ADVANCES IN MATHEMATICS 33, 109–143 (1979)

A New Look at Interpolation Theory for Entire Functions of One Variable

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INTRODUCTION

The existence of solutions of the inhomogeneous Cauchy–Riemann equations as a powerful tool in the study of analytic functions of several complex variables is well demonstrated in the monograph [22] of L. Hörmander. The main objective of this expository paper is to show how this technique can be used to unify and simplify the study of interpolation problems for entire functions of one complex variable.

The problems we consider are typified by the following model. Let $\{z_k\}$ be a sequence of complex numbers diverging to ∞, $\{m_k\}$ a sequence of positive integers, and $\{a_{k,j}\}$ a doubly-indexed sequence of complex numbers satisfying the growth condition

$$| a_{k,j} | \leqslant A \exp(B \mid z_k \mid), \qquad 0 \leqslant j < m_k, \qquad k = 1, 2,... \tag{1}$$

for some constants $A, B > 0$. Under what conditions does there exist an entire function λ such that

(i) $\lambda^{(j)}(z_k)/j! = a_{k,j}$; and

(ii) λ is of exponential type, i.e., for some constants $A', B' > 0$, we have

$$| \lambda(z)| \leqslant A' \exp(B' \mid z \mid)\,?$$

Various intersting problems are obtained by imposing conditions on the z_k, m_k, $a_{k,j}$, or by changing the growth condition (1). We give here both new proofs of some old results, (due primarily to A. F. Leont'ev and others [27, 9, 10, 11] and some sharper, more general results on various interpolation problems.

0001-8708/79/080109–35$05.00/0

For example, concerning the model problem just stated Theorem 4 implies the following result.

THEOREM. *Let f be an entire function of exponential type whose zero set is precisely the sequence z_k with multiplicities m_k. The necessary and sufficient condition that the problem has a solution for every sequence $a_{k,j}$ satisfying* (1) *is that*

$$\frac{|f^{(m_k)}(z_k)|}{m_k!} \geqslant \epsilon \exp(-C \mid z_k \mid), \qquad k = 1, 2,...$$

for some constants ϵ, $C > 0$.

Theorem 4 also applies to growth conditions which are not functions of $| z |$.

The application of the existence of solutions of the Cauchy–Riemann equations to interpolation problems is to prove a "semi-local" interpolation theorem. For $p(z)$ a nonnegative, continuous subharmonic function on the complex plane, let A_p denote the algebra of all entire functions f on $\mathbb{C}$ such that $| f(z)| \leqslant A \exp(Bp(z))$, $z \in \mathbb{C}$, for some constants A, $B > 0$ (which depend on f). If $f_1 ,..., f_m \in A_p$ and ϵ, $C > 0$ let $|f_1|^2 \cdots + | f_m |^2$ and

$$S(f; \epsilon, C) = \{z \in \mathbb{C}: |f(z)| < \epsilon \exp(-Cp(z))\}. \tag{2}$$

Intuitively, $S(f; \epsilon, C)$ is a "small neighborhood" of the zeros $| z_k |$ of $f = (f_1 ,..., f_m)$. A standard argument (see Section 1) then will show the following result is true.

SEMI-LOCAL INTERPOLATION THEOREM. *Let $\tilde{\lambda}(z)$ be analytic on $S(f; \epsilon, C)$ and satisfy $| \tilde{\lambda}(z)| \leqslant A \exp(Bp(z))$ for $z \in S(f; \epsilon, C)$. Then (provided p satisfies the conditions 4i and 4ii of Section 1), there exists an entire function $\lambda \in A_p$, constants ϵ_1, C_1, A', $B' > 0$ and functions $\alpha_1 ,..., \alpha_m$ analytic on $S(f; \epsilon_1 , C_1)$ such that for all $z \in S(f; \epsilon_1 , C_1)$,*

$$\lambda(z) = \tilde{\lambda}(z) + \sum_{i=1}^{m} \alpha_i(z) f_i(z) \tag{3}$$

and

$$| \alpha_i(z)| \leqslant A_1 \exp(B_1 p(z)).$$

Thus, if the f_i have common zeros $\{z_k\}$ with multiplicites m_k, then from (3)

$$\lambda^{(j)}(z_k) = \tilde{\lambda}^{(j)}(z_k) \qquad \text{for all} \quad 0 \leqslant j < m_k , \qquad k = 1, 2,... .$$

Consequently, the study of interpolation problems is reduced to finding a solution $\tilde{\lambda}$ "semi-locally"; i.e., on the set $S(f; \epsilon, C)$. This approach is the one variable version of the method used by Ehrenpreis [15] and Palamadov [32] in their studies of interpolation on algebraic varieties in $\mathbb{C}^n$.

The study of the semi-local interpolation problem leads immediately to several interesting questions. For example:

(1) Given $\{z_k, m_k\}$, can a "good" choice be found for $f_1, \ldots, f_m \in A_p$ with common zeros at the z_k of multiplicity m_k?

(2) How big are the sets $S(f; \epsilon, C)$?

(3) How does the structure of $S(f; \epsilon, C)$ affect the $a_{k,j}$ for which the interpolation problem has a solution?

Our treatment is far from complete, mostly because with general growth conditions complete solutions to these problems are not known.

We have tried to make the paper as self-contained as possible. Hence, we have included a proof of the basic existence theorem for the $\bar{\partial}$-operator in one variable, and "new" proofs of Weierstrass' theorem on the existence of analytic functions with prescribed zeros and the related theorem on interpolation by entire functions ("new" in that we have not seen them in the literature, though they are no doubt well-known to workers in the area).

Interpolation problems have been studied for a long time due to their applications to number theory, harmonic analysis, approximation theory, etc. We give here (Section 4 below) a representative application, a derivation of a major result in abelian harmonic analysis, the Fourier representation of mean-periodic functions of one variable obtained by L. Schwartz in [35]. In this sense we fulfill a second aim of this paper, which is to present a background to our recent work on interpolation in several variables [6], where we obtain a generalization of Schwartz' theorem to mean-periodic functions of n variables.

1. Preliminaries

We start by recalling some standard notation. If Ω denotes an open subset of the complex plane $\mathbb{C}$, $C^k(\Omega)$ (respectively $C_0{}^k(\Omega)$) denotes the space of complex-valued functions in Ω with continuous partial derivatives of order $\leqslant k$ (respectively, those functions in $C^k(\Omega)$ with compact support in Ω), $1 \leqslant k \leqslant \infty$. (We will often omit Ω if $\Omega = \mathbb{C}$.) The coordinates in Ω are denoted $z = x + iy$ and $\bar{z} = x - iy$. If $u \in C^1(\Omega)$, we can define two differential operators

$$\frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{1}{i}\frac{\partial u}{\partial y}\right), \qquad \frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{1}{i}\frac{\partial u}{\partial y}\right),$$

so that its differential can be written $du = (\partial u/\partial z)\, dz + (\partial u/\partial \bar{z})\, d\bar{z}$. The holomorphic functions in Ω, are precisely those $u \in C^1(\Omega)$ satisfying the homogeneous Cauchy–Riemann equation: $\partial u/\partial \bar{z} = 0$. It is in fact enough for u to be a distribution in Ω and satisfy that equation in the sense of the space of distributions $\mathscr{D}'(\Omega)$ (see [36] for notation and definitions). The space of holomorphic functions in Ω is denoted $A(\Omega)$. Our main interest resides on the spaces of holomorphic functions with growth conditions, $A_p(\Omega)$ defined below. We will mainly restrict ourselves to the case $\Omega = \mathbb{C}$.

DEFINITION. For $p(z)$ a subharmonic function on Ω, $p \not\equiv -\infty$, in every component of Ω, let $A_p(\Omega)$ be the space of all functions $f \in A(\Omega)$ such that for some constants A, B, which depend on f, satisfy

$$f(z)| \leqslant A \exp(Bp(z)) \qquad z \in \Omega.$$

Similarly, let $W_p(\Omega)$ be the space of all measurable functions g in Ω such that for some constant $c = c(g)$ we have

$$\int_\Omega |g(z)|^2 \exp(-cp(z))\, d\lambda(z) < \infty,$$

where $d\lambda$ denotes the Lebesgue measure. Note that given $g \in L^2_{\text{loc}}(\Omega)$, there exists p such that $g \in W_p(\Omega)$. To see this take $p(z) = \psi(-\log d(z))$, where $d(z)$ is the distance from z to the complement of Ω (or $|z|$ if $\Omega = \mathbb{C}$) and ψ is a convex function that increases very rapidly (see [22, p. 94]).

We are actually interested in the space $A_p = A_p(\mathbb{C})$ where p satisfies the following two extra conditions:

(4i) $p(z) \geqslant 0$ and $\log(1 + |z|^2) = O(p(z))$;

(4ii) there exist constants C, $D > 0$ such that $|\zeta - z| \leqslant 1$ implies $p(\zeta) \leqslant Cp(z) + D$.

It is not the conditions on p which are important, but rather the following consequences.

(a) All polynomials belong to A_p.

(b) A_p is closed under differentiation; i.e. $f \in A_p$ implies $f' \in A_p$.

(c) $W_p(\mathbb{C}) \cap A(\mathbb{C}) = A_p(\mathbb{C})$.

The conditions also imply that $p(z) = O(\exp(A|z|))$ for some $A > 0$, and the stronger condition

(b′) If $f \in A_p$, then there are constants A, $B > 0$ such that

$$\sum_{k=0}^{\infty} \left| \frac{f^{(k)}(z)}{k!} \right| \leqslant A \exp(Bp(z)).$$

The conditions (4i) and (4ii) can be weakened considerably and most of the theorems presented here will still work. Furthermore, we can essentially remove them by replacing the space A_p by a space $A_{\mathscr{P}}$ = direct limit of spaces A_p, $p \in \mathscr{P}$, under some suitable conditions on $\mathscr{P}$ (see [21, 24]).

EXAMPLE 1. $p(z) = |z|$. Then A_p is the space of all entire functions of exponential type,

$$|f(z)| \leqslant A \exp(B|z|), \quad \forall z \in \mathbb{C}$$

for some $A, B > 0$. This example is the space in which interpolation problems were originally studied by A. F. Leont'ev [27, 28], and where the most complete results are known.

EXAMPLE 2. $p(z) = |\operatorname{Im} z| + \log(1 + |z|^2)$, ($\operatorname{Im} z$ = imaginary part of z). In this case A_p is the space $\hat{\mathscr{E}}'$ of Fourier transforms of distributions with compact support in the real line (see [15, 36]), i.e. all entire functions such that for some constants $A, B, C > 0$

$$|f(z)| \leqslant A(1 + |z|)^B \exp(C|\operatorname{Im} z|).$$

Even for this example a *complete* solution to the interpolation problems is not known.

EXAMPLE 3. $p(z) = \log(1 + |z|^2)$. Then A_p is the space of all polynomials.

EXAMPLE 4. $p(z) = p(x + iy) = |x|^\alpha + |y|^\beta$, $\alpha, \beta \geqslant 1$.

EXAMPLE 5. $p(z) = |\operatorname{Im} z| + |z|^q$, $0 < q < 1$.

EXAMPLE 6. $p(z) = |z|^\rho$, $0 < \rho < \infty$. Then A_p is the space of all entire functions of order $\leqslant \rho$ and finite type, i.e. for some $A, B > 0$,

$$|f(z)| \leqslant A \exp(B|z|^\rho).$$

The space of all functions of order $< \rho$ is not an A_p space, but rather one of the more general $A_{\mathscr{P}}$ mentioned above. In the same way, there are many spaces considered in analysis where the growth conditions in (2) are not given by subharmonic functions, but rather by an "arbitrary" function $\psi(z)$. By introducing the function $p(z)$ = largest subharmonic minorant of ψ, which will be $\not\equiv -\infty$ if the space is non-trivial, one can reduce problems to the spaces $A_{\mathscr{P}}$.

The space A_p carries a natural topology as an inductive limit of Banach spaces, and will be considered endowed with it without further explanation (see [39]).

In what follows the inhomogeneous Cauchy–Riemann equation

$$\frac{\partial u}{\partial \bar{z}} = v \tag{5}$$

will play a central role. The main point is that this equation is always solvable and a priori estimates are available for the solution. We follow the treatment in

[22]. The proof is simpler due to the fact that we are only considering functions of one complex variable so the boundary of Ω introduces no difficulty. First, let us observe that if $v \in C^k(\Omega)$, $k \geqslant 1$, then a distribution solution u to (5) is at least of class C^{k+1}. In fact, we have

$$\frac{1}{4}\Delta u = \frac{\partial^2 u}{\partial z\, \partial \bar{z}} = \frac{\partial v}{\partial z},$$

and Weyl's lemma does the rest [36]. Furthermore, it is very easy to solve (5) explicitly if $v \in C_0^1$. Namely, a solution u is given by

$$u(z) = \frac{1}{\pi} \int \frac{v(\zeta)}{z - \zeta}\, d\lambda(\zeta) = \frac{1}{2\pi i} \iint \frac{v(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} \tag{6}$$

where the domain of integration is the whole plane. The proof of this fact is an immediate consequence of the following variant of Cauchy's integral formula. (Just take ω a sufficiently big disk in the next lemma.)

LEMMA 1 (Pompeiu's formula, see [22, p. 3]). *Let ω be a bounded open set in $\mathbb{C}$, with boundary $\partial\omega$ the union of a finite number of C^1 Jordan Curves. If $u \in C^1(\bar{\omega})$ then for all $\zeta \in \omega$*

$$u(\zeta) = \frac{1}{2\pi i} \left\{ \int_{\partial\omega} \frac{u(z)}{z - \zeta}\, dz + \iint_{\omega} \frac{\partial u}{\partial \bar{z}}(z)\, \frac{dz \wedge d\bar{z}}{z - \zeta} \right\}. \tag{7}$$

Proof. It is enough to apply Green's theorem on the set $\omega \backslash \Delta(\zeta; \epsilon)$, $\epsilon > 0$ and let $\epsilon \to 0$. Here $\Delta(\zeta; \epsilon) = \{z \in \mathbb{C}: |z - \zeta| < \epsilon\}$. We will keep this notation throughout the paper. ∎

The reason we have introduced the spaces $W_p(\Omega)$ is to be able to solve equation (5) with an arbitrary smooth or even L^2_{loc} function on the right hand side. As pointed out above, if $v \in L^2_{\text{loc}}(\Omega)$ then we can always find p such that $v \in W_p(\Omega)$. We first prove the following fundamental a priori estimate (see [22, Chap. 4] for the case of several variables).

LEMMA 2. *Let Ω be an open subset of C, p subharmonic in Ω, and $f \in C_0^2(\Omega)$. Then*

$$\int_\Omega |f|^2 e^p\, d\lambda \leqslant \frac{1}{2} \int_\Omega \left| \frac{\partial f}{\partial \bar{z}} \right|^2 (1 + |z|^2)^2 e^p\, d\lambda. \tag{8}$$

Proof. Let us first assume $p \in C^2(\Omega)$. Then the following identity can be verified by direct computation.

$$\begin{aligned} &e^p \left| \frac{\partial f}{\partial z} + f \frac{\partial p}{\partial z} \right|^2 + e^p |f|^2 \frac{\partial^2 p}{\partial z\, \partial \bar{z}} \\ &\quad = \frac{\partial}{\partial \bar{z}} \left[e^p \left(\bar{f} \frac{\partial f}{\partial z} + |f|^2 \frac{\partial p}{\partial z} \right) \right] - \frac{\partial}{\partial z} \left(e^p \bar{f} \frac{\partial f}{\partial \bar{z}} \right) + e^p \frac{\partial \bar{f}}{\partial z} \frac{\partial f}{\partial \bar{z}}. \end{aligned} \tag{9}$$

Since f has compact support, by integrating (9) we obtain

$$\begin{aligned}\int_\Omega \left|\frac{\partial f}{\partial \bar z}\right|^2 e^p\,d\lambda &= \int_\Omega |f|^2 \frac{\partial^2 p}{\partial z\,\partial \bar z} e^p\,d\lambda + \int_\Omega \left|\frac{\partial f}{\partial z} + f\frac{\partial p}{\partial z}\right|^2 e^p\,d\lambda \\ &\geqslant \int_\Omega |f|^2 \frac{\partial^2 p}{\partial z\,\partial \bar z} e^p\,d\lambda.\end{aligned} \tag{10}$$

As p is subharmonic and C^2, we have that $p_1(z) = p(z) + 2\log(1 + |z|^2)$ is also subharmonic and C^2 and, moreover $\partial^2 p_1/\partial z\,\partial \bar z \geqslant 2/(1 + |z|^2)^2$. Replacing p by p_1 in (10) and using the last inequality on the Laplacian of p_1 we get, for $f \in C_0^2(\Omega)$ and p at least C^2,

$$\int_\Omega \left|\frac{\partial f}{\partial \bar z}\right|^2 (1 + |z|^2)^2 e^p\,d\lambda \geqslant 2\int_\Omega |f|^2 e^p\,d\lambda. \tag{11}$$

It is easy to remove the assumption that $p \in C^2(\Omega)$ (see [22], p. 94). Let Ω_ϵ denote the set of points in Ω whose distance to $\mathbb{C}\backslash\Omega$ is at least ϵ (if $\Omega = \mathbb{C}$, then $\Omega_\epsilon = \mathbb{C}$ also). For a given $f \in C_0^2(\Omega)$, $\operatorname{supp} f \subseteq \Omega_\epsilon$ for all ϵ sufficiently small. Let χ be a C^∞ function with support in $\{|z| \leqslant 1\}$, $\chi \geqslant 0$, $\chi(z) = \chi(|z|)$ and $\int \chi\,d\lambda = 1$. As usual define $\chi_\epsilon(\zeta) = \epsilon^{-2}\chi(\zeta\epsilon^{-1})$ and $p_\epsilon(z) = (p * \chi_\epsilon)(z) = \int p(z - \zeta)\,\chi_\epsilon(\zeta)\,d\lambda(\zeta)$. Then $p_\epsilon \in C^\infty(\Omega_\epsilon)$, p_ϵ is subharmonic and $p_\epsilon(z) \downarrow p(z)$ as $\epsilon \to 0+$. (See e.g. [22, p. 19].) Hence (11) holds for p_ϵ for all ϵ small. By the monotone convergence theorem, it also holds for p.

We are ready now to state and prove Hörmander's theorem [22, Th. 4.4.2].

THEOREM 1. *Let Ω be an open subset of $\mathbb{C}$, p subharmonic in Ω and v a function in $W_p(\Omega)$, such that*

$$\int_\Omega |v|^2 \exp(-p)\,d\lambda = M < +\infty.$$

Then there exists a function u satisfying

$$\int_\Omega \frac{|u|^2 \exp(-p)}{(1 + |z|^2)^2}\,d\lambda \leqslant M/2$$

and,

$$\frac{\partial u}{\partial \bar z} = v$$

in the sense of distributions. (Both functions are necessarily in $L^2_{\mathrm{loc}}(\Omega)$.) If $v \in C^\infty(\Omega)$ so does u.

Proof. The proof is a simple application of functional analysis and the basic a priori estimate (8). For $j = 0, 1$, let H_j be the Hilbert space of all functions f on Ω such that

$$\| f \|_j^2 := \int_\Omega | f |^2 \exp(p_j)\, d\lambda < \infty,^1$$

where $p_j(z) := p(z) + 2j \log(1 + | z |^2)$. The dual space H_j' to H_j can be identified to the space of all locally square integrable functions g such that

$$\||\, g \,\||_j^2 := \int_\Omega | g |^2 \exp(-p_j)\, d\lambda < \infty$$

with the pairing

$$\langle f, g \rangle_j := \int_\Omega fg\, d\lambda, \qquad f \in H_j, \quad g \in H_j'.$$

Since p is locally bounded above, it follows easily that $C_0^\infty(\Omega)$ is dense in H_j. Consider now the operator $D : H_0 \to H_1$ with domain $C_0^\infty(\Omega)$, defined by

$$Df = -\frac{\partial f}{\partial \bar{z}}.$$

Since the domain of D is dense in H_0, its adjoint D^* is well defined with domain in H_1' and range in H_0'. Let us compute D^* explicitly. If $u \in H_1'$ and $D^*u = v \in H_0'$, then

$$\langle f, v \rangle_0 = \langle f, D^*u \rangle_0 = \langle Df, u \rangle_1, \qquad \forall f \in C_0^\infty(\Omega).$$

This just means

$$\int_\Omega fv = -\int_\Omega \frac{\partial f}{\partial \bar{z}} u, \qquad \forall f \in C_0^\infty(\Omega);$$

that is, $\partial u / \partial \bar{z} = v$ in the sense of distributions. What we have to prove, therefore, is that D^* is onto.

To see this, let $v \in H_0'$ and consider the linear functional L defined on the range of D in H_1 by

$$L(Df) = \langle f, v \rangle_0, \qquad f \in C_0^\infty(\Omega).$$

$L(Df)$ is well defined since D is injective on $C_0^\infty(\Omega)$. By applying the Schwartz inequality and the a-priori estimate (8) we obtain

$$\begin{aligned} | L(Df) |^2 &\leqslant \left(\int_\Omega | f |^2 e^p \right) \left(\int_\Omega | v |^2 e^{-p} \right) \\ &\leqslant \frac{1}{2} \left(\int_\Omega \left| \frac{\partial f}{\partial \bar{z}} \right|^2 (1 + | z |^2)^2 e^p \right) \||\, v \,\||_0^2 \\ &= \frac{1}{2} \||\, v \,\||_0^2 \| Df \|_1^2 . \end{aligned}$$

[1] Here we are following the notation, $A := B$, means that A is being defined by the right hand side of this expression.

Hence, by the Hahn-Banach theorem, L may be extended to a continuous linear functional on all of H_1 with norm $\leqslant ||| v |||_0/2^{1/2}$. Thus, there exists $u \in H_1'$ such that

$$||| u |||_1^2 \leqslant \tfrac{1}{2} ||| v |||_0^2,$$

and

$$\langle f, v\rangle_0 = L(Df) = \langle Df, u\rangle_1 \qquad \forall f \in C_0^\infty(\Omega),$$

which provides the desired solution u. ∎

Corollary 1. *If $v \in L^2_{\mathrm{loc}}(\Omega)$, then there is another function $u \in L^2_{\mathrm{loc}}(\Omega)$ such that $\partial u/\partial \bar{z} = v$. Furthermore, if $v \in C^\infty(\Omega)$ then $u \in C^\infty(\Omega)$.*

Proof. It was pointed out after the definition of $W_p(\Omega)$ that p can always be chosen so that $v \in W_p(\Omega)$. Thus, the corollary follows from the theorem. ∎

We can now give very short proofs of Weierstrass' theorem on the existence of analytic functions with given zeroes and a corresponding interpolation result (see [33, p. 298]). If $f \in A(\Omega)$, $f \not\equiv 0$, then its *multiplicity variety* $V(f)$ is the set of pairs (z_k, m_k), $z_k \in \Omega$, m_k integer $\geqslant 1$ where z_k runs over all the zeroes of f in and m_k denotes the multiplicity of that zero. (By abuse of language, we will also write $z_k \in V(f)$.) Clearly, the z_k have no accumulation point in Ω, the theorem of Weierstrass is the converse to this statement.

Theorem 2. *Given a set $V = \{(z_k, m_k)\colon z_k \in \Omega,\ m_k \geqslant 1\}$, where the z_k are distinct and have no accumulation point in Ω, then there is a function $f \in A(\Omega)$ with $V(f) = V$.*

Proof. The proof follows what in several variables is called Oka's principle [22, p. 138]. Namely, first construct such a function within the class $C^\infty(\Omega)$ and then modify it to get a function in $A(\Omega)$. Assume that we have a function $\varphi \in C^\infty(\Omega)$ and pairwise disjoint open neighborhoods $\Delta_k \subseteq \Omega$ such that $z_k \in \Delta_k$, $\varphi(z) = 0$ only if $z = z_k$ for some k, and for some nonvanishing $h_k \in C^\infty(\Delta_k)$,

$$\varphi(z) = (z - z_k)^{m_k} h_k(z) \qquad \text{in } \Delta_k. \tag{12}$$

The desired function f will be of the form $f := \varphi e^\psi$, $\psi \in C^\infty(\Omega)$. We only have to look for ψ such that $f \in A(\Omega)$, because then it is clear from (12) and the fact that φ vanishes only at the z_k that $V(f)$. Now,

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial \varphi}{\partial \bar{z}} e^\psi + \varphi \frac{\partial \psi}{\partial \bar{z}} e^\psi = 0$$

is the condition of analyticity. Hence, ψ must satisfy,

$$\frac{\partial \psi}{\partial \bar{z}} = -\frac{1}{\varphi} \frac{\partial \varphi}{\partial \bar{z}}. \tag{13}$$

The right hand side is in $C^\infty(\Omega)$ since (12) implies that $\partial\varphi/\partial\bar{z} = (z - z_R)^{m_R}(\partial h_R/\partial_{\bar{z}})$ in Δ_k. By Corollary 1, a C^∞ solution to (13) does in fact exist, and this concludes the proof of Theorem 2 modulo the construction of the function φ, which we carry out below by a procedure that can be simplified somewhat if $\Omega = \mathbb{C}$ or if Ω is bounded. We can clearly assume that $0 \in \Omega$ and $z_k \neq 0$ for every k. Let us call Ω^* the open set in the Riemann sphere obtained from Ω by the inversion $z \to w := 1/z$. If $b\Omega^*$ denotes the boundary of Ω^*, then it is located in a bounded region of the finite plane $\mathbb{C}$, since now $\infty \in \Omega^*$. We can order the points w_k, $w_k = 1/z_k$, so that $\operatorname{dist}(w_k, b\Omega^*) \geqslant \operatorname{dist}(w_{k+1}, b\Omega^*)$, and choose $w_k' \in b\Omega^*$ such that $| w_k - w_k' | = \operatorname{dist}(w_k, b\Omega^*)$. Let U_k be the open disk with center w_k' and radius $r_k = 2 | w_k - w_k' |$. Then U_k contains the line segment $[w_k, w_k']$ joining w_k to w_k'. Further, the open sets $U_k \cap \Omega^*$ form a locally finite family in Ω^* since $r_k \to 0$ and $w_k' \notin \Omega^*$. Choose C^∞ functions μ_k with compact support in U_k, $0 \leqslant \mu_k \leqslant m_k$, and $\mu_k(w) = m_k$ on some neighborhood of $[w_k, w_k']$. Let $\log[(w - w_k)/(w - w_k')]$ be any branch of the logarithm analytic outside $[w_k, w_k']$. Then the functions

$$g_k(w) = \exp\left(\mu_k(w) \log \frac{w - w_k}{w - w_k'}\right) = \left(\frac{w - w_k}{w - w_k'}\right)^{\mu_k(w)}$$

are single valued, C^∞ functions on the Riemann sphere except at w_k', analytic on a neighborhood of w_k, $g_k \equiv 1$ outside of U_k, and $g_k = 0$ only at $w = w_k$. The multiplicity of vanishing is m_k since $\mu_k \equiv m_k$ on a neighborhood of w_k. The function

$$g(w) = \prod_k g_k(w)$$

is in $C^\infty(\Omega^*)$, being a locally finite product of functions in $C^\infty(\Omega^*)$. It is clear that $\varphi(z) := g(1/z)$ is in $C^\infty(\Omega)$ and satisfies the required conditions. ∎

THEOREM 3. *Let $V = \{(z_k, m_k)\}$ be a multiplicity variety in Ω. Let $a_{k,l}$ be any sequence of complex numbers, where $0 \leqslant l < m_k$, and k runs over the same set of indices as the pairs in V. Then there is a function $g \in A(\Omega)$ whose lth Taylor coefficient at z_k is precisely $a_{k,l}$.*

Proof. Let $f \in A(\Omega)$ be such that $V(f) = V$. Let $\epsilon_k > 0$ be chosen so that the disks $\Delta_k = \Delta(z_k; \epsilon_k)$ are disjoint and let $P_k(z)$ be the polynomials

$$P_k(z) := \sum_{l=0}^{m_k-1} a_{k,l}(z - z_k)^l. \tag{14}$$

Further, choose $\chi_k \in C_0^\infty(\Delta_k)$, $0 \leqslant \chi_k \leqslant 1$, $\chi_k \equiv 1$ on the disk $\Delta(z_k; \frac{1}{2}\epsilon_k)$, and define, for $\psi \in C^\infty(\Omega)$.

$$g(z) := \sum_k P_k(z)\, \chi_k(z) + \psi(z) f(z), \tag{15}$$

which clearly is in C^∞ and has the correct Taylor coefficients. This time we just have to choose ψ satisfying

$$\frac{\partial\psi}{\partial\bar{z}} = -\sum P_k(z)\,(\partial/\partial\bar{z})\chi_k(z)/f(z). \tag{16}$$

Since the right hand side of (16) is C^∞, we have found the desired function ψ again by Corollary 1. ∎

Let us just point out the obvious fact that to be able to interpolate any set of values $\{a_{k,l}\}$ it is necessary that V be a multiplicity variety, i.e. if $V(f) = V$ implies $f \equiv 0$, then it is easy to construct a set $\{a_{k,l}\}$ for which no interpolation is possible.

Associated to a multiplicity variety V in Ω there is a unique closed ideal in $A(\Omega)$,

$$I = I(V) = \{F \in A(\Omega)\colon F \text{ vanishes at } z_k \text{ with multiplicity} \geqslant m_k\} \tag{17}$$

Two functions h, $g \in A(\Omega)$ can be identified modulo I if and only if

$$\frac{h^{(l)}}{l!}(z_k) = \frac{g^{(l)}}{l!}(z_k) = a_{k,l}\,, \qquad 0 \leqslant l < m_k\,, \quad k = 1, 2,\ldots \tag{18}$$

Hence, Theorem 3 above states that the quotient space $A(\Omega)/I$ can be identified to the space of all sequences $\{a_{k,l}\}$. We will describe them as "analytic functions on V", and denote that space by $A(V)$. The map $\rho_V = \rho$,

$$\rho\colon A(\Omega) \to A(V), \tag{19}$$

$\rho(g) = \{a_{k,l}\}$ as defined by (18), is called the *restriction* map. Let us also point out that there is an obvious inclusion relation between multiplicity varieties in Ω. If $V = \{(z_k\,, m_k)\}$ and $V' = \{(z'_j\,, m'_j)\}$, we say that $V \subseteq V'$ if for every index k in V there is an index j in V' with

$$z_k = z'_j \qquad \text{and} \qquad m_k \leqslant m'_j\,.$$

Clearly, $V \subseteq V'$ if and only if $I(V') \subseteq I(V)$.

In the remainder of the paper it will be assumed that $\Omega = \mathbb{C}$ and the question to study is to find the image of A_p under the restriction map ρ. Theorems 2 and 3 provide the models for this work. Along the lines of Theorem 2, the only interesting case occurs when the multiplicity variety V satisfies $V \subset V(f)$ for some $f \in A_p$. Otherwise, V is a set of uniqueness for A_p and it is not even possible to interpolate the sequence which is 1 at one point and 0 at the other points of V. If $V \subset V(f)$ for some $f \in A_p$, then it is not possible, in general, to find $f_1 \in A_p$ with $V = V(f_1)$. For example, if $V = \{1, 2, 3,\ldots\}$ and $p(z) = |z|$, then V is a subset of the zeros of $\sin \pi z \in A_p$ but, it is a consequence of Lindelöf's theorem

([29], p. 45) that V is not the zero set of an entire function of exponential type. On the other hand, if $V \subset V(f)$, $f \in A_p$, then it is easy to find $f_1 \in A_p$ with V the common zeros of f and f_1 by the simple device of "jiggling" the zeros of f. Namely, if $W = \{(w_k, n_k)\} \subset V(f)$ then it is possible to choose numbers $\epsilon_k \neq 0$ so small that

$$f_1(z) = f(z) \prod_k \left(\frac{z - w_k + \epsilon_k}{z - w_k}\right)^{n_k}$$

belongs to A_p. Clearly, the common zeros of f and f_1 are those of f with W removed.

Thus, it is no loss of generality to take V as the common zeros of $f_1, ..., f_m$ in A_p. However, the trivial choice f, f_1 just outlined provides no more information than $V \subset V(f)$. Consequently, the interesting problem in the direction of Theorem 2 is:

> What conditions on V insure that V is the common zeros of a "good" set of functions $f_1, ..., f_m \in A_p$?

This question will not be discussed here; we will always assume that $V = V(f)$, or $V = V(f_1, ..., f_m)$ is the common zeros of $f_1, ..., f_m \in A_p$ (with multiplicities). However, we remark that in case $p(z) = p(|z|)$ classical results (going back to E. Borel and J. Hadamard) on the distribution of zeros of entire functions allow one to give a "good" solution to the problem. A thorough discussion of these ideas may be found in the book of B. Ja. Levin [29].

Along the lines of Theorem 3, a natural extension is given by the semi-local interpolation theorem stated in the introduction. We only outline the proof, since it is essentially the same as that of Theorem 3. Of course, Theorem 3 may be deduced from the semi-local interpolation theorem in much the same way that Corollary 1 follows from Theorem 1.

Outline of proof of the semi-local interpolation theorem. Since $\partial f_i/\partial z \in A_p$ we have that for suitable $\epsilon_1 < \epsilon$, $C_1 > C$, the distance from $z \in S(f; \epsilon_1, C_1)$ to the complement of $S(f; \epsilon, C)$ is at least $\exp(-Ap(z) - B)$, where A, B are (large) positive constants. We can therefore choose $\chi \in C^\infty$ such that $0 \leqslant \chi \leqslant 1$, $\chi = 1$ on $S(f; \epsilon_1, C_1)$, χ has support in $S(f; \epsilon, C) \supset S(f; \epsilon_1, C_1)$ and $|\partial\chi/\partial\bar{z}| \leqslant A \exp(Bp(z))$. ($A$ and B denote constants which may be different at different occurrences.) Then $\chi\tilde{\lambda}$ and $v_i = -(\bar{f}_i/|f|^2)[\partial(\chi\tilde{\lambda})/\partial\bar{z}]$, where $|f|^2 = |f_1|^2 + \cdots + |f_m|^2$, are C^∞ functions since $\partial(\chi\tilde{\lambda})/\partial\bar{z}$ vanishes on $S(f; \epsilon_1, C_1)$. Further, $|v_i(z)| \leqslant A \exp(Bp(z))$. Therefore, by Theorem 1 there exists $u_i \in W_p \cap C^\infty$ such that $\partial u_i/\partial\bar{z} = v_i$. The function $\lambda = \chi\tilde{\lambda} + u_1 f_1 + \cdots + u_m f_m \in W_p$ and $\partial\lambda/\partial\bar{z} = 0$. Hence, $\lambda \in A_p$. Because $v_i = 0$ on $S(f; \epsilon_1, C_1)$, the functions α_i obtained by restricting u_i to $S(f; \epsilon_1, C_1)$ are analytic. The estimate for α_i given in the Theorem may be obtained in a standard way (see e.g., [22, Theorem 1.2.4]), by further decreasing ϵ_1 and increasing C_1. ∎

2. Interpolation with Bounds, I

We first define the space of analytic functions with growth conditions on a multiplicity variety V. This will be the candidate for the range of the restriction map, $\rho(A_p)$, $\rho = \rho_V$ as in (19). It turns out there are at least two natural candidates for such a role. The first one is given by the natural condition corresponding to one of the consequences of (4i), (4ii) in Section 1, namely (b′) there.

Definition. Let $V = \{(z_k, m_k)\}$ be a multiplicity variety. Then $A_p(V)$ is the space of all functions $\{a_{k,l}\} \in A(V)$ such that for some constants $A, B > 0$

$$\sum_{l=0}^{m_k-1} | a_{k,l} | \leqslant A \exp(Bp(z_k)), \qquad k = 1, 2, \ldots \tag{20}$$

Note that when $m_k = O(\exp(Bp(z_k)))$, then (20) is equivalent to

$$| a_{k,l} | \leqslant A_1 \exp(B_1 p(z_k)), \qquad k = 1, 2, \ldots \tag{21}$$

It is clear from (b′) that $\rho(A_p) \subseteq A_p(V)$, but in general, the space $A_p(V)$ is too large. One reason for this is that the conditions (20) on the $a_{k,l}$ were deduced from purely local growth conditions; i.e., the growth in $| \zeta - z_k | \leqslant 1$. Allowing more global conditions will give a sharper bound which is more appropriate in some cases. For $r > 0$, let $p(z; r) = \max\{p(z + \zeta): | \zeta | \leqslant r\}$. Then, exactly the same argument that proves (b′) (Cauchy's formula), yields for every $g \in A_p$ and every $r > 0$ the inequality

$$\sum_{j=0}^{\infty} \left| \frac{g^{(j)}(z)}{j!} \right| r^j \leqslant A \exp(Bp(z; r)) \qquad \forall z \in \mathbb{C}, \tag{22}$$

where $A, B > 0$ are constants depending only on g.

Therefore it is natural to make the following definition.

Definition. The space $A_{p,\infty}(V)$ consists of those $\{a_{k,l}\} \in A(V)$ such that for some $A, B > 0$ and all $r > 0$

$$\sum_{l=0}^{m_k-1} | a_{k,l} | r \leqslant A \exp(Bp(z_k; r)), \qquad k = 1, 2, \ldots \tag{23}$$

The following propositions then hold.

Proposition 1. $A_{p,\infty}(V) \subseteq A_p(V)$.

Proof. It is an obvious consequence of $p(z; 1) \leqslant Cp(z) + D$ (see (4ii)). ∎

Proposition 2. *The restriction map* ρ: $A(\mathbb{C}) \to A(V)$ maps A_p *into* $A_{p,\infty}(V)$.

Proof. A consequence of (22). ∎

EXAMPLE 7. If the m_k's are bounded $A_{p,\infty}(V) = A_p(V)$.

EXAMPLE 8. Consider $p(z) = |z|$. There are functions $f \in A_p$ with zero sets $V = \{(z_k, m_k)\}$ where m_k is of the same order of magnitude as $|z_k|$ (cf. [29], chapter 1). Then (23) gives an estimate

$$|a_{k,l}| \leqslant A_0 \frac{\exp(B_0 |z_k|)}{|z_k|^l},$$

by taking $r = |z_k|$. If l is large, say $l \approx |z_k|$, then $|a_{k,l}| \to 0$ as $k \to \infty$, while (20) allows $|a_{k,l}|$ to be unbounded.

EXAMPLE 9. Let $p(z) = |z|^\rho$, where $\rho > 0$. Then $\{a_{k,l}\} \in A_{p,\infty}(V)$ if and only if for some $A, B > 0$

$$|a_{k,l}| \leqslant A \frac{\exp(B |z_k|^\rho)}{(1 + |z_k|)^l}, \qquad 0 \leqslant l < m_k, \quad (z_k, m_k) \in V. \tag{24}$$

Further, $A_{p,\infty}(V) = A_p(V)$ if and only if $m_k = O(|z_k|^\rho / \log(1 + |z_k|))$. The inequality (24) holds for some $A, B > 0$ if and only if for some $A_1, B_1 > 0$

$$|a_{k,l}| \leqslant \frac{A_1 \exp(B_1 |z_k|^\rho)}{(l!)^{1/\rho}}.$$

We omit the standard calculations verifying these facts.

EXAMPLE 10. Let $p(z) = |\operatorname{Im} z| + \log(1 + |z|^2)$. Then $\{a_{k,l}\} \in A_{p,\infty}(V)$ if and only if for some $A, B > 0$,

$$|a_{k,l}| \leqslant A^{l+1} \frac{\exp(Bp(z_k))}{l!}, \qquad 0 \leqslant l < m_k, \quad (z_k, m_k) \in V.$$

Thus, $A_{p,\infty}(V) = A_p(V)$ if and only if $m_k = O(p(z_k)/\log p(z_k))$. In this case, we see that $A_{p,\infty}(V) = A_p(V)$ requires $m_k = O(\log |z_k| / \log \log |z_k|)$ when z_k is, say, on the real axis.

DEFINITION. If ρ maps A_p onto $A_p(V)$, we will say that V is an *interpolating variety* (for A_p). If ρ maps A_p onto $A_{p,\infty}(V)$ then we will say that V is a *weak-interpolating* variety (for A_p).

Much of the work in the literature refers to varieties of the form $V = V(f)$, $f \in A_p$, $p(z) = |z|$ or, more generally $|z|^\rho$. Leont'ev [27, 28] has studied extensively the case $p(z) = |z|$, $m_k = 1$. Others, e.g. [11] impose severe restrictions on the multiplicities m_k. There is also a characterization of interpolating varieties for $p(z) = |z|^\rho$, in [9, 10] but involving extraneous conditions on derivatives of f of order higher than m_k (cf. Sect. 3 of [9] and Theorem 4 below).

In case $p(z)$ is not a function of $|z|$, the only work we know of is that of Ehrenpreis and Malliavin [16] where $p(z) = |\operatorname{Im} z| + \log(1 + |z|^2)$ as in Example 2 is considered, under the assumptions that $m_k = 1$ and f is invertible (see Definition below). This last condition on f, which we also have to impose in general, is what precludes us from solving completely the problem of characterizing interpolating or weak-interpolating varieties of the form $V(f)$, $f \in A_p$. In case $p(z) = |z|^\rho$, any $f \not\equiv 0$ is automatically invertible (see next section).

DEFINITION. If $f_1, \ldots, f_m \in A_p$, then $I_{\text{loc}}(f_1, \ldots, f_m)$, the *local ideal generated* by $f_1, \ldots, f_m$ is the set of all functions $g \in A_p$ such that, for any $z \in \mathbb{C}$ there is an open neighborhood U of z and functions $g_1, \ldots, g_m \in A(U)$ with the property

$$g = \sum_{j=1}^{m} f_j g_j \qquad \text{in } U.$$

It is easy to see that if $V = V(f_1, \ldots, f_m)$ is the multiplicity variety of common zeroes of $f_1, \ldots, f_m$ (i.e. common zeroes, counted with multiplicities), then $I_{\text{loc}}(f_1, \ldots, f_m) = I(V) \cap A_p$. Since $I(V)$ is closed in $A(\mathbb{C})$ and $A_p \to A(\mathbb{C})$ is continuous, it follows that $I_{\text{loc}}(f_1, \ldots, f_m)$ is closed in A_p. By $((f_1, \ldots, f_m))$ we will denote the ideal generated in A_p by those same functions.

DEFINITION. We say that $f_1, \ldots, f_m$ as above are *jointly invertible* if $I_{\text{loc}}(f_1, \ldots, f_m) = ((f_1, \ldots, f_m))$. For a single function $f \in A_p$, we say f is invertible if $I_{\text{loc}}(f) = ((f))$; i.e. the principal ideal generated by f is closed.

Hence, f invertible in A_p means that if $g \in A_p$ and $g/f \in A(\mathbb{C})$ then $g/f \in A_p$. It also means that $((f))$ is closed and, consequently, the map $g \to fg$ is an open map from A_p into itself.

Remark 1. No confusion should arise between f being invertible and f being a unit in the ring A_p. The last concept means that $1/f \in A_p$ and, in particular, f has no zeroes, while invertible functions might have lots of zeroes.

EXAMPLE 11 (Ehrenpreis [13, p. 523]). If $p(z) = |\operatorname{Im} z| + \log(1 + |z|^2)$, then $f \in A_p$ is invertible if and only if there are positive constants A, ϵ with the property

$$\forall x \in \mathbb{R},\ \max\{|f(x')|: x' \text{ real},\ |x - x'| \leqslant A \log(1 + |x|^2)\} \geqslant \epsilon(1 + |x|)^{-1/\epsilon}.$$

Examples of invertible functions in the space of Example 11 are the polynomials and, more generally, the exponential polynomials with pure imaginary frequencies, i.e.

$$f(z) = \sum_{k=1}^{m} q_k(z) \exp(-i\alpha_k z),$$

where the q_k are polynomials and the α_k real numbers. On the other hand, if $\varphi \in C_0^\infty(\mathbb{R})$, then its Fourier transform $\hat{\varphi}$ is not invertible.

EXAMPLE 12. If $p(z) = p(|z|)$, $p(2z) = O(p(z))$ then every $f \in A_p$, $f \not\equiv 0$ is invertible (see next section, Propositions 3 and 4).

There is a relationship between $f_1, ..., f_n$ being jointly invertible and interpolation as shown by the following result.

THEOREM 4. *Let $f_1, ..., f_n \in A_p$ and $V = V(f_1, ..., f_n)$. If for some $\epsilon > 0$, $C > 0$ we have for all $(z_k, m_k) \in V$*

$$\sum_{j=1}^{n} \frac{|f_j^{(m_k)}(z_k)|}{m_k!} \geqslant \epsilon \exp(-Cp(z_k)) \tag{25}$$

then V is an interpolating variety. In the converse direction, if V is an interpolating variety and the functions $f_1, ..., f_n$ are jointly invertible, then the inequality (25) *holds for some ϵ, $C > 0$ at every point $(z_k, m_k) \in V$.*

Proof. To prove the necessity of (25) assume that $f_1, ..., f_n$ are jointly invertible and the map $\rho: A_p \to A_p(V)$ is onto. We claim there exist constants C_1, $C_2 > 0$ and functions $h_k \in A_p$ such that

$$|h_k(z)| \leqslant C_1 \exp(C_2 p(z)), \tag{26}$$

and, for all $(z_j, m_j) \in V$

$$h_k^{(l)}(z_j) = 0 \qquad 0 \leqslant l < m_j, \tag{27}$$

unless, $j = k$ and $l = m_k - 1$ when

$$h_k^{(m_k-1)}(z_k) = (m_k - 1)! \tag{28}$$

holds. The existence of the h_k may be proved either by appealing to a version of the open mapping theorem (cf. [23, p. 294]) or, by what is essentially the same thing, the following direct argument. Let S denote the space of all sequences $\{a_{k,l}\} \in A_p(V)$ such that $\sum_{l=0}^{m_k-1} |a_{k,l}| \leqslant 1$. The space S is complete under the metric induced by the norm

$$\sup \left\{ \sum_{l=0}^{m_k-1} |a_{k,l}| : k = 1, 2, ... \right\}.$$

If $S_n := \{\rho(f) : f \in A_p, \ |f(z)| \leqslant n \ \exp(np(z)), \ \rho(f) \in S\}$ then it is readily verified that S_n is a closed subset of S, and, since ρ is onto, $\bigcup_n S_n = S$. It follows from the Baire Category Theorem that some S_n has non empty interior, from which the conclusion follows easily. Note that this part of the argument uses only that ρ is onto.

Next, we claim that there exist functions $g_{k,1},..., g_{k,n} \in A_p$ and positive constants C_3, C_4 such that

$$(z - z_k)\, h_k(z) = \sum_{i=1}^{n} g_{k,i}(z) f_i(z) \tag{29}$$

and

$$| g_{k,i}(z)| \leqslant C_3(1 + | z_k |) \exp(C_4 p(z)). \tag{30}$$

To see this, we use the invertibility of $f_1,..., f_n$. By (27) the functions $(z - z_k)h_k(z)$ belong to $I_{\text{loc}}(f_1,..., f_n)$, and hence, to the ideal generated by $f_1,..., f_n$ in A_p. The uniform bound (30) follows from (26) either by the version of the open mapping theorem just mentioned, or by a direct argument like the one given in the first part of the proof.

The lower bound (25) of the theorem now follows. Equate the leading terms of the power series expansions about z_k of both sides of (29) to obtain

$$1 = \sum_{i=1}^{n} g_{k,i}(z_k) \frac{f_i^{(m_k)}(z_k)}{m_k!},$$

so that (25) follows from the upper bound (30) on the $g_{k,i}$ and property (4i) of the weight function p.

To prove the converse, we need to bypass the fact that the multiplicities m_k could be quite large. The key is provided by the following lemma.

LEMMA 3. *Let $G(\zeta)$ be analytic in $| \zeta | \leqslant 1$ and satisfy $| G(\zeta)| \leqslant M$. Suppose further that G has a zero of order q at $\zeta = a$, $0 < | a | < 1$, and a zero of order m at $\zeta = 0$ with*

$$\left| \frac{G^{(m)}(0)}{m!} \right| \geqslant \delta > 0. \tag{31}$$

Then,

$$| a |^q \geqslant \delta/M. \tag{32}$$

Proof. We define the analytic function ψ in $| \zeta | \leqslant 1$ by

$$G(\zeta) = \zeta^m \left(\frac{\zeta - a}{1 - \bar{a}\zeta}\right)^q \psi(\zeta). \tag{33}$$

Clearly, $| \psi(\zeta)| \leqslant M$ for $| \zeta | \leqslant 1$, $\psi(0) \neq 0$ and $\psi(a) \neq 0$. Differentiating m times and evaluating at $\zeta = 0$, the identity (33) yields

$$\delta \leqslant \frac{| G^{(m)}(0)|}{m!} = | a |^q \, | \psi(0)| \leqslant M \, | a |^q. \quad \blacksquare$$

Now we are ready to continue the proof of Theorem 4. Let $z_k \in V$ have multiplicity m_k, and let d_k be the minimum of 1 and the distance from z_k to $V \backslash \{z_k\}$. If $d_k < 1$, choose $z_i \in V$ such that $d_k = | z_k - z_i |$. Because of (25), there exists ϵ, $C > 0$ and j, $1 \leqslant j \leqslant n$ such that

$$\frac{| f_j^{(m_i)}(z_i)|}{m_i!} \geqslant \frac{\epsilon}{n} \exp(-Cp(z_i)).$$

By Lemma 3, applied to $G(\zeta) = f_j(z_i + \zeta)$, we have (recall (4ii))

$$d_k^{m_k} \geqslant \epsilon_1 \exp(-C_1 p(z_k)) \tag{34}$$

for some ϵ_1, $C_1 > 0$ (independent of k). This equality clearly still holds for $d_k = 1$ if $\epsilon_1 \leqslant 1$. It follows that d_k also satisfies (since $m_k \geqslant 1$)

$$d_k \geqslant \epsilon_1 \exp(-C_1 p(z_k)). \tag{35}$$

We claim that for suitable ϵ_1, $C_1 > 0$, the component U_k of $S(f; \epsilon_1, C_1)$ (see (2)) containing z_k is contained in $\Delta(z_k; d_k/2)$. To see this, choose j with $|f_j^{(m_k)}(z_k)| \geqslant (m_k!\epsilon/n) \exp(-Cp(z_k))$. Such a j exists by hypothesis (25). For $| \zeta | \leqslant 1$, let $f_j(z_k + d_k\zeta) = \zeta^{m_k}\psi(\zeta)$. Then $| \psi(0)| = |f^{(m_k)}(z_k)| d_k{}^m/m_k! \geqslant \epsilon_2 \exp(-C_2 p(z_k))$ and for $| \zeta | \leqslant 1$, $| \psi(\zeta)| \leqslant M = \max\{| f_j(z_k + \tau)|: | \tau | \leqslant 1\} \leqslant A \exp(Bp(z_k))$. Then from Caratheodory's inequality (cf. [29, p. 19]) we obtain $| \psi(\zeta)| \geqslant \epsilon_3 \exp(-C_3 p(z_k))$ for all $| \zeta | \leqslant 1/2$. Consequently,

$$| f_j(z_k + \tau)| \geqslant \epsilon_3 \frac{| \tau |^{m_k}}{d_k} \exp(-C_3 p(z_k)), \qquad | \tau | \leqslant d_k/2 \tag{36}$$

Also, $\log | \psi(\zeta)|$ is harmonic in $| \zeta | < 1$ so

$$\log | \psi(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log | \psi(e^{i\theta})| \, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log | f_j(z_k + d_k e^{i\theta})| \, d\theta - m_k .$$

Thus $m_k \leqslant \log M - \log | \psi(0)| \leqslant A_1 + A_2 p(z_k)$. Let $| \tau | = d_k/2$ in (36) to obtain

$$| f_j(z_k + \tau)| \geqslant \epsilon_3 2^{-m_k} \exp(-C_3 p(z_k)) \geqslant \epsilon_4 \exp(-C_4 p(z_k)).$$

Taking account of (4ii), we therefore have proved that $U_k \subset \Delta(z_k; d_k/2)$, provided ϵ_1, C_1 are suitably chosen.

The proof is easily completed by applying the semi-local interpolation theorem. Define $\tilde{\lambda}$ on $S(f; \epsilon_1, C_1)$ as follows. On U_k, the component of $S(f; \epsilon_1, C_1)$ containing z_k, set $\tilde{\lambda}(z) = \sum_j a_{k,j}(z - z_k)^j$. (Note that no other $z_j \in V$ is in U_k.) If U is a component of $S(f; \epsilon_1, C_1)$ which contains no z_k, then set $\tilde{\lambda}(z) \equiv 0$. Then $| \tilde{\lambda}(z)| \leqslant A \exp(Bp(z))$, $z \in S(f; \epsilon_1, C_1)$, and $\tilde{\lambda}$ solves the interpolation problem on $S(f; \epsilon_1, C_1)$. The theorem then follows from the semi-local interpolation theorem. This concludes the proof.

Observe that in the proof of Theorem 4, the zeros z_k were actually trapped inside the small disks Δ_k, with $|f|$ fairly large near the boundary of Δ_k.

COROLLARY 2. *With the same hypotheses of joint invertibility as in Theorem 4, the multiplicity variety $V = V(f_1, ..., f_n)$ is an interpolating variety if and only if there exist $\epsilon > 0$, $C > 0$ such that*

(i) *each $z_k \in V$ is contained in a bounded component of $S(f; \epsilon, C)$ whose diameter is at most one*; *and*

(ii) *no two distinct points of V lie in the same component of $S(f; \epsilon, C)$.*

Remark 2. Even when we are dealing with a single function f, the condition (25) is not enough to ensure that f is invertible for general growth functions p. In fact, there are p such that $e^z \in A_p$ and $e^{-z} \notin A_p$ (see [24]). It might seem that if $f \in A_p$ and $V(f)$ is an interpolating variety, then there is an $F \in A_p$ with $V(F) = V(f)$ and F invertible or, at least, satisfying (25). Nevertheless there is a $\varphi \in C_0^\infty(\mathbb{R})$, $p(z) = |\operatorname{Im} z| + \log(1 + |z|^2)$, where $V(\hat{\varphi})$ is an A_p-interpolating variety. (W. E. Squires, personal communication).

If we have more than one function then the situation is quite complicated, as the following two examples show.

EXAMPLE 13 [15, p. 319]. Let λ be a Liouville number, $p(z) = |z|$, $f_1(z) = (\sin \pi z)/z$ and $f_2(z) = (\sin \pi/\lambda)z$, Then $1 \in I_{\text{loc}}(f_1, f_2)$ but $1 \notin ((f_1, f_2))$, though we know, by the Spectral Synthesis Theorem from [35], that $I_{\text{loc}}(f_1, f_2) = \overline{((f_1, f_2))}$ = closure of $((f_1, f_2))$ in A_p. Of course, (25) holds for this example.

EXAMPLE 14. It is also easy to show an example where (25) fails to hold but $V(f_1, f_2)$ is interpolating. Again we set $p(z) = |z|$ and choose two sequences of numbers $-\frac{1}{2} < \epsilon_1(n) < \epsilon_2(n) < \frac{1}{2}$ such that $\epsilon_1(n) \to 0$, $\epsilon_2(n) \to 0$ very fast when $n \to \infty$. Now, we let f_j be a function in A_p with simple zeroes exactly at the integers and at the points $\pm(n + \epsilon_j(n))$, $n = 1, 2, ...$ We have then $V(f_1, f_2) = V(\sin \pi z)$ which is interpolating by Theorem 4. On the other hand (25) fails. Of course, the reason is that $I_{\text{loc}}(f_1, f_2) \neq ((f_1, f_2))$.

A slight modification of Example 13 provides some insight into the geometric nature of an interpolating variety of the form $V = V(f)$.

EXAMPLE 15. One obvious obstruction to (25) occurs when the zeroes of f are very close together. The function $f(z) = \sin \pi z \cdot (\sin \pi/\lambda)z$, $0 \neq \lambda \in \mathbb{R}$, is invertible for $p(z) = |\operatorname{Im} z| + \log(1 + |z|^2)$ and $p(z) = |z|$, as pointed out in Example 11. If λ is a Liouville number then the zeroes of the form λm, $m \in \mathbb{Z}$ and those of the form $n \in \mathbb{Z}$, get too close together, violating (34) for either p.

On the other hand, (25) is more subtle. Namely, one can find an even function $f \in A_p$, $p(z) = |z|$, for which (25) does not hold, but which has simple zeroes at all Gaussian integers in squares of the form $|x - n_k^2| \leqslant n_k$, $|y| \leqslant n_k$ for a sequence of integers $n_k \uparrow \infty$.

We now consider weak-interpolating varieties. It does not seem possible to give a result as strong as Theorem 4 in this case. However, we can give some sufficient conditions and some necessary conditions. For some weight functions, it turns out these will coincide, as we point out in the examples following Theorem 6.

Recall that

$$p(z; r) = \max\{p(z + \zeta): |\zeta| \leqslant r\}. \tag{37}$$

For $B \geqslant 0$, $l \geqslant 0$ and $\{(z_k, m_k)\} = V$, let

$$\gamma_{k,l} = \gamma_{k,l}(B) = \inf\{r^{-l} \exp(Bp(z_k; r)): r > 0\}. \tag{38}$$

and

$$\gamma_k = \gamma_k(B) = \gamma_{k,m_k-1}(B). \tag{39}$$

Notice that the numbers γ_k come basically from (23); i.e., $(a_{k,l}) \in A_{p,\infty}(V)$ implies $|a_{k,l}| \leqslant A\gamma_{k,l}(B)$ for some $B > 0$. The results which follow are also valid with $A_{p,\infty}(V)$ replaced by spaces defined by $|a_{k,l}| \leqslant A\delta_{k,l} \exp(Bp(z_k))$, subject only to some mild regularity conditions on the $\delta_{k,l}$.

The following theorem gives a necessary condition for $V = V(f_1, ..., f_n)$ to be a weak interpolating variety when $f_1, ..., f_n$ are jointly invertible. Other more general, more technical conditions can be given without the hypothesis of joint invertibility.

THEOREM 5. *Let $V = V(f_1, ..., f_n)$, $f_i \in A_p$. Suppose that V is a weak-interpolating variety and that $f_1, ..., f_n$ are jointly invertible. Then for each $B > 0$, there exist constant ϵ, $C > 0$ such that*

$$\sum_{j=1}^{n} \frac{|f_j^{(m_k)}(z_k)|}{m_k!} \geqslant \epsilon\gamma_k(B) \exp(-Cp(z_k)). \tag{40}$$

Proof. The proof is the same as the necessity part of the proof of Theorem 4. The only difference is that the functions h_k are now chosen with (28) replaced by $h_k^{(m_k-1)}(z_k) = (m_k - 1)!\, \gamma_k(B)$. We omit the argument. ∎

Next we give sufficient conditions. For each $B > 0$, let $R_k = R_k(B) \geqslant 1$ denote a point at which $R_k^{-m_k+1} \exp(Bp(z_k; R_k))$ is close to γ_k, say $\leqslant 2\gamma_k$ (recall the definition of γ_k, in (39)).

THEOREM 6. *Let $f_1, ..., f_n \in A_p$ and $V = V(f_1, ..., f_n)$. Suppose that for each $B > 0$, there exist constants ϵ, C_1, C_2, $C_3 > 0$ such that for all $(z_k, m_k) \in V$*

(i) $m_k \leqslant C_1 p(z_k) + C_2$

(ii) $p(z_k; 2R_k) \leqslant C_1 p(z) + C_2$, $\forall z\colon |z - z_k| \leqslant 2R_k$

(iii) $\sum_{j=1}^{n} \frac{|f_j^{(m_k)}(z_k)|}{m_k!} \geqslant \epsilon \gamma_k(B) \exp(-C_3 p(z_k))$.

Then V is a weak-interpolating variety.

Proof. The proof follows the argument of the proof of the sufficiency part of Theorem 4, so we only sketch it. Note that in the proof of Theorem 4, hypothesis (i) was derived from the condition analogous to (iii), while the numbers R_k were taken $\equiv 1$ so that (ii) was vacuous.

By applying Lemma 3 to one of the functions $G(\zeta) = f_j(z_k + \zeta R_k)$ we can deduce from (iii) and (ii) that

$$|f| = (|f_1|^2 + \cdots + |f_n|^2)^{1/2} \geqslant \gamma_k \frac{|z - z_k|^{m_k}}{R_k^{m_k}} \eta \exp(-Cp(z_k))$$

for $|z - z_k| < \delta R_k$, where $\gamma_k = \gamma_k(B)$ and η, δ, C are suitable positive constants (say $\delta = 1/4$). Then the proof proceeds exactly as in Theorem 4.

There is also an analogue of Corollary 2 (with the same notation).

COROLLARY 3. *If the hypotheses of Theorem 6 hold, then for some $\epsilon, C, C_1, C_2 > 0$ we have*:

(i) *Each $z_k \in V$ belongs to a bounded component of $S(f; \epsilon, C)$, and the weight function satisfies*

$$p(z) \leqslant C_1 p(\zeta) + C_2$$

for any two points z, ζ of that component;

(ii) *No two distinct points of V lie in the same bounded component of $S(f; \epsilon, C)$.*

While the necessary conditions of Theorem 5 do not coincide with the sufficient conditions of Theorem 6, there are many weight functions $p(z)$ for which they do. Here are some examples.

EXAMPLE 16. Let $p(z) = |z|^\rho$, $0 < \rho < +\infty$. Then $m_k = O(p(z_k))$ follows from Jensen's formula. It is readily checked that $R_k = O(|z_k|)$ so that hypothesis (ii) of Theorem 6 always holds. According to Example 9, condition (iii) is equivalent to

$$\sum_{j=1}^{n} \frac{|f_j^{(m_k)}(z_k)|}{m_k!} \geqslant \frac{\epsilon \exp(-C|z_k|^\rho)}{(1 + |z_k|)^{m_k}}. \tag{41}$$

Thus, for $f_1, \ldots, f_n$ jointly invertible, the condition (41) is equivalent to V being a weak-interpolating variety. We remark that if $V \subset V(f)$, $f \in A_p$, then $V = V(f_1, f_2)$ where f_1, f_2 are jointly invertible.

EXAMPLE 17. Let $p(z) = |\operatorname{Im} z| + \log(1 + |z|^2)$, and suppose $V = V(f_1, ..., f_n)$ where $f_1, ..., f_n$ are jointly invertible. It can be shown that $m_k = O(p(z_k))$ (see next section). Then a further direct calculation shows $R_k = O(m_k) = O(p(z_k))$, so that (ii) of Theorem 6 also holds. Consequently, from Theorems 5 and 6, and Example 10, we see that V is a weak-interpolating variety if and only if for some ϵ, $C > 0$

$$\sum_{j=1}^{n} |f_j^{(m_k)}(z_k)| \geqslant \frac{\epsilon \exp(-C\,|\operatorname{Im} z_k|)}{(1 + |z_k|)^C}.$$

This example answers the question posed by Ehrenpreis and Malliavin in [16, p. 180].

3. INTERPOLATION WITH BOUNDS. II. GROUPINGS

As we have seen interpolation does not hold in general even for $V = V(f)$ where $f \in A_p$. Hence, it is natural to try to find a description of the subspace of $A(V)$ which is the range of the restriction map $\rho\colon A_p \to A(V)$. Because of the semi-local interpolation theorem, it is only necessary to construct an extension on each component of $S(f; \epsilon, C)$. It is natural to expect that the classical polynomial interpolation formulas, such as the Newton interpolation formula, will provide the appropriate extension. In fact, they do for radial growth rates, as has been shown by Borisevich and Lapin [9] (see also [10]). For nonradial growth rates, we prove a similar result in case $V = V(f)$ where $f \in A_p$ is slowly decreasing, a concept inspired by Example 11, which is often an equivalent but more precise formulation of invertibility.

DEFINITION. A function $f \in A_p$ is called *slowly decreasing* if the following two conditions hold.

(42i) There exist $\epsilon > 0$, $A > 0$ such that each connected component S_α of the set

$$S(f; \epsilon, A) = \{z\colon |f(z)| < \epsilon \exp(-Ap(z))\}$$

is relatively compact. And

(42ii) There exists a constant $B > 0$ (independent of α) such that

$$p(\zeta) \leqslant Bp(z) + B \qquad \text{for any} \quad z, \zeta \in S_\alpha, \quad \text{any} \quad \alpha.$$

PROPOSITION 3. *If f is slowly decreasing, then f is invertible.*

Proof. Suppose $g \in A(\mathbb{C})$ and $gf \in A_p$. To show $g \in A_p$: for some constants $C, D > 0$ we have

$$|g(z)f(z)| \leqslant C \exp(Dp(z)), \qquad \text{for all} \quad z \in \mathbb{C}.$$

On the boundary of the set S_α we have, by (42i)

$$|g(\zeta)| \leqslant C_1 \exp(D_1 p(\zeta)) \leqslant C_1 \exp(D_1 \max_{S_\alpha} (p(w))), \qquad \zeta \in \partial S_\alpha, \tag{43}$$

where $C_1 = C/\epsilon$, $D_1 = D + A$. Since S_α is bounded we can apply the maximum principle. Hence (42ii) and (43) yield the estimate

$$|g(z)| \leqslant C_2 \exp(D_2 p(z)), \; z \in S_\alpha, \quad \text{any} \quad \alpha,$$

for some C_2, $D_2 > 0$. Outside $S(f; \epsilon, A)$, the estimate is immediate. ▮

PROPOSITION 4. *If $p(z) = p(|z|)$ and $p(2z) = O(p(z))$, then any $f \in A_p$, $f \not\equiv 0$, is slowly decreasing.*

Proof. It is enough to show that for some $\epsilon > 0$, $A > 0$ and any $r > 0$ sufficiently large there are r_1, r_2 such that $r/2 \leqslant r_1 < r < r_2 \leqslant 2r$ and

$$\min\{|f(z)|: |z| = r_1 \quad \text{or} \quad |z| = r_2\} \geqslant \epsilon \exp(-Ap(r)).$$

This last inequality is an immediate consequence of our hypotheses and a standard minimum modulus theorem (see [29, p. 21] or [7]). ▮

Remark 3. As an application of the same minimum modulus theorem it can be shown that for $p(z) = |\operatorname{Im} z| + \log(1 + |z|^2)$, the converse of Proposition 3 also holds.

We will need below the following lemmas whose proofs are immediate.

LEMMA 4. *If $f \in A_p$ is slowly decreasing, then there are rectifiable Jordan curves Γ_α with the following properties*

(a) *The curves Γ_α are disjoint and $V = V(f) \subset \bigcup_\alpha U_\alpha$, where $U_\alpha = \operatorname{int} \Gamma_\alpha$.*

(b) *For some constant $A > 0$ we have for all α,*

$$|f(z)| \geqslant \frac{1}{A} \exp(-Ap(z)), \qquad \textit{for} \quad z \in \Gamma_\alpha.$$

(c) *For some constant $B > 0$ we have, for any α and any pair $z, \zeta \in \overline{U}_\alpha$*

$$p(z) \leqslant Bp(\zeta) + B.$$

(d) *If d_α = diameter of Γ_α, then for some constant $C > 0$ we have*

$$d_\alpha \leqslant C \exp(Cp(z)), \qquad \text{for any} \quad z \in \overline{U}_\alpha.$$

(e) *For some constant* $D > 0$,

$$\text{length}(\Gamma_\alpha) \leqslant D \exp(Dp(z)), \qquad \textit{for any} \quad z \in \overline{U}_\alpha .$$

And finally,

(f) *If* n_α *denotes the number of points in* $V_\alpha := V \cap U_\alpha$, *counted with multiplicities, then*

$$n_\alpha \leqslant N \exp(Np(z)),$$

for some constant $N > 0$ *and any* $z \in \overline{U}_\alpha$.

The only thing needed for the proof is to notice that (42ii) and (4i) imply (c) for the diameters of the sets S_α obtained in (42i).

LEMMA 5. *If* $p(z) = |\operatorname{Im} z| + \log(1 + |z|^2)$ *or* $p(z) = p(|z|)$ *and* $p(2z) = O(p(z))$, *then* (c) *in Lemma* 5 *can be replaced by*:

(c′) *Let* $W_\alpha = \{z \in \mathbb{C}\colon \operatorname{dist}(z, U_\alpha) \leqslant 2d_\alpha\}$ *then for some constant* $B > 0$ *we have*

$$p(z) \leqslant Bp(\zeta) + B$$

for any α, *any* $z, \zeta \in W_\alpha$.

Lemma 4 allows us to introduce a certain subspace $A_{p,g}(V)$ of $A(V)$ in which groupings are taken into account. Let $\{a_{k,l}\} \in A(V) \cong A(\mathbb{C})/I(V)$ (see (17)), and take any representative $\varphi \in A(\mathbb{C})$. Then we can construct a sequence of functions $\varphi_\alpha \in A(U_\alpha)$ by the formula

$$\varphi_\alpha(z) = \frac{1}{2\pi i}\int_{\Gamma_\alpha} \frac{\varphi(\zeta)}{f(\zeta)}\frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta, \qquad z \in U_\alpha . \tag{44}$$

Clearly, for $z \in U_\alpha$ we have

$$\varphi_\alpha(z) = \varphi(z) - \frac{f(z)}{2\pi i}\int_{\Gamma_\alpha} \frac{\varphi(\zeta)}{f(\zeta)}\frac{d\zeta}{\zeta - z} = \varphi(z) + f(z)\,\psi_\alpha(z),$$

$\psi_\alpha \in A(U_\alpha)$. Hence, $\rho_{V_\alpha}(\varphi_\alpha) = \rho_{V_\alpha}(\varphi) = \{a_{k,l}^{(\alpha)}\} = \{a_{k,l};\ z_k \in V_\alpha\}$. Moreover, the φ_α's do not depend on the representative φ. In fact, if one replaces φ in (44) by $\varphi + f\psi$, $\psi \in A(\mathbb{C})$, then the φ_α do not change. This shows that we have a linear map $\eta\colon A(V) \to \prod_\alpha A(U_\alpha)$, such that $\rho_V \circ \eta = id$. It is also true that (44) defines a map $\eta_\alpha : A(V_\alpha) \to A(U_\alpha)$ such that $\rho_{V_\alpha} \circ \eta_\alpha = id$. If $\{a_{k,l}\} = \rho_V(\varphi)$, $\varphi \in A_p$, then, for some constants $K_1, K_2 > 0$ we have

$$|\varphi_\alpha(z)| \leqslant K_1 \exp(K_2 p(z)), \qquad z \in U_\alpha . \tag{45}$$

As matter of fact, we can define a norm in the finite dimensional vector space $A(V_\alpha)$ by

$$\| a_{k,l}^{(\alpha)} \|_\alpha = \inf\{\| \varphi \|_\infty : \varphi \in A(U_\alpha), \rho_{V_\alpha}(\varphi) = \{a_{k,l}^{(\alpha)}\}\}, \tag{46}$$

where $\| \varphi \|_\infty = \sup\{| \varphi(z)| : z \in U_\alpha\}$. Then, one can see without difficulty, using the properties (a)–(f) in Lemma 4, that given $C_1, C_2 > 0$ we can find $K_1, K_2 > 0$ such that

$$\| a_{k,l}^{(\alpha)} \|_\alpha \leqslant C_1 \exp(C_2 p(z)) \qquad \text{for any} \quad z \in U_\alpha, \tag{47}$$

implies (45) holds. Conversely, if (45) holds, then for some $C_1', C_2' > 0$ (47) also holds. The relation between these sets of constants is independent of α. It is also clear that if (47) holds we can replace, $\| \varphi \|_\infty$ by $\| \varphi(z) \exp(-C_2'' p(z))\|_\infty$ in (42) for some $C_2'' > 0$, and get an equivalent norm.

Definition. $A_{p,g}(V)$ consists of those functions $\{a_{k,l}\} \in A(V)$ such that (47) holds for some constants $C_1, C_2 > 0$ and for every α.

Application of the semi-local interpolation theorem, as in Section 2, yields the following theorem.

Theorem 7. *If the junction $f \in A_p$ is slowly decreasing, the map ρ_V induces a linear topological isomorphism between the spaces $A_p/((f))$ and $A_{p,g}(V)$, $V = V(f)$.*

Theorem 7 shows also that, although $A_{p,g}(V)$ was defined in terms of a specific family of curves Γ_α, it is actually a subspace of $A(V)$ independent of the family $\{\Gamma_\alpha\}$.

Using the calculus of residues the formula (44) can be written explicitly in terms of the $\{a_{k,l}^{(\alpha)}\}$, for instance, if all the multiplicities $m_k = 1$ for $z_k \in V$ we have, $(a_k = a_{k,0})$, the Lagrange interpolation formula,

$$\varphi_\alpha(z) = \sum_{z_k \in V_\alpha} \frac{a_k}{f'(z_k)} \frac{f(z_k) - f(z)}{z_k - z}. \tag{48}$$

Hence, Theorem 7, in conjunction with Proposition 4, yields the result obtained in [9] for the weight $p(z) = |z|^\rho$. The disadvantage of (48), and the more complicated ones obtained for arbitrary multiplicities m_k, is that they involve not only $\{a_{k,l}^{(\alpha)}\}$ but also the function f. One would expect a characterization of $A_{p,g}(V)$ in terms of polynomial interpolation, and this is the case if the conclusion of Lemma 5 holds for f. In order to show this, we have to recall some well-known facts about the Newton interpolation formula and divided differences (see [18, 31, 41]).

Let $\zeta_1, \ldots, \zeta_n$, $n = n_\alpha$, stand for the points in V_α repeated according to multiplicity. Then the polynomials $P_0 \equiv 1$, $P_1(z) = (z - \zeta_1), \ldots, P_{n-1}(z) = \prod_{j=1}^{n-1}(z - \zeta_j)$ form a basis of the space of polynomials of degree $n - 1$. There is a unique polynomial $Q = Q_\alpha$ of degree at most $n-1$ such that $\rho_{V_\alpha}(Q) = \{a_{k,l}^{(\alpha)}\}$, and it can be written as

$$Q(z) = \sum_{j=0}^{n-1} \Delta^{(j)} P_j(z).$$

The coefficients $\Delta^{(j)} = \Delta^{(j)}(\{a_{k,l}^{(\alpha)}\})$ are the jth divided differences of the $a_{k,l}^{(\alpha)}$'s. They can be computed recursively. For instance, if $\zeta_1 = z_k$ then $\Delta^{(0)} = a_{k,0}$. Similarly, if $\zeta_1 = \cdots = \zeta_{l+1} = z_k$ then $\Delta^{(l)} = a_{k,l}$. If this polynomial Q, the *Newton interpolation polynomial*, satisfies the estimate (45), the above discussion shows that (47) holds and hence there is a function $\varphi \in A_p$ such that $\rho_{V_\alpha}(\varphi) = \{a_{k,l}^{(\alpha)}\}$. (This can be done even if we know the estimate for a single α since $\{b_{k,l}\} \in A(V)$, defined by $b_{k,l} = a_{k,l}$ if $z_k \in V_\alpha$ and $b_{k,l} = 0$ if $z_k \notin V_\alpha$, is in $A_{p,g}(V)$.) We can now estimate the $\Delta^{(j)}$ using the following lemma, whose proof we omit.

LEMMA 6 [31, 41]. *Let φ be holomorphic in the open set $W \subseteq \mathbb{C}$, $|\varphi(z)| \leq M$ in W, and $\zeta_1, \dots, \zeta_n$ be given such that for some $\delta > 0$, $\bigcup_{j=1}^n \Delta(\zeta_j; \delta) \subseteq W$, then*

$$|\Delta^{(j)}| \leq 2^j \delta^{-j} M, \qquad 0 \leq j \leq n-1. \tag{49}$$

(Here $\Delta^{(j)}$ is computed with respect to $\rho_V(\varphi)$, V the multiplicity variety associated to $\zeta_1, \dots, \zeta_n$ in the obvious way.)

Hence, assuming (c′) holds, if either $Q = Q_\alpha$ satisfies the estimate (45) or (47) holds, we have, for some constants C_3, $C_4 > 0$,

$$|||\{a_{k,l}^{(\alpha)}\}|||_\alpha = \max_{0 \leq j \leq n-1} |\Delta^{(j)}(\{a_{k,l}^{(\alpha)}\})\, d_\alpha^{\,j}| \leq C_3 \exp(C_4 p(z)),$$
$$\forall z \in U_\alpha, \quad n = n_\alpha, \tag{50}$$

as follows from Lemma 6 applied to $W = W_\alpha$, $\delta = 2d_\alpha$. In particular, if $\{a_{k,l}\} \in A_{p,g}(V)$, then (50) holds for every α with C_3, C_4 independent of α. Conversely, if (50) holds for a given α then it is obvious from the definition of the polynomials P_j and Q_α that, for every $z \in U_\alpha$ and some new constants C_5, $C_6 > 0$,

$$|Q_\alpha(z)| \leq C_3 \exp(C_4 p(z)) \sum_{j=0}^{n-1} d_\alpha^{-j} |(z-\zeta_1) \cdots (z-\zeta_j)|$$
$$\leq n_\alpha C_3 \exp(C_4 p(z)) \leq C_5 \exp(C_6 p(z)).$$

The last inequality follows from Lemma 4, (*f*). Hence, if (50) holds with constants independent of α, $\{a_{k,l}\} \in A_{p,g}(V)$. These remarks are collected in Theorem 8 for future reference.

THEOREM 8. *Let $f \in A_p$ be slowly decreasing and the norms $|||\{a_{k,l}^{(\alpha)}\}|||_\alpha$ of $\{a_{k,l}\} \in A(V)$ be defined with respect to some grouping $\{\Gamma_\alpha\}$ satisfying* (a)–(c′)–(f) *of*

Lemmas 4 *and* 5. *Then* $A_p/((f))$ *is isomorphic under the restriction map* ρ *to the subspace of* $A(V)$ *of those* $\{a_{k,l}\}$ *such that* (50) *holds for some* C_3, C_4 *independent of* α.

Remark 4. It is an immediate consequence of Proposition 2 and Theorem 7 that, if f is slowly decreasing, then $A_{p,g}(V) \subseteq A_{p,\infty}(V)$. It is also true for weights like $p(z) = |z|^\rho$ or $p(z) = |\operatorname{Im} z| + \log(1 + |z|^2)$ that, if there is a grouping $\{\Gamma_\alpha\}$ for $V = V(f)$ such that every V_α contains a single point of V, then $A_{p,\infty}(V) \subseteq A_{p,g}(V)$ and hence V is weak-interpolating (see Corollaries 2 and 3 of the previous section).

We conclude this section with an example for which the groupings can be made more explicit.

EXAMPLE 18. As already pointed out, for the weight $p(z) = |\operatorname{Im} z| + \log(1 + |z|^2)$, the concept of invertible function and slowly decreasing function are synonymous, and the exponential polynomials with pure imaginary frequencies are invertible in A_p. A more precise statement about the groupings can be made by using the sharper lower bounds known for such functions [5, 19] ans also sharper bounds on the number of zeroes they can have in disks of fixed radii [40, ch. 6]. Namely, let

$$f(z) = \sum_{k=1}^{m} q_k(z) \exp(-i\alpha_k z), \tag{51}$$

where the q_k are non-zero polynomials of degree ν_k, and the $\alpha_k \in \mathbb{R}$. Let $K =$ smallest closed interval containing all the frequencies α_k. Then if $y = \operatorname{Im} z$, we have

$$h(z) = \max\{\operatorname{Re}(-i\alpha_k z) \colon 1 \leqslant k \leqslant m\} = h_K(z) = \max\{\xi y \colon \xi \in K\}. \tag{52}$$

From [19] there is a constant $C > 0$ and $r = r(z)$, $0 \leqslant r \leqslant 1$ such that

$$\min_{0 \leqslant \theta \leqslant 2\pi} |f(z + re^{i\theta})| \geqslant C \exp h(z), \qquad \forall z \in \mathbb{C}. \tag{53}$$

Hence, the d_α's can be taken $\leqslant 1$. Therefore, (50) can be replaced by

$$|||\{a_{k,l}^{(\alpha)}\}|||'_\alpha = \max_{0 \leqslant j \leqslant n-1} |\Delta^{(j)}(\{a_{k,l}^{(\alpha)}\})| \leqslant C_3 \exp(C_4(p(z)) \quad \forall z \in U_\alpha,\ n = n_\alpha. \tag{50'}$$

Furthermore, from [40] we can get an upper bound for n_α,

$$n_\alpha \leqslant \nu + \frac{2(1+\nu)}{\log \nu}(1 + \Omega), \tag{54}$$

where $\nu = \sum_k(\nu_k + 1)$, $\Omega = \max_k |\alpha_k|$.

In fact, the same situation still is valid if we replace f by $F(z) = f(z) + g(z)$, where $g(z) = \hat{\mu}(z)$ and μ is either an integrable function with sup $\mu \subseteq K$ or μ is an arbitrary distribution in $\mathscr{E}'(\mathbb{R})$ with supp $\mu \subset\subset K$. If we use the weight $p(z) = |z|$, then we can allow complex frequencies α_k in (51). In this case (53) and (54) still hold when (52) is replaced by

$$h(z) = \max\{\operatorname{Re}(-i\alpha_k z): 1 \leqslant k \leqslant m\} = h_K(z) = \max\{\langle \zeta, z\rangle: \zeta \in K\} \quad (52')$$

where $\langle \zeta, z\rangle = \xi x + \eta y$, $(\zeta = \xi + i\eta,\ z = x + iy)$ and $K =$ convex hull of the set $\{i\bar{\alpha}_k: 1 \leqslant k \leqslant m\}$. We can replace f by $F = f + g$, $g = \hat{\mu} =$ Fourier–Borel transform of a Radon measure μ with sup $\mu \subset\subset K$,

$$g(z) = \int e^{-i\zeta z}\, d\mu(\zeta).$$

4. Mean-Periodic Functions

Let $f \in \mathscr{E} = \mathscr{E}(\mathbb{R})$, the space of all C^∞ functions on the real line. We say that f is *mean-periodic* if there is a nonzero distribution of compact support, $\mu \in \mathscr{E}'$, such that the function $\mu * \check{f}(x)$ $(= \int f(t - x)\, d\mu(t))$ vanishes identically. Here we are following the standard notation for $\check{f}(x) = f(-x)$ and convolution as presented in [35] or [36]. Similarly, if $f \in A(\mathbb{C})$, we say that f is *analytic mean-periodic* if there is a nontrivial analytic functional μ, which can be represented by a Radon measure of compact support, such that $\mu * \check{f} \equiv 0$.

Probably the oldest problem in harmonic analysis has been the representation of mean-periodic functions in terms of the simplest mean-periodic functions, i.e. exponential-monomials $x^l e^{-izx}$. Since $\mu * e^{izx} = \hat{\mu}(z)\, e^{izx}$, where $\hat{\mu}$ is the Fourier transform of μ, it follows that the pairs (z, l) have to be taken from $V = V(\hat{\mu})$. When μ represents an ordinary differential operator with constant coefficients, L. Euler [17] showed that the finite sums

$$f(x) = \sum_{(z,l)\in V} c_{z,l} x^l e^{-izx}, \qquad V = V(\hat{\mu}), \tag{55}$$

represent all the solutions to the equation $\mu * \check{f} = 0$. Similarly, a periodic function in $\mathscr{E}$ (or $A(\mathbb{C})$) has a Fourier series that converges in the topology of $\mathscr{E}$ (or $A(\mathbb{C})$) resrpectively). For solutions of difference-differential equations, partial solutions to the representation problem go at least as far back as [20, 34]. In [26] Leont'ev pointed out the need to group terms in order that a series of the type (55) converge at all—this corresponds to Example 15 above (see also [37]). Finally, following earlier work of Delsarte [12], L. Schwartz solved completely this problem in [35], proving that the representation (55) holds for any mean-periodic function with respect to an arbitrary μ. What happens is that one needs both grouping of terms and an Abel-summation procedure to make (55) convergent, even in the pointwise sense. After that, it will converge in $\mathscr{E}$. In [13] it was observed that if one assumes that $\mu \in \mathscr{E}'$ is slowly decreasing then the Abel-summation procedure can be dispensed with. In the case of several variables, i.e. $\varphi \in \mathscr{E}(\mathbb{R}^n)$ or $\varphi \in A(\mathbb{C}^n)$, Ehrenpreis' Fundamental Principle [14, 15, 32] consisted in extending the Fourier-type representation (55) to solutions of partial differential equations with constant coefficients. There was a further extension of this Fundamental Principle to certain difference-differential operators in $\mathbb{R}^n$ in [2, 4]. The representation question is settled in [3, 6] for the case in which either $\varphi \in \mathscr{E}(\mathbb{R}^n)$ is mean-periodic with respect to μ, $\hat{\mu}$ slowly decreasing, or $\varphi \in A(\mathbb{C}^n)$ is analytic mean-periodic (no restriction on μ). The method of [6] is based on the relationship existing between the Fourier-representation (55) and the interpolation theorems presented above. In the case of one variable, this idea appears in [16] in the case the series (55) converges absolutely without any grouping and it is completely settled there under the assumption that all the multiplicities $m_k = 1$.

To start with, we recall that $\mathscr{E}$ (resp. $A(\mathbb{C})$) is a reflexive Frechet space whose dual $\mathscr{E}'$ (resp. $A'(\mathbb{C})$) is linearly isomorphic, via the Fourier transform, to the space $A_p(\mathbb{C})$, $p(z) = |\operatorname{Im} z| + \log(1 + |z|^2)$ (resp. $p(z) = |z|$), and this isomorphism is also topological ([15], Ch. 5). Given $\hat{\mu} \in \mathscr{E}'$ we will say it is slowly decreasing if that is true for $\mu \in A_p$. From here on we will assume $\mu \in \mathscr{E}'$ is slowly decreasing. (No restriction is needed if we consider $\mu \in A'(\mathbb{C})$, beyond $\mu \neq 0$). We will only refer to mean-periodic functions in $\mathscr{E}$, since everything carries over verbatim to analytic mean-periodic functions. Let $\mathscr{M}$ be the closed subspace of $\mathscr{E}$ defined by

$$\mathscr{M} = \{f \in \mathscr{E} : \mu * \check{f} = 0\}$$

and $\mathscr{J}$ the ideal in $\mathscr{E}'$ given by

$$\mathscr{J} = \{\mu * \nu : \nu \in \mathscr{E}'\}.$$

Since μ is slowly decreasing, i.e. $\hat{\mathscr{J}} = ((\hat{\mu}))$ is closed in $\hat{\mathscr{E}}' = A_p$ (see Proposition 3), it follows that $\mathscr{J}$ is also closed. Hence $\mathscr{M}$ is the dual space to $\mathscr{E}'/\mathscr{J} \simeq \hat{\mathscr{E}}'/\hat{\mathscr{J}} = A_p/((\hat{\mu}))$. By Theorem 7, $A_p/((\hat{\mu})) \simeq A_{p,g}(V)$, $V = V(\hat{\mu})$. We use Theorem 8 to characterize $(A_{p,g}(V))'$. An element $\varphi \in A_{p,g}(V)$ is a sequence $\{\varphi_\alpha\}$ of vectors in the finite dimensional vector spaces $A(V_\alpha)$, which for some $C > 0$ satisfy

$$\sup_\alpha |||\varphi_\alpha|||_\alpha e^{-Cp_\alpha} < \infty$$

where $P_\alpha = \min\{p(z) : z \in U_\alpha\}$. Hence, $F \in (A_{p,g}(V))'$ is given by a sequence of vectors $\{F_\alpha\}$, $F_\alpha \in A(V_\alpha)^*$, such that for every $C > 0$ we have

$$\sum_\alpha |||F_\alpha|||_\alpha^* e^{Cp_\alpha} < \infty,$$

and

$$F(\varphi) = \sum_{\alpha} F_\alpha(\varphi_\alpha). \tag{56}$$

(Here, $A(V_\alpha)^*$ denotes the dual of the finite dimensional space $A(V_\alpha)$ and $|||\cdot|||_\alpha^*$ the dual norm.) Hence, the series (56) converges absolutely and uniformly over any bounded set of φ's for a fixed F. Given $f \in \mathscr{M}$, we have then a corresponding element $F \in (A_{p,g}(V))'$ given by

$$T(f) = F(\rho_V(\hat{T})), \qquad \forall T \in \mathscr{E}', \tag{57}$$

since ρ_V establishes the isomorphism in Theorem 8. By specializing (57) to $T = \delta_x =$ Dirac mass at x, we get

$$f(x) = F(\rho_V(e^{-izx})) = \sum_{\alpha} \left(\sum_{(z_k, l) \in V_\alpha} c_{k,l} x^l e^{-iz_k x} \right). \tag{55'}$$

The outside series now converges absolutely and uniformly over compact subsets of $\mathbb{R}$. By taking $T = \delta_x^{(j)}$ we obtain the same statement for all formal derivatives.

THEOREM 9 [35, 13, 48]. *If $f \in \mathscr{E}(\mathbb{R})$ is mean-periodic with respect to a slowly decreasing distribution μ, then f has a Fourier representation* (55′) *convergent in $\mathscr{E}$. If $f \in A(\mathbb{C})$ is mean-periodic with respect to $\mu \in A'(\mathbb{C})$, $\mu \neq 0$, the same statement holds* ($x \in \mathbb{C}$).

The weak-interpolating case is the case where the series (55′) becomes

$$f(x) = \sum_k c_k(x)\, e^{-iz_k x},$$

c_k a polynomial of degree $< m_k$, the summation extended over the distinct roots of the equation $\hat{\mu}(z) = 0$. This series converges in the topology of $\mathscr{E}$ without any groupings, see Remark 4 (compare also with [16]).

The case of other distribution spaces, e.g. $f \in \mathscr{D}'(\mathbb{R})$, can be handled similarly.

In the above proof of the Fourier expansion for mean-periodic functions, the dual norms $|||\cdot|||^*$ are used but there is no need to compute them explicitly. We proceed to do that computation here, since the explicit expression we obtain allows us to solve a problem posed by Ehrenpreis [13, problem 9]. For the sake of simplicity we will assume throughout that the multiplicities are always one, but the result extend to the general case without difficulty.

LEMMA 7. *Let $z_0, \ldots, z_n$ be $n+1$ distinct points in $\mathbb{C}$, J the linear isomorphism of $\mathbb{C}^{n+1}$ into itself given by mapping an $n+1$-tuple $(a_0, \ldots, a_n)$ to the $n+1$ divided differences $\Delta^{(j)}$ of the a_i with respect to $(z_0, \ldots, z_n)$:*

$$J\colon a = (a_0, \ldots, a_n) \to \Delta = (\Delta^{(0)}, \ldots, \Delta^{(n)}),$$

where $\Delta^{(0)} = a_0$, $\Delta^{(1)} = (a_1 - a_0)/(z_1 - z_0)$, etc. Then the matrix representing J^{-1} is the following:

$$J^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & (z_1 - z_0) & 0 & \cdots & 0 \\ 1 & (z_2 - z_0) & (z_2 - z_0)(z_2 - z_1) & 0 \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \vdots \end{bmatrix}$$

For $d \geqslant 1$ fixed, let

$$|||\, a \,||| := \left(\sum_{k=0}^{n} | \Delta^{(k)} d^k |^2 \right)^{1/2}$$

Then, its dual norm, with respect to the bilinear form $\langle a, b \rangle := a_0 b_0 + \cdots + a_n b_n$, is

$$|||\, b \,|||^* = \left(\sum_{k=0}^{n} | c_k d^{-k} |^2 \right)^{1/2},$$

where $c := (J^{-1})^t b$. Furthermore, if $|||\, b \,|||^* \leqslant 1$ and diameter $\{z_0, \ldots, z_n\} \leqslant d$, then

$$| b_k | \prod_{j \neq k} | z_j - z_k | \leqslant n 4^n d^n, \qquad k = 0, \ldots, n.$$

Proof. The computation of the matrix J^{-1} is an immediate consequence of the fact that, for $P(z) = \sum_{k=0}^{n} \Delta^{(k)} \prod_{0 \leqslant j \leqslant k-1} (z - z_j)$ we have $P(z_j) = a_j$ when $\Delta = Ja$. The determination of $|||\cdot|||^*$ is standard. It only remains to prove the last estimate, which is obvious for $k = n$ since $| c_n | \leqslant |||\, b \,|||^* d^n$ and

$$| c_n | = | b_n | \prod_{0 \leqslant j < n} | z_j - z_n |$$

as follows from our explicit computation of J^{-1}. To obtain the general case we only have to observe that from Lemma 6 it follows that the maps

$$\pi_k(a_0 \cdots a_k \cdots a_n) = (a_0 \cdots a_n \cdots a_k)$$

have norm $\leqslant n 4^n$, when $\mathbb{C}^{n+1}$ is endowed with the norm $|||\cdot|||$. ∎

The problem we want to discuss is the following. Assume $\mu, \nu \in \mathscr{E}'$, μ is slowly decreasing and $\hat{\mu}, \hat{\nu}$ have no common zeroes. When is it possible to solve the system

$$\begin{aligned} \check{\mu} * f &= g \\ \check{\nu} * f &= h, \end{aligned} \tag{58}$$

subject to the compatibility condition $\check{\mu} * h = \check{\nu} * g$. We look for $f \in \mathscr{E}(\mathbb{R})$ and assume that $g, h \in \mathscr{E}(\mathbb{R})$, but the same reasoning works in $\mathscr{D}'(\mathbb{R})$.

A small amount of functional analysis shows that the necessary and sufficient condition for the existence of a solution f for every pair g, h which satisfies the compatibility condition, is that the algebraic ideal generated by μ and ν be closed in $\mathscr{E}'$, which, by the spectral synthesis theorem [35], and the assumption that $\hat{\mu}$ and $\hat{\nu}$ have no common zeroes, is itself equivalent to the existence of two distributions μ_1, $\nu_1 \in \mathscr{E}'$ such that

$$\check{\mu} * \mu_1 + \check{\nu} * \nu_1 = \delta_0 . \tag{59}$$

(See [21] for an analytic condition equivalent to (59).)

The question then arises of characterizing all g, h for which (58) is solvable when (59) fails. Since μ is slowly decreasing, the convolution operator $\check{\mu}*\colon \mathscr{E} \to \mathscr{E}$ is onto ([13]), and hence (58) can be reduced to the equivalent system

$$\begin{aligned} \check{\mu} * F &= 0 \\ \check{\nu} * F &= H \end{aligned} \tag{58$'$}$$

with compatibility condition $\check{\mu}*H = 0$.

Hence, both H and the function F we are looking for are mean-periodic with respect to the slowly decreasing distribution μ. We can apply Theorem 9 so they have expansions

$$F(x) = \sum \varphi_{k,l} x^l e^{-iz_k x}$$

and

$$H(x) = \sum \gamma_{k,l} x^l e^{-iz_k x}$$

convergent in the topology of $\mathscr{E}$ (after groupings), and the coefficients are uniquely determined by F, H (see (55′) or use the Spectral Synthesis Theorem.) Let us assume the multiplicities of the zeroes of $\hat{\mu}$ are always one. Writing $\varphi_k := \varphi_{k,0}$, $\gamma_k := \gamma_{k,0}$, the system (58′) is equivalent to the equations

$$\hat{\nu}(z_k)\, \varphi_k = \gamma_k \qquad z_k \in V = V(\hat{\mu}), \tag{60}$$

which can be solved uniquely, since $\hat{\nu}(z_k) \neq 0$. The only problem is to decide whether the φ_k can be the Fourier coefficients of a C^∞ function. The answer lies in Theorem 8 (see also proof of Theorem 9), i.e. for any $C > 0$

$$|||\, \gamma_k/\hat{\nu}(z_k) |||_\alpha^* = O(\exp(-Cp_\alpha)). \tag{61}$$

These inequalities are fairly explicit, due to Lemma 7 above. In particular, a necessary condition on H is the existence of constants $K = K(C) > 0$ such that

$$|\, \gamma_k \,| \prod_{j \neq k} |\, z_j - z_k \,| \leqslant K d_\alpha^{n_\alpha} \,|\, \hat{\nu}(z_k)| \exp(-Cp(z_k))$$

where the product is taken over those z_j lying in the same group α as z_k. The other constants involved have been absorbed by the exponential term, using $n_\alpha = O(p_\alpha)$.

The case in which the multiplicities are allowed to be arbitrary may be treated similarly. The condition (60) is replaced by triangular systems of linear equations with $\hat{\nu}(z_k)$ as the diagonal elements, and the modifications needed in the Lemma 7 are direct.

Let us illustrate this result with the same very simple example from the classical theory of Fourier series which we have used repeatedly (which, of course, could be studied without using Theorem 9). If $\hat{\mu}(z) = \sin \pi z$, then F, H must be periodic functions of period 2π, there are no groupings, $d \leqslant 1$, $z_k = k \in \mathbb{Z}$. Theorem 8 reduces to the well-known fact that, for all $C > 0$,

$$\varphi_k, \gamma_k = O((1 + |k|)^{-C}).$$

If $\hat{\nu}(z) = 1/z \sin(\pi/\lambda)z$, $\lambda \notin \mathbb{Q}$, then (61) imposes conditions on γ_k depending on how fast can λ be approximated by rational numbers, namely

$$\gamma_k = O((1 + |k|)^{-C} \operatorname*{dist}_{s\in\mathbb{Z}} (k, s\lambda)).$$

We have included at the end of our bibliography a few references not mentioned in the text which touch upon deeper properties of mean-periodic functions, Dirichlet series and other related subjects. These references are not complete, since there is a vast amount of literature in these areas.

Acknowledgments

The authors gratefully acknowledge the support received from the National Science Foundation in the preparation of this paper, and also wish to thank Professor G.-C. Rota for his encouragement to write it.

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