}(u_1) \simeq K$ or an extension of Z_3 by S_7. Since $C_G(u_1) \subseteq N_G(L)$, t_1 centralizes $C_{L_1}(u_1) \simeq K$, whence $C_G(\langle u_1, t_1\rangle) = C_G(u_1) \subseteq C_G(t_1)$. More precisely, $t_1 \in t\langle u_1\rangle$ and $t_1 \neq t$, so that $t_1 = tu_1$. This argument proves that if t_1 centralizes $u_1 \in u^L$, then $t_1 = tu_1$. But this is clearly false, since $\{u_1\} \neq u^L \cap C_L(u_1) \subset C_L(t_1)$. This contradiction proves that G does not exist and completes the proof of Theorem 1.1.

4. The 2-Balanced Functor

Throughout this section and the following section, G will be a minimal counterexample to Theorem 1.2. By Lemma 2.23 and Theorem 1.1, there is no $L \in \mathscr{L}(G)$ with $\tilde{L} \simeq \mathrm{He}$. Thus by Theorem 2.22, if (b, y, K) is an unbalancing triple in G then one of the following holds:

(1) $K/O(K) \simeq L_2(q)$ for some odd $q \geqslant 49$ and $C_G(K/O(K))$ has a cyclic Sylow 2-subgroup.

(2) $K \twoheadrightarrow J \in \mathscr{L}^*(G)$ with $\tilde{J} \simeq L_3(4)$ and (a, c, J) is an unbalancing triple in G for some (a, x).

If (1) holds, then we are done. Thus we shall assume that (1) does not hold. Then every unbalancing 2-component K has $\tilde{K}$ isomorphic to $L_2(5)$, $L_2(7)$, or $L_3(4)$. In particular, by Lemma 2.23, G is locally 2-balanced. Moreover, since G is a counterexample to Theorem 1.2, there exist unbalancing triples (a, x, J) with $\tilde{J} \simeq L_3(4)$. We fix one such. By Corollary 2.17, G is simple and $J \in \mathscr{L}^*(G)$. We fix $S \in \mathrm{Syl}_2(N_G(J))$ with $\langle a, x\rangle \subseteq S$ and set $Z = S \cap Z^*(J)$. Our aim in this section will be to prove the following proposition.

Proposition 4.1. *$W_A = 1$ for all $A \subseteq JC_G(\tilde{J})$ with $A \simeq E_8$.*

Thus for the remainder of this section we shall assume that $W_A \neq 1$ for some $A \subseteq JC_G(\tilde{J})$ with $A \simeq E_8$. We fix this A and W_A for the remainder of the argument and we set $M = N_G(W_A) < G$.

Lemma 4.2. *Suppose that $Z \neq \langle 1\rangle$ and $m(C_S(x)) = 3$. Then $a^G \cap S \neq \{a\} \cup x^S$.*

Proof. Suppose that $a^G \cap S = \{a\} \cup x^S$. Then $Z(S) \cap x^S = \varnothing$ implies that $S \in \mathrm{Syl}_2(G)$. By Lemma 2.16, we may assume that $x^g = a$ for some $g \in N_G(H)$ for some $C_S(x) \subseteq H \subseteq S$. As $a \in Z(H)$, $H = C_S(x)$. We claim that $m(Z(H)) = 3$. Since $m(H) = 3$, Z is cyclic, by Proposition 2.2(3), and $C_{S \cap J}(x) = \langle a, y\rangle \trianglelefteq H$, $|y| = 4$, $\langle y\rangle \cap Z = 1$ by Proposition 2.2(3). Thus $\Phi(\langle a, y\rangle) = \langle y^2\rangle \subseteq \Omega_1(Z(H))$, which implies that $\langle a, x, y^2\rangle = \Omega_1(Z(H))$. Now $x^S \cap \langle a, x, y^2\rangle = \langle a, y^2\rangle x$. Thus $x^{N_G(H)} = \{a\} \cup \langle a, y^2\rangle x$. But $N_G(H)/C_G(\langle a, x, y^2\rangle)$ is isomorphic to a subgroup of $GL(3, 2)$ and $5 \nmid |GL(3, 2)|$, a contradiction.

Lemma 4.3. *$S \cap JC(\tilde{J})$ is not 3-connected. In particular,* $\exp Z = 4$.

Proof. Suppose $S \cap JC(\tilde{J})$ is 3-connected. In particular, $m_2(JC(\tilde{J})) \geqslant 4$. Recall that $M = N_G(W_A) < G$. Then $N_G(B) \subseteq M$ for all abelian 2-subgroups B of $JC(\tilde{J})$ of rank at least 3 and M controls G-fusion of such subgroups. So $N(J) \subseteq M$. By induction $\langle J^M\rangle/O(\langle J^M\rangle) \simeq J/O(J)$ or $F^*(M/O(M)) \simeq \mathrm{He}$. As $\langle a, x, J\rangle \subseteq M$, $M/O(M) \simeq \mathrm{Aut\,He}$ in the latter case.

Suppose that $M/O(M) \cong \mathrm{Aut\,He}$. Let $S \subseteq T \in \mathrm{Syl}_2(M)$. By Lemma 3.1, T is 3-connected. So, $T \in \mathrm{Syl}_2(G)$. By Corollary 2.26, M controls G-fusion in T. But then $|G : G'|$ is even, contrary to Proposition 2.16.

We now have $K = O(M)J \trianglelefteq M$. Then $S \in \mathrm{Syl}_2(M)$ and $S_1 = S \cap JC(\tilde{J}) = S \cap KC(\tilde{K})$. By Lemma 2.18 and Corollary 2.26, M controls G-fusion in S_1 and $S \in \mathrm{Syl}_2(G)$. Also $C_G(a) \subseteq M$. Suppose that $a^G \cap M = a^M$. Then a fixes only the point M in the transitive representation of G on the right cosets of M. Thus M satisfies the hypothesis of Theorem 2.20(2). But J is a 2-component of $C_M(a)$, a contradiction. Thus $M = G$. But this contradicts the simplicity of G'.

Thus $a^G \cap M \neq a^M$. We let $x_1 \in (a^G - a^M) \cap S$. By Lemma 2.18 and the first paragraph of this proof, $m(C_{S_1}(x_1)) \leqslant 2$. Then by Proposition 2.2, one of the following holds:

(1) $|Z| = 16$ and x_1 induces a field automorphism on $\tilde{J}$.

(2) Z is cyclic and x_1 induces a unitary automorphism on $\tilde{J}$.

In either case, we may find $h \in G - M$ with $(x_1)^h = a$, $C_S(x_1)^h \subseteq S$. In Case 1, $a \in \Phi(C_S(x_1))$. Thus $a^h \in \Phi(S) - \{a\} \subseteq S_1 - \{a\}$. But then $h \in M$, a contradiction. Thus Case 2 holds. Now $m_2(C_G(\langle a, x_1\rangle)) = 3$ and $x_1 \notin \Phi(C_G(\langle a, x_1\rangle))$. Thus by Propositions 2.2 and 2.3, a induces a unitary automorphism on $\tilde{J}_1$ where $J_1 = J^{h^{-1}} \trianglelefteq C_G(x_1)$. In particular, a is not a square in $C_G(x_1)$. Thus $\langle a\rangle \in \mathrm{Syl}_2(C_S(x_1) \cap C_S(\tilde{K}))$. If $Z = \langle 1\rangle$, then $C_S(x_1) = \langle a, x_1, Q\rangle$ where $Q \subseteq J$, $Q \cong Q_8$. Let $\langle z\rangle = Z(Q)$. Then $z \in C_{J_1}(a)$ and $J_1/O(J_1) \cong L_3(4)$. Thus $a \sim az$ in $N_G(J_1)$. But $az \notin a^M$, a contradiction. Thus $a \in Z$ and $a^M \cap S_1 = \{a\}$. As every element of $(a^G \cap S) - \{a\}$ induces a unitary automorphism on $\tilde{K}$, we may repeat the argument of Lemma 4.2 to obtain a contradiction.

Lemma 4.4. *Suppose that Z is non-cyclic. Let $B = \langle C_{S\cap J}(x), x\rangle \cong E_{16}$ and let C be an elementary subgroup of $S \cap J$ of order 16 containing $C_{S\cap J}(x)$ and mapping onto $Z(S \cap J/Z)$. Then $J = \langle N_J(E) \mid E \subseteq B$ or $E \subseteq C$ and $E \cong E_8\rangle$.*

Proof. We first observe that B is indeed elementary of order 16; this follows from Proposition 2.2(3). The existence of such a C follows from Proposition 2.3.

Let $Y = \langle N_J(E) \mid E \subseteq B$ or $E \subseteq C$ and $E \cong E_8\rangle$. Observe that $O(J) \subseteq Y$. Fix a complement H to S in $N_J(S)$. Then H normalizes precisely three groups E_1, E_2, E_3 which contain $[C, H]$ and have index 2 in C. For each i, $|S : N_S(E_i)| = 1$ or 4 and, for $i \neq j$, $\langle N_S(E_i), N_S(E_j)\rangle = S$. It follows that $N_J(S) \subseteq Y$. Now consider $E = \langle x, \Omega_1(Z)\rangle \subseteq B$. Then $N_J(E)$ covers $C_{\tilde{J}}(x) \cong PSU(3, 2)$. Since no parabolic subgroup of $\tilde{J}$ has order divisible by 9, we get

$$\langle N_{\tilde{J}}(\tilde{S}), C_{\tilde{J}}(x)\rangle = \tilde{J}.$$

The Lemma follows.

Lemma 4.5. *Suppose $M_1 \subseteq G$ with $J_1 = O(M_1)J \trianglelefteq M_1$ and $S \in \mathrm{Syl}_2(M_1)$ with $a \in Z(S)$. Suppose that $N_G(S \cap J) \subseteq M_1$. Then $a^G \cap C_{M_1}(\tilde{J}_1) = a^{M_1}$.*

Proof. Suppose that some $a^g \in C_{M_1}(\tilde{J}_1)$. Let $J_0 = L(C_{J_1}(a^g))$. Then $J_0^{g^{-1}} \subseteq L(C_G(a))$. Since J is maximal in G, Theorem 2.9 implies that $J_0^{g^{-1}} \subseteq J$. We may assume that $a^g \in S$. Hence $S \cap J \subseteq J_0$ and so $(S \cap J)^{g^{-1}} = (S \cap J)^m$ for some $m \in J \subseteq M_1$. Since $mg \in N_G(S \cap J) \subseteq M_1$, we get $g \in M_1$ as required.

LEMMA 4.6. *Z is cyclic of order* 4.

Proof. We shall assume that $m(Z) = 2$ and work for a contradiction.

We claim that if $E \subseteq JC(\tilde{J})$ with $E \simeq E_8$, then $W_E \neq 1$. Since $m(Z) = 2$, if $E, F \subseteq J$ with $E \simeq F \simeq E_8$, then E is 3-connected to a conjugate of F in J. Thus it suffices to show that A is 3-connected to some eights group in J. First we consider the case that $| A \cap Z | = 4$. Let $uv \in A \backslash Z$, $| uv | = 2$, $u \in J$, $v \in C(\tilde{J})$. Then A is connected to $\langle A \cap Z, u_1 \rangle$ where $u_1 \in Zu$, $| u_1 | = 2$. In case $| A \cap Z | \leqslant 2$, A is 3-connected to some $A_1 \subseteq \Omega_1(Z)A$ with $| A_1 \cap Z | = 4$. The claim follows.

Let B and C be as in Lemma 4.4. Set $W = W_A = W_B = W_C$. Then $M = N_G(W)$. By Corollary 2.26 and Lemma 4.4, $J \subseteq M$. Since $W \neq 1$, $M < G$. Since M is unbalanced, we "know" $M_1 = \langle J^M \rangle$ by induction, i.e. $M_1 = O(M_1)J$ or $M_1/O(M_1) \simeq \text{He}$. But $M_1/O(M_1) \not\simeq \text{He}$, since otherwise $\exp Z = 2$, against Lemma 4.3. So $M_1 = O(M_1)J \trianglelefteq M$.

Again, any two eights-groups in J containing $\Omega_1(Z)$ are conjugate. In particular, for any such eights group E, $W = W_E$ and if $g \in G$ satisfies $E^g \subseteq J$, then $g \in M = N_G(W)$. Now if we show that $J_0 = C_J(Z) \trianglelefteq N_G(Z)$, we will get $N_G(Z) \subseteq M$. We proceed to do this. Now $J \trianglelefteq C_G(a)$ implies that $J_0 = C_J(Z) \trianglelefteq C_G(Z)$. Assume that $J_0 \ntrianglelefteq N_G(Z)$ and take $k \in N_G(Z)$ with $J_0{}^k \neq J_0$. Then

$$[J_0, J_0{}^k] \subseteq O(J_0) \cap O(J_0{}^k).$$

Since $SCN_4(S \cap J_0) \neq \varnothing$, it follows that $SCN_5(\langle J_0^{N_G(Z)} \rangle \cap S) \neq \varnothing$, which contradicts Lemmas 2.29 and 4.3. Therefore $J_0 \trianglelefteq N_G(Z) \subseteq M$.

Again, Lemma 4.3 implies that $\exp Z = 4$, whence $Z(S) \cap J = Z$. Also, $C = \Omega_1(Z_2(S \cap J))$, so that $N_G(S \cap J) \subseteq N_G(C) \subseteq M$. By Lemma 4.5 and the fact that $Z(S) \subseteq C_M(\tilde{J}_0)$, we get $N_G(S) \subseteq M$ and $S \in \mathrm{Syl}_2(G)$.

By Lemma 2.18(iii), we get that if $b \in S$, $b \sim_G a$ and $m_2(C_{S \cap J}(b)) \geqslant 3$, then $b \sim_M a$. Take $b \in (S \cap a^G) - Z$. Then $m_2(C_{S \cap J}(b)) \leqslant 2$, whence b induces a field automorphism and $Z \simeq \mathbf{Z}_4 \times \mathbf{Z}_4$. Take $g \in G$ with $C_S(b)^g \subseteq S$ and $b^g = a$. Then $a \in \Phi(C_S(b))$ implies that $a^g \in \Phi(S) \subseteq (S \cap J)\, C_S(\tilde{J})$, so that $m_2(C_{S \cap J}(a^g)) \geqslant 3$. But then $g \in M$, a contradiction. Thus $S \cap a^G \subseteq Z$, whence $\langle a^G \cap S \rangle$ is a strongly closed abelian subgroup of S (lying in Z). This contradicts Theorem 2.20(1). The proof is complete.

LEMMA 4.7. *$C_S(\tilde{J})$ is cyclic and $\Omega_1(S \cap J)\, C_S(\tilde{J}) \subseteq S \cap J$.*

Proof. By Lemma 4.6, $Z \simeq \mathbf{Z}_4$. We assume that $m_2(C_S(\tilde{J})) \geqslant 2$ and obtain a contradiction to Lemma 4.3.

Let $A_0 \subseteq S \cap J$ be an eights group with $a \in A_0$ and $A_0Z/Z = Z(S \cap J/Z)$. Take an eights group $B \neq A_0$ in $S_1 = (S \cap J) C_S(\tilde{J})$. Assume that B and A_0 are not connected and that $B \not\subseteq J$. We shall derive a contradiction.

We have $| B \cap J | \leqslant 4$ and we may assume that $a \in B$. Since A_0 and B induce commuting elementary abelian groups of automorphisms in $\tilde{J}$, we may take $E \subseteq S \cap J$ with $ZA_0 \subseteq E$, $E/Z \cong E_{16}$ and $[E, B] \subseteq Z$. Since $[Z, B] = 1$ and since every $b \in B$ inverts each $[c, b]$ for $c \in E$, we get $[E, B] \subseteq \langle a \rangle$. We have $E \cong D_8 \circ D_8 \circ \mathbf{Z}_4$.

We claim that $| C_B(A_0)| = 2$. Suppose false and let $B_1 = C_B(A_0) > \langle a \rangle$. Also $A_1 = C_{A_0}(B)$ has index 2 in A_0 and, as A_0 and B are not connected, A_1B_1 cannot be an eights group. So $A_1 = B_1$ is a four-group. Take $uv \in B - A_0$, $u \in S \cap J$, $v \in C_S(J)$. Since $| uv | = 2$ and $u^2 \in \langle a \rangle$, we get $v^2 \in \langle a \rangle$. Thus there is an involution $v_1 \in Zv$ and $\langle A_1, v_1 \rangle$ is connected to both B and A_0, a contradiction. So, $| C_B(A)| = 2$.

Now, we have $A_0B \cong D_8 \circ D_8$. Since $(EB)' = \langle a \rangle$, $| C_{EB}(A_0B)| = 2^2 | B : B \cap J | \geqslant 2^3$. Since $C_{EB}(A_0B)/Z \cong E_{2^r}$, $1 \leqslant r \leqslant 2$, we get $C_{EB}(A_0B)$ isomorphic to $\mathbf{Z}_4 \times \mathbf{Z}_2$, $\mathbf{Z}_4 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ or $\mathbf{Z}_4 \circ D_8$. If any one of these possibilities were to hold, we could connect B to some eights group B_1 with $| C_{B_1}(A)| \geqslant 4$, against the previous paragraph. So, we have proven that $B \subseteq J$.

If possible, take some $y \in C_S(\tilde{J}) - Z$ with $| y | = 2$. Then B is connected to $\langle B_1, y \rangle$, where B_1 is a hyperplane of B, whence B and A_0 are connected, a contradiction. So no such y exists, whence $m_2(C_S(\tilde{J})) = 1$.

Since $C_S(\tilde{J})$ acts trivially on Z, $C_S(\tilde{J})$ is not quaternion, hence is cyclic. The last statement in our Theorem follows from the fact that a generator for Z has no square root in $S \cap J$. Our proof is now complete.

We can now complete the proof of Proposition 4.1.

Proof of Proposition 4.1. By Lemma 4.7, $Z \cong \mathbf{Z}_4$ and $C_S(\tilde{J})$ is cyclic. Also $S \cap J$ contains every involution of $(S \cap J) C_S(\tilde{J})$. Since $C_Z(x) = \langle a \rangle$, $Z(S) = \langle a \rangle$, whence $S \in \mathrm{Syl}_2(G)$.

We claim that $a^G \cap S \cap J \not\subseteq \{a\}$. Assume false. By the Z^*-Theorem and Lemma 2.18, we may take $b \in (a^G \cap S) - \{a\}$ and $g \in N_G(C_S(b))$ with $b^g = a$. Let $X = C_S(b)$. As $b \notin \Phi(X)$, $a \notin \Phi(X)$. If b induces a field or unitary automorphism on $\tilde{J}$, then $\Omega_1(Z(X)) = \langle b \rangle \times Z_1$ where $Z_1 = \Omega_1(Z(X)) \cap J$, $| Z_1 | = 4$ and $b^S \supseteq bZ_1$. But then $| a^{N_G(X)} | = 5$ and we have a contradiction as in Lemma 4.2. Thus b induces a graph automorphism on $\tilde{J}$ and $\Omega_1(Z(X)) = \langle b \rangle \times Z_1$ with $Z_1 = \Omega_2(Z(X)) \cap J$, $| Z_1 | = 8$ and $b^S \supseteq bZ_1$. Then $| a^{N_G(X)} | = 3^2$. But $3 \mid | N_G(X) \cap C_G(a)/C_G(X)|$. So, $3^3 \mid | N_G(X)/C_G(X)|$. But $N_G(X)/C_G(X)$ is isomorphic to a subgroup of $GL(4, 2)$ and $3^3 \nmid | GL(4, 2)|$, a contradiction. The claim follows.

Take $B \subseteq S \cap J$ with $B \cong E_8$, $| B \cap Z | = 2$ and $BZ/Z = Z(S \cap J/Z)$. By Corollary 2.5, $S \cap J$ induces on B the stability group of the chain $B \supset B \cap Z \supset 1$. Also, $S_0 = C_{S \cap J}(B)$ satisfies $S_0 \supseteq Z$ and $S_0/Z \cong \mathbf{Z}_4 \times \mathbf{Z}_4$. Thus

$B \subseteq \Phi(S_0)$ and $B^{\#} \subseteq a^G$. We now claim that $N_G(B) \cap C_G(b)$ induces the stability group of the chain $B \supset \langle b \rangle \supset 1$ for all $b \in B^{\#}$. Namely, let $T = C_S(B)$ and let $g \in G$ satisfy $b^g = a$ and $T^g \subseteq S$. Then $B^g \subseteq \Phi(T^g) \subseteq \Phi(S)$, so that B^g induces a group of inner automorphisms on J. By Lemma 4.7, $B^g \cap C_S(\tilde{J}) = \langle a \rangle$. The claim follows. Therefore, $N_G(B)/C_G(B) \cong GL(3, 2)$. By considering the action of a 7-element on $C_S(B) \in \mathrm{Syl}_2(C_G(B))$, we get $Z = C_S(\tilde{J})$ and $C_S(B) = C_{S \cap J}(B)$ or $C_S(B) = C_{S \cap J}(B)\langle c \rangle$, where c induces a graph automorphism on J and $|c| = 2$.

Let $L = N_G(T)$. Then $L/O(L)T \cong GL(3, 2)$. Since $S \in \mathrm{Syl}_2(L)$ and $c \notin \Phi(S)$, we get $T = T_1\langle c \rangle$, where T_1 is invariant under a Hall $2'$-subgroup of L. The structure of the indecomposable $F_2[GL(3, 2)]$-modules and the fact that $c \notin \Phi(S)$ imply that $c \notin L'$ and that $T_1 \trianglelefteq L$. So, $L = L_1\langle c \rangle$, where $L_1 = L' \cdot O(L)$ and $T_1 \subseteq L_1$. If $T_1' \neq 1$, the action of a 7-element on T_1 implies that T_1 is isomorphic to a Sylow 2-group of $Sz(8)$ [22]. However, then Aut T_1 does not involve $GL(3, 2)$, contradiction. Therefore $T_1 \cong \mathbf{Z}_4 \times \mathbf{Z}_4 \times \mathbf{Z}_4$. Now set $F = C_{L_1}(a)$, $R_0 = O_{2',2}(F)$, and take $R \in \mathrm{Syl}_2(R_0)$. Take a 3-element $h \in N_F(R)$ with $R = [R, h]$. The structure of J implies that $R = S \cap J$. Since $a^L \cap R \not\subseteq \{a\}$, conjugacy in J implies that every involution in R is in a^G. Since $|S \cap L_1 : R| = 2$ and $L_1 = O^2(L_1)$, the Thompson transfer lemma implies that every involution on L_1 fuses to a in G.

If $C_S(B) = C_{S \cap J}(B)$, then G has a Sylow 2-group isomorphic to that of HiS or O'Nan's group, according to whether $L_1/O(L_1)$ contains a copy of $GL(3, 2)$ or not. Since these simple groups have been characterized by their Sylow 2-groups [14, 26], we may identify G. But then G is a balanced group, a contradiction.

Thus $C_S(B) = C_{S \cap J}(B)\langle c \rangle$, $|c| = 2$ and c induces a graph automorphism on J. As $G = G'$, G has one class of involutions and $C_G(a)/O(C_G(a)) \cong \mathbf{Z}_4 \cdot L_3(4) \cdot E_4$. Since c has order 2 and induces a graph automorphism on $\tilde{J}$, we get that $C_{L_1}(c)/O(C_{L_1}(c)) \cong E_8$. Frob(21) and that c inverts $O_{2',2}(L_1)/O(L_1) \cong \mathbf{Z}_4 \times \mathbf{Z}_4 \times \mathbf{Z}_4$. Let $E \in \mathrm{Syl}_2(C_L(c))$, $E \cong E_{16}$ and let $Y = N_G(E)$. Since $E \subseteq C_G(a)$, we get $E \in \mathrm{Syl}_2(C_G(E))$. Denote by $*$ images under $Y \to \mathrm{Aut}(E) \cong GL(4, 2)$. Then $(Y \cap L)^* \cong E_8 \cdot \mathrm{Frob}(21)$. Now take $g \in G$ so that $c^g = a$ and $E^g \subseteq S$. By considering $N_S(E^g)$, we see that there is a 2-element $y \in Y$ with $y^* \neq 1$ and $c^y = c$. Thus $2^4 \cdot 3 \cdot 7 \mid |Y^*|$ and Y^* is irreducible on E. Since $Y^* \hookrightarrow GL(4, 2)$, irreducibility and $|\mathrm{Syl}_7(Y^*)| \equiv 1 \pmod 7$ imply that $Y^* \cong GL(4, 2)$. It follows that $N_Y(\Omega_1(T_1))$ covers $L/O(L)$. However $E \trianglelefteq N_Y(\Omega_1(T_1))$, whereas $E \cdot O(L)/O(L) \not\trianglelefteq L_1/O(L_1)$; in fact, $[E, L_1] = T_1$. This contradiction establishes Proposition 4.1.

5. The Proof of Theorem 1.2.

In this section we complete the proof of Theorems 1.2 and 1.3. Thus we assume throughout this section that G is a minimal counterexample to Theorem

1.2. In the next proposition we collect the properties of G which we have established.

PROPOSITION 5.1. (1) *$F^*(G)$ is simple and $|G : F^*(G)| \leqslant 2$.*

(2) *If (a, x, J) is an unbalancing triple in G, then either $J/O(J) \cong L_2(q)$ for some $q \in \{5, 7\}$ or $J/Z^*(J) \cong L_3(4)$.*

(3) *If J is a maximal unbalancing 2-component in G, then $\tilde{J} \cong L_3(4)$ and J is a maximal 2-component in G. Moreover $W_A = \langle 1 \rangle$ for all $A \subseteq JC_G(\tilde{J})$, $A \cong E_8$. Also $G = \langle G', t \rangle$ for any $t \in I(C_G(J))$.*

(4) *Let A be as in (3), $L \in \mathscr{L}(G)$, $E \cong E_8$ and E 3-connected to A. Suppose that either $E \subseteq C_G(\tilde{L})$ or $m_2(C_E(L)) \geqslant 2$. Then $O(L) \subseteq Z(L)$.*

(5) *There exists an unbalencing triple (a, x, J) and an involution t in $C_G(\tilde{J})$ such that $L = \langle (L(C_J(t))^{L(C_G(t))} \rangle$ has the following properties:*

(a) *$\tilde{L} \cong L_3(4)$.*

(b) *L is maximal in $\mathscr{L}(G)$.*

(c) *$O(L) \nsubseteq Z(L)$.*

(d) *$W_A = \langle 1 \rangle$ for all $A \subseteq \langle L, t \rangle$ or $A \subseteq C_G(\tilde{L})$ with $A \cong E_8$.*

Proof. Proposition 2.16 gives (1). Theorem 2.22 and Theorem 1.1 yield (2). Corollary 2.17 and Proposition 4.1 yield (3). Using (3) together with Lemma 2.28 we get (4). As G is a minimal counterexample to Theorem 1.2, G has a maximal unbalancing triple (a, x, J). By (3), $\tilde{J} \cong L_3(4)$ and J is maximal in G. By Theorem 2.13, J is not standard in G. By Corollary 2.11, there exists an involution $t \in C_G(\tilde{J})$ with $O(L) \nsubseteq Z(L)$ where $L = \langle L(C_J(t))^{L(C_G(t))} \rangle$ and $L = O(L)L(C_J(t))$. Thus L is maximal in G and $L/O(L) \cong J/O(J)$. Since J covers $L/O(L)$, $W_A = \langle 1 \rangle$ for all $A \cong E_8$, $A \subseteq L$ by (3). If $A \cong E_8$ and either $A \subseteq \langle L, t \rangle$ or $A \subseteq C_G(\tilde{L})$, then A is 3-connected to some $B \subseteq L$, $B \cong E_8$. So $W_A = W_B = \langle 1 \rangle$. This proves (5).

We remark that by (3), G is also a counterexample to Theorem 1.3.

Notation 5.2. Let $\mathscr{L}$ be the set of all members L of $\mathscr{L}(G)$ such that

(1) There exists (a, x, J) an unbalancing triple and $t \in I(C_G(\tilde{J})$ such that $L = \langle L(C_J(t))^{L(C_G(t))} \rangle$.

(2) L satisfies properties (a)–(d) of 5.1(5).

By 5.1(5), $\mathscr{L} \neq \varnothing$. Among all $L_1 \in \mathscr{L}$, pick L with $|Z^*(L)|_2$ maximal and, subject to this, with $|C_G(\tilde{L})|_2$ maximal. Let $N = N_G(L)$, $S \in \mathrm{Syl}_2(N)$, $S \subseteq T \in \mathrm{Syl}_2(G)$, $Q = S \cap C_G(\tilde{L})$, $Z = Q \cap L$ and $X = O(L)$. As $L \in \mathscr{L}$, Lemma 2.28 implies that $m(Q) \leqslant 2$ and $m(C_Q(X)) = 1$. Let t be the involution in $C_Q(X)$.

LEMMA 5.3. *Let $u \in I(S - Q)$ with $u \in \langle L, t \rangle$. Then $C_X(u) \neq \langle 1 \rangle$ and for some $u_1 \in \{u, ut\}$, $C_X(u_1) \nsubseteq O(C(u_1))$.*

Proof. If $C_X(u) = \langle 1 \rangle$, then u inverts X. Thus $u \in Z(N/C_N(X))$, whence $u \in C_N(\tilde{L})$. But $u \in I(S - Q)$. So $C_X(u) \neq \langle 1 \rangle$.

If $C_X(u_1) \subseteq O(C(u_1))$ for $u_1 \in \{u, ut\}$, then $C_X(u) = C_X(u_1) \subseteq W_{\langle u,t \rangle} = \langle 1 \rangle$, a contradiction.

Notation 5.4. Let $S_0 = S \cap L$. Let $\mathscr{U} = \{u \in I(\langle S_0, t \rangle) \mid C_X(u) \nsubseteq O(C_G(u))\}$. For $u \in \mathscr{U}$ let

$$K(u) = \langle (C_X(u) \cap L(C_G(u)))^{L(C_G(u))} \rangle.$$

$K(u)$ is a nontrivial product of unbalancing 2-components of $C_G(u)$. Moreover, $K(u)$ is normalized by $C_G(\langle u, t \rangle)$.

LEMMA 5.5. (1) *Let $z \in I(Q)$. If K is a 2-component of $C_G(z)$ with either $K/O(K) \cong L_2(q)$ or $K/Z^*(K) \cong L_3(4)$. then $K = \langle L(C_L(z))^{C_G(z)} \rangle$ and $\tilde{K} \cong L_3(4)$.*

(2) *If $z \in I(Q) - \{t\}$, then z inverts X.*

(3) *Let $u \in \mathscr{U}$. Then $\Omega_1(Q)$ normalizes every 2-component K_1 of $K(u)$ and $C_Q(\tilde{K}_1) = \langle 1 \rangle$.*

(4) *$\Omega_1(Q)$ is isomorphic to a subgroup of $\mathbf{Z}_2 \times \mathbf{Z}_2$.*

Proof. (1) Let $K_0 = \langle L(C_L(z))^{L(C_G(z))} \rangle$. As $m(Q) \leqslant 2$, $K_0 \trianglelefteq C_G(z)$. Suppose that $K_0 \neq K$. If $z = t$, then $m_2(\langle K, t \rangle) \geqslant 3$, which contradicts the fact that $m(Q) \leqslant 2$. If $\tilde{K} \cong L_3(4)$, then $m_2(C_{\langle K,t,z \rangle}(t)) \geqslant 3$, again a contradiction. Finally if $z \neq t$ and $K/O(K) \cong L_2(q)$, then $m_2(C_{\langle K,t,z \rangle}(t)) \geqslant 3$, again a contradiction.

(2) Suppose that $z \in I(Q) - \{t\}$ and $C_X(z) \neq \langle 1 \rangle$. As $W_{\langle z,t \rangle} = \langle 1 \rangle$, we may assume that $C_X(z) \nsubseteq O(C_G(z))$. This violates (1).

(3) As $u \in \langle S_0, t \rangle$, $O(C_G(\langle u, t \rangle)) \subseteq X$. Thus for each 2-component, K_1, of $K(u)$, $X \cap K_1 = O(C_{K_1}(t)) \nsubseteq O(K_1)$. If $z \in I(Q) - \{t\}$, then z inverts X. Thus z normalizes K_1 and does not centralize $\tilde{K}_1$.

(4) Let K_1 be as in (3). By (3), $\Omega_1(Q)$ is isomorphic to a subgroup of a Sylow 2-subgroup of $N^* = N_G(K_1) \cap C_G(t)/C_G(\tilde{K}_1) \cap C_G(t)$. If $\tilde{K}_1 \cong L_2(q)$, then a Sylow 2-subgroup of N^* is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$ and the conclusion is clear. If $\tilde{K}_1 \cong L_3(4)$, then a Sylow 2-subgroup of N^* is isomorphic to a subgroup of $\mathbf{Z}_2 \times D_8$ and $t^* \notin (N^*)'$. As $\Omega_1(Q)$ is generated by involutions and $m_2(\Omega_1(Q)) \leqslant 2$, $\Omega_1(Q)$ is isomorphic to a subgroup of $\mathbf{Z}_2 \times \mathbf{Z}_2$.

LEMMA 5.6. $S \notin \mathrm{Syl}_2(G)$.

Proof. Suppose that $S \in \mathrm{Syl}_2(G)$. As $\langle t \rangle = \Omega_1(Q) \cap C_G(L)$, $t \in Z(S)$. Pick $u_0 \in I(S_0)$ with $|C_S(u_0)| \geqslant |C_S(v)|$ for all $v \in I(S - \Omega_1(Q))$. As $\langle u_0, t \rangle \cap \mathscr{U} \neq \varnothing$, we may pick $u \in \{u_0, u_0 t\} \cap \mathscr{U}$. Now $|S : C_S(u)| \leqslant 4$ and if $C_S(u) \notin$

$\mathrm{Syl}_2(C_G(u))$, then $u^G \cap Q \neq \varnothing$. Suppose that $u^G \cap Q \neq \varnothing$. By 5.5(1), $K(u) = K^g$ for some $g \in G$. As $|S : C_S(u)| \leqslant 4$, $S \cap K(u)$ has index at most 4 in a Sylow 2-subgroup of $K(u)$. But this contradicts the fact that t induces a unitary automorphism on $\widetilde{K(u)}$. Thus $u^G \cap Q = \varnothing$, whence $C_S(u) \in \mathrm{Syl}_2(C_G(u))$. Let K_1 be a 2-component of $K(u)$. Then $S \cap K_1 \in \mathrm{Syl}_2(K_1)$. But t centralizes $S \cap K_1$ and induces an unbalancing automorphism on $\tilde{K}_1$, which is impossible.

Notation 5.7. If $m_2(Q) = 2$, let $\langle t, z\rangle = \Omega_1(Q)$. Moreover, if $Z(T) \cap \Omega_1(Q) \neq \langle 1\rangle$, let $z \in Z(T)$. Let $S_0 = S \cap L$.

LEMMA 5.8. (1) *If* $\Omega_1(Q) = \langle t\rangle$, *then* $Q = \langle t\rangle$.

(2) *If* $Z \neq \langle 1\rangle$, *then* $Q = \Omega_1(Q)$; *in particular*, $\exp Z \leqslant 2$.

Proof. (1) Let $r \in N_T(S) - S$. Then $Q^r \cap Q = \langle 1\rangle$. Thus Q^rQ/Q is a normal subgroup of S/Q of 2-rank 1. Then by Corollary 2.4(3), $|Q^rQ/Q| = 2$. Thus $Q = \langle t\rangle$.

(2) By (1), we may assume that $\Omega_1(Q) = \langle t, z\rangle$.

Assume first that $Z = \mathbf{Z}_2 \times \mathbf{Z}_4$ or $\mathbf{Z}_4 \times \mathbf{Z}_4$. By Proposition 2.3, $\Omega_1(Z(S)) \subseteq \langle t, z\rangle$. As $S < T \in \mathrm{Syl}_2(G)$, we conclude (using Notation 5.7) that $\Omega_1(Z(S)) = \langle t, z\rangle$ and $\Omega_1(Z(T)) = \langle z\rangle$. Let $L_1 = \langle (L(C_L(z)))^{L(C_G(z))}\rangle$. As L is maximal in $\mathscr{L}(G)$, L covers $L_1/O(L_1)$ and $L_1 \trianglelefteq C_G(z)$. As $S_0 \in \mathrm{Syl}_2(L_1)$, $S_0 \trianglelefteq T$ and $t^T = \{t, zt\}$. Pick $u \in \mathscr{U}$ with $|C_T(u)|$ maximal. As before, we see that either $u^G \cap \{t, z\} \neq \varnothing$ or $C_T(u) \in \mathrm{Syl}_2(C_G(u))$. Suppose that the former case holds. By Lemma 5.5(1), $K(u)/O(K(u)) \cong L/O(L)$. Let bars denote homomorphic images in $\bar{C} = C_G(u)/C_G(u) \cap C_G(\widetilde{K(u)})$. A second application of 5.5(1) shows that $Z \cong \bar{Z} \subseteq \overline{C_G(\langle u, t\rangle)}$. Now t induces a unitary automorphism on $K(u)/O(K(u))$ and $Z^*(K(u))$ has 2-rank 2. Then Proposition 2.2 implies that a Sylow 2-subgroup of $\overline{C_G(\langle u, t\rangle)}$ is isomorphic to Z_4 or E_8, contradicting the fact that $\bar{Z} \subseteq \overline{C_G(\langle u, t\rangle)}$. Thus $u^G \cap \{t, z\} = \varnothing$, whence $C_T(u) \in \mathrm{Syl}_2(C_G(u))$. As $|T : S| = 2$, $|C_T(u) : C_S(u)| \leqslant 2$. It follows that $K(u)/O(K(u)) \cong L_2(5)$. But again $Z \cong \bar{Z} \subseteq \overline{C_G(\langle u, t\rangle)}$ and, in this case, $\overline{C_G(\langle u, t\rangle)} \cong \mathbf{Z}_2 \times S_3$, contradicting the fact that Z has exponent 4.

We have now established that Z is either cyclic or elementary abelian. We shall assume henceforth that $Z \neq \langle 1\rangle$. We fix $u \in \mathscr{U}$ with $|C_S(u)|$ maximal. Thus $|S : C_S(u)| \leqslant 2$. First we argue that $C_{S_0}(u)' \cap Z \neq \langle 1\rangle$. If Z is elementary, then $C_{S_0}(u) = S_0$ and $Z \subseteq S_0'$. Hence we may assume that $Z \cong \mathbf{Z}_4$. In paragraph 3 of the proof of Lemma 4.7 it is shown that if $Z \subseteq E \subseteq S_0$ with $E/Z \cong E_{16}$, then $E \cong D_8 \circ D_8 \circ \mathbf{Z}_4$. As $E/Z \supseteq Z(S/Z)$, $u \in E$ and $C_E(u) \cong \mathbf{Z}_2 \times D_8 \circ \mathbf{Z}_4$. Thus $\Omega_1(Z) \subseteq C_E(u)'$.

Now suppose that $Q \neq \langle t, z\rangle$. Then there exists $y \in C_Q(\langle t, z\rangle)$ with $y^2 \in \langle t, z\rangle^{\#}$ and $[S, y] \subseteq \langle t, z\rangle$. Suppose that y normalizes a 2-component, K_1, of $K(u)$. Then $\langle t, z, y\rangle$ projects faithfully into $\bar{N} = N_G(K_1) \cap C_G(t)/C_G(\tilde{K}_1) \cap$

$C_G(t)$. Let bars denote homomorphic images in $\bar{N}$. As $\bar{t}$ induces an outer involutary automorphism on $\bar{K}_1$ and $\bar{y}$ has order 4, we must have $\bar{K}_1 \simeq L_3(4)$. Then by Theorem 2.9, $\bar{K}_1$ is invariant under $C_S(u)$. As $\bar{t}$ induces a unitary automorphism of $\bar{K}_1$, $\overline{C_S(u)}$ is isomorphic to a subgroup of $\mathbf{Z}_2 \times SD_{16}$ by Proposition 2.2. As $\bar{y}$ has order 4, $\overline{C_S(\langle u, y\rangle)}$ is abelian. Thus $\langle t, z\rangle \cap C_S(\langle u, y\rangle)' = \langle 1\rangle$. But $C_{S_0}(u) \subseteq C_S(\langle u, y\rangle)$, contradicting the fact that $\langle t, z\rangle \cap C_{S_0}(u)' \neq \langle 1\rangle$.

Thus $\langle y\rangle$ permutes the 2-components of $K(u)$ in orbits of length 2. Let $K(u) = K_1K_2 \cdots K_r$. We deduce from Theorem 2.9 first that, for all i, $\tilde{K}_i \simeq L_2(q_i)$ for some $q_i \in \{5, 7\}$. Now $\langle t, z, u\rangle$ normalizes each K_i by Lemma 5.5(2). It follows easily that $\langle t, z, u\rangle$ is 3-connected to every eights subgroup of $T_1 \cap K_1C_G(\tilde{K}_1)$. As $m(C_{T_1}(\tilde{K}_i) \geqslant 3$ for all i, it follows by Lemma 2.28 that $O(K_i) = \langle 1\rangle$ for all i. Choose notation so that $K_2 = (K_1)^y$. Let $\langle z_i\rangle = K_i \cap S$. Then $z_1z_2 = z_1z_1^y \in [S, y] \subseteq \langle t, z\rangle$. As $z_1z_2 \in K_1K_2$ and t induces an outer automorphism on each K_i, $z_1z_2 \in \{z, tz\}$. In particular, z_1z_2 inverts $C_X(u)$ by Lemma 5.5(2). Thus $C_X(u) \subseteq K_1 \times K_2$. As $K(u)$ is the normal closure of $C_X(u) \cap L(C_G(u))$ in $L(C_G(u))$, it follows that $K(u) = K_1 \times K_2$. In particular, $C_S(u)$ normalizes $K_1 \times K_2$. Let $*$ denote homomorphic images in $N^* = N_G(K_1 \times K_2) \cap C_G(t)/C_G(\langle K_1 \times K_2, t\rangle)$. Then $C_S(u)^*$ is isomorphic to a subgroup of $E_4 \wr \mathbf{Z}_2$ with y^* inducing a wreathing automorphism. Thus $| C_S\langle u, y\rangle^* | = 8$, whence $C_S(\langle u, y\rangle)^* = \langle t^*, y^*, z^*\rangle$ is central-by-cyclic, hence abelian. This again contradicts the fact that $\langle t, z\rangle \cap C_S(\langle u, y\rangle)' \neq \langle 1\rangle$. The proof is complete.

LEMMA 5.9. *$\Omega_1(Q) = \langle t, z\rangle$ and $m(\langle S_0, t, z\rangle) = m(S) = 6$.*

Proof. Clearly the second statement follows from the first and Lemma 5.8(2). Thus we may assume, using Lemma 5.8(1), that $Q \simeq \langle t\rangle$. Thus, by Condition 1 in the definition of $\mathscr{L}$, (t, x, L) is an unbalancing triple for some $x \in I(C_G(t))$. Thus $Z(S) = \langle t, u\rangle$ for some $u \in I(\langle S_0, t\rangle) - \{t\}$. As $t \notin Z(T)$, $t \sim tu$ in $N_T(S)$. Then $C_X(ut) \subseteq O(C_G(ut))$. Hence $C_X(u) \cap O(C_G(u)) = \langle 1\rangle$. As $\langle u\rangle = Z(T)$, $u \in Z^*(K_1)$ for any 2-component, K_1, of $K(u)$. Thus $\tilde{K}_1 \simeq L_3(4)$. As K_1 is not standard in G by Corollary 2.11, either $O(K_1) \nsubseteq Z(K_1)$ or there exists $v \in I(C_G(K_1)$ with $u \neq v$ and $O(\langle K_1^{C_G(v)}\rangle) \nsubseteq Z(\langle K_1^{C_G(v)}\rangle)$. As (u, t, K_1) is an unbalancing triple, $K_1 \in \mathscr{L}$ in the former case and $K_2 = \langle K_1^{C_G(v)}\rangle$ is in $\mathscr{L}$ in the latter case. Suppose first that the latter case holds. Then $u \in Z^*(K_2)$ and $\langle u, v\rangle \subseteq C_G(\tilde{K}_2)$. But then $| C_G(\tilde{K}_2)|_2 > | C_G(L)|_2$, violating the choice of L. Thus $K_1 \in \mathscr{L}$ and, by the choice of L, $\langle u\rangle \in \mathrm{Syl}_2(C_G(\tilde{K}_1))$. But then as S normalizes K_1 and t induces a unitary automorphism on $\tilde{K}_1$, we have $| S | \leqslant 2^5$, contradicting the fact that $| S | \geqslant 2^8$.

LEMMA 5.10. *Let $u \in \mathscr{U}$ with $| C_X(u)|$ maximal. Then $K(u) = K_1K_2 \cdots K_r$ with $\tilde{K}_i \simeq L_2(q_i)$ for some $q_i \in \{5, 7\}$. Moreover, $1 \leqslant r \leqslant 4$.*

Proof. As $\exp Z \leqslant 2$ by Lemma 5.8(2), $u^L \cap Z(S) \neq \varnothing$ for all $u \in \mathscr{U}$. Thus we may assume that $u \in Z(S)$, whence S normalizes $K(u)$.

Suppose first that $K_1 \trianglelefteq K(u)$ with $\tilde{K}_1 \cong L_3(4)$. Then S normalizes K_1 by Theorem 2.9. As $t \in Z(S)$ and t induces a unitary automorphism of $\tilde{K}_1$, $m(S/C_S(\tilde{K}_1)) \leqslant 3$. Thus $m(\langle S_0, t, z\rangle) \geqslant 6$ implies that $m(C_{\langle S_0, t, z\rangle}(\tilde{K}_1)) \geqslant 3$. Then by Lemma 2.28, $O(K_1) \subseteq Z(K_1)$. By Theorem 2.13(1), K_1 is not standard in G. Thus by Corollary 2.11, there exists $v \in I(C_G(K_1))$ with $L_1 = \langle (K_1)^{L(C_G(v))}\rangle$ satisfying $O(L_1) \nsubseteq Z(L_1)$. Then $L_1 = O(L_1)\, K_1$ and $L_1 \in \mathscr{L}$, since (u, t, K_1) is unbalancing. By the choice of L and Lemma 5.8, $\exp(O_2(K_1)) \leqslant 2$ and if $O_2(K_1) \neq \langle 1\rangle$, then $| C_G(\hat{L}_1)|_2 \leqslant 4$. Thus if $O_2(K_1) \neq \langle 1\rangle$, then $\langle u, v\rangle \in \mathrm{Syl}_2(C_G(\hat{L}_1))$. But then $\langle u, v\rangle$ is self-centralizing in a Sylow 2-subgroup of $C_G(\tilde{K}_1)) \leqslant 2$ contradicting the fact that $m(C_{\langle S_0, t, z\rangle}(\tilde{K}_1) \geqslant 3$.

We have shown thus far that if $K_1 \trianglelefteq K(u)$ with $\tilde{K}_1 \cong L_3(4)$, then $O(K_1) \subseteq Z(K_1)$ and $O_2(K_1) = \langle 1\rangle$. As $S \subseteq N_G(K_1)$, $Z(S) \cap K_1 \neq \langle 1\rangle$. Let $z_1 \in I(Z(S) \cap K_1)$ and let $z_1 \in Z(R_1)$ with $R_1 \in \mathrm{Syl}_2(K_1)$. As z_1 and z invert $C_X(u) \cap K_1/C_X(u) \cap Z(K_1)$, zz_1 either centralizes K_1 or induces a unitary automorphism on K_1. Thus, for some $z_0 \in \{z, tz\}$ and $w \in C_G(K_1)$, we have $z_0 = z_1 w$. As t induces a unitary automorphism on K_1, there exists $z_2 \in Z(R_1)$ with $[z_2, t] = z_1$. Thus $t \sim tz_1$ in $C_G(z_0)$. It follows that $z_1 \in Z(S) \cap C_S(\tilde{L})$. Thus $z_1 \in \{z, tz\}$. In any case z_1 inverts $C_X(u)$. Thus $C_X(u) \subseteq K_1$. Let A be an elementary abelian subgroup of $\langle S_0, t, z\rangle$ of rank 6 and let f be a 3-element of L normalizing but not centralizing A. Then $A = A_0 \times \langle t, z\rangle$ where $A_0 = [A_0, f] \subseteq L$. As $\{a, at\} \cap \mathscr{U} \neq \varnothing$ for all $a \in A_0$, the choice of u implies that $| C_X(a)| \leqslant | C_X(u)|$ for all $a \in \langle A_0, t\rangle - \langle t\rangle$. Let $B = C_{\langle A_0, t\rangle}(K_1)$. As $m(S/C_S(K_1)) \leqslant 3$, $m(B) \geqslant 2$ and $C_X(u) \subseteq C_X(b)$ for all $b \in B^{\#}$. Thus $C_X(u) = C_X(b)$ for all $b \in B^{\#}$. But $X = \langle C_X(b) \mid b \in B^{\#}\rangle$. Thus $X \subseteq Z(L)$, contrary to the choice of L.

We have now established that $K(u) = K_1 K_2 \cdots K_r$ with $\tilde{K}_i \cong L_2(q_i)$ for some $q_i \in \{5, 7\}$. As $m(S) = 6$ and $\langle u, t\rangle \cap K(u) = \langle 1\rangle$, we have $r \leqslant 4$, as desired.

LEMMA 5.11. *Let $u \in \mathscr{U}$ with $| C_X(u)|$ maximal. Then $K(u) \neq K_1 K_2 \cdots K_r$ with $\tilde{K}_i \cong L_2(q_i)$ for some $q_i \in \{5, 7\}$ and $1 \leqslant r \leqslant 4$.*

Proof. As in Lemma 5.10, we may assume that $u \in Z(S)$. Suppose that $K(u) = K_1 K_2 \cdots K_r$ with $\tilde{K}_i \cong L_2(q_i)$ for some $q_i \in \{5, 7\}$. Let $E = S \cap K(u)$. Then E is elementary of rank r and $\langle t, u\rangle \cap E = \langle 1\rangle$. If $r \geqslant 3$, then $\langle E, t, z\rangle/\langle t, z\rangle$ is a normal elementary subgroup of $S/\langle t, z\rangle$ of 2-rank at least 2 not containing $Z(S/\langle t, z\rangle)$. This contradicts Corollary 2.4(4). Thus $r \leqslant 2$ and, by Corollary 2.4(3), if $r = 2$, then $E \subseteq Z(S)$. Thus $S/C_S(\widetilde{K(u)})$ is abelian. Now $Z(S) \subseteq \langle Z(S_0), t, z\rangle$ and $Z(S_0) \subseteq S'$. Thus $| Z(S)/Z(S) \cap S' | \leqslant 4$. As $\langle E, t\rangle \subseteq Z(S)$ and $\langle E, t\rangle \cap S' = \langle 1\rangle$, $| E | = 2$. Thus $r = 1$. As $SCN_6(S) \neq \varnothing$, $W_A = \langle 1\rangle$ for all eights-subgroups A of S by Lemma 2.29. As $m_2(C_S(\tilde{K}_1)) = 4$, $O(K_1) = \langle 1\rangle$ by Lemma 2.28. Now we may repeat, almost verbatim, the third paragraph of the proof of Lemma 5.10 to derive a contradiction and complete the proof.

As Lemmas 5.10 and 5.11 contradict each other, we have completed the proof of Theorems 1.2 and 1.3.

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