1.D.3:
6.B

Nuclear Physics **A313** (1979) 77–94; © *North-Holland Publishing Co., Amsterdam*

THE BARGMANN-SEGAL TRANSFORM, SU(3) SYMMETRY, AND NUCLEAR CLUSTER NORMS †

K. T. HECHT and W. ZAHN

Department of Physics, The University of Michigan, Ann Arbor, Michigan 48109, USA

Received 16 June 1978
(Revised 29 August 1978)

Abstract: By combining the use of the Bargmann-Segal integral transform with SU(3) recoupling techniques a new method is developed for the calculation of matrix elements in a basis of cluster model wave functions. An analytic expression is derived for the norm of an α+heavy fragment "cluster" function. Norms in p and sd shell nuclei can be calculated by a simple formula involving a few tabulated coefficients and readily available SU(3) Racah coefficients. Specific tabulations are given for nuclei with $12 \leqq A \leqq 24$.

1. Introduction

With the development of the nuclear cluster model [1]) in the last decade sophisticated computational methods in the framework of a refined resonating group theory have been developed [2]). The use of integral transforms for the evaluation of matrix elements was found to be a powerful tool [3]). Recently, various integral transforms have successfully been applied in different but similar approaches to nuclear problems [4–9]). Detailed applications of the refined resonating group method, however, have been limited to very light nuclei ($A \leqq 12$); and the evaluation of matrix elements in an angular-momentum coupled basis has proved to be difficult for cluster functions involving more than two fragments [10]). The extension of integral transform techniques to heavier cluster systems may be possible by exploiting the SU(3) symmetry properties of the relative motion and internal cluster harmonic oscillator functions, particularly if cluster functions are expressed in terms of an SU(3) coupled basis in which SU(3) recoupling techniques can be used to advantage. Of the many integral transforms employed in microscopic cluster model calculations the Bargmann-Segal (BS) transform [6]) is ideally suited to the exploitation of SU(3) recoupling techniques [11]) since oscillator functions have very simple SU(3) coupling properties in Bargmann space. Kernels for norms and two-body operators (expanded in terms of Gaussians) have BS transforms of simple Gaussian form for cluster functions built from 0s cluster components, each with mass number $\leqq 4$. For problems involving heavier cluster fragments (e.g. ^{19}F as a ^{15}N$+\alpha$ "cluster" system), the BS transforms of these

† Work supported by the National Science Foundation and the Deutsche Forschungsgemeinschaft, Bonn-Bad Godesberg.

kernels are products of Gaussians with polynomials in the Bargmann-space variables for the relative motion Jacobi vectors. The evaluation of these polynomials is particularly simple if they are expressed in an SU(3) coupled basis.

The technique to be illustrated in this paper can be exploited for the evaluation of kernels for both norms and two-body operators, and for cluster functions with fragments of equal and unequal oscillator size parameters. The special case of the norm kernel for a cluster system with fragments described by oscillator functions of equal size, however, forms a particularly simple example since such a norm kernel is an SU(3) scalar [12]). Although the restriction to this simple problem is made mainly for purposes of illustration, a cluster function built from cluster components with oscillator functions of equal size may nevertheless be very useful for p and sd shell nuclei since a physically realistic description of such nuclei may involve the combination of a fairly rich shell model (valence) basis with core excitations described in terms of such cluster wave functions [13]). A recent "extended" shell model calculation of this type for ^{20}Ne has proved to be very successful [13]). Norm kernels for cluster wave functions built from oscillator functions of equal size are also useful for multinucleon transfer spectroscopy if norms can be calculated for relative motion functions of an excitation high enough to construct radial functions of realistic shape. A number of recent tabulations of α-spectroscopic amplitudes [14]), in particular, have included results up to high oscillator excitation [12,13,15]). For realistic applications the relative motion functions in a cluster basis built from oscillator functions of equal size must thus be able to carry excitations up to a high number of oscillator quanta. In the present method the polynomials in the Bargmann-space variables, from which the needed kernels are built, can be evaluated from known matrix elements in the space of functions corresponding to the lowest Pauli-allowed relative motion excitations which have a 100 % overlap with simple (valence) shell model wave functions. Since the BS transform of such a kernel then contains relative motion functions of arbitrarily higher excitation an expansion of such a kernel will lead directly to the evaluation of the needed matrix elements. The technique thus is one which propagates information from the space of lowest Pauli-allowed excitations to arbitrarily high excitations of the relative motion functions of the cluster basis.

The formulation of the general technique will be given in sect. 2 which will also serve to establish the notation. The details of the technique will be illustrated with the evaluation of the BS transforms of a few norm kernels in sect. 3. The cluster problems chosen for this purpose will be a series of $A = 4n+k$ nuclei ($k = 1, 2, 3, 4$) described in terms of a cluster model basis made up of a heavy fragment and an α-cluster where the heavy fragment is assumed to have an internal function of highest possible space and SU(3) symmetry. Detailed results are tabulated for nuclei with $12 \leqq A \leqq 24$ from which the $\alpha+(A-4)$ fragment norms can be calculated by a simple formula involving readily available SU(3) Racah coefficients. The limitation to these simple $\alpha+(A-4)$ fragment systems serves to illustrate the technique with problems which involve the minimum of SU(3) recoupling. The present investigation is also limited to the evaluation of norm kernels which are SU(3) scalars. Kernels for more complicated operators (or for norms of cluster functions with fragments of unequal oscillator size parameters) will be built from SU(3) tensors. Their BS transforms can be constructed by similar techniques which will, however, involve more SU(3) irreducible tensor calculus and more recoupling of the SU(3) tensors from which these transforms are built.

2. Formulation of the technique

The use of the combined BS integral transform and SU(3) recoupling technique will be illustrated by the simplest possible example, the calculation of the norm for the cluster wave functions of a series of nuclei described by cluster wave functions with cluster components consisting of a heavy $[A = 4(n-1)+k]$ fragment (n = integer, k = 1, 2, 3 or 4), and an α-cluster

$$\Psi = \mathscr{A}[\varphi_{\alpha_{n+1}}\Phi^{(\lambda_c\mu_c)}\chi(\boldsymbol{R}_n)^{(Q_n0)}]^{(\bar{\lambda}\bar{\mu})}_{\kappa LM}. \tag{1a}$$

Here, $\mathscr{A}$ denotes total antisymmetrization. The internal wave function of the heavy fragment, $\Phi^{(\lambda_c\mu_c)}$, is itself built from a superposition of $n-1$ α-cluster internal wave functions, a single k-cluster internal wave function, and relative motion functions for $n-1$ Jacobi vectors $\boldsymbol{R}_1, \ldots, \boldsymbol{R}_{n-1}$

$$\Phi^{(\lambda_c\mu_c)}\ldots = (\prod_{i=1}^{n}\varphi_i)[\ldots[\chi(\boldsymbol{R}_1)^{(Q_10)}\times\chi(\boldsymbol{R}_2)^{(Q_20)}]^{(\lambda_{12}\mu_{12})}\ldots\chi(\boldsymbol{R}_{n-1})^{(Q_{n-1}0)}]^{(\lambda_c\mu_c)}\ldots, \tag{1b}$$

where the internal function for the ith cluster, φ_i, is built from 0s oscillator functions in the α- or k-cluster internal degrees of freedom and includes the spin-isospin functions, which are SU(4) scalars for the $n-1$ α-clusters and of SU(4) symmetry $[1^k]$ for the k-particle cluster. The relative motion functions, χ, are constructed in terms of SU(3) coupled oscillator functions carrying a number of oscillator quanta Q_i which correspond to the minimum Pauli-allowed values for $i = 1, \ldots, n-1$. The superscripts indicate SU(3) quantum numbers $(\lambda\mu)$, in Elliott's notation; and the square brackets denote SU(3) coupling. Although the internal function for the heavy fragment could be built from a superposition of several $(\lambda_c\mu_c)$, it will be assumed that this function is adequately described by a single SU(3) representation corresponding to the highest possible oscillator quanta symmetry in the function of highest space symmetry; e.g., $(\lambda_c\mu_c) = (04)$ for a ^{12}C fragment, or $(\lambda_c\mu_c) = (02)$ for an $A = 14$ fragment.

If the cluster wave function of eqs. (1a) and (1b) is abbreviated by

$$\Psi = \mathscr{A}\prod_{i=1}^{n}\varphi_i(\xi_i)\chi(\boldsymbol{R}), \tag{1c}$$

where $\boldsymbol{R}$ stands collectively for $\boldsymbol{R}_1, \boldsymbol{R}_2, \ldots, \boldsymbol{R}_n$, the kernel $\mathscr{K}(\bar{\boldsymbol{R}}, \boldsymbol{R})$ for an operator,

O, is defined by

$$\int \mathrm{d}\xi \prod_{i=1}^{n} \varphi_i^* O \mathscr{A} \prod_{i=1}^{n} \varphi_i \chi(\boldsymbol{R}) = \int \mathrm{d}\bar{\boldsymbol{R}} \mathscr{K}(\bar{\boldsymbol{R}}, \boldsymbol{R}) \chi(\bar{\boldsymbol{R}}). \tag{2}$$

The operator may be the Hamiltonian, or the unit operator for the norm. If O is an operator of complicated SU(3) or spherical tensor rank, this equation is schematic since it implies considerable SU(3) or angular momentum coupling. However, it is precise for the unit operator, or any SU(3) scalar operator.

It will be advantageous to introduce the BS transform of the kernel

$$H(\bar{\boldsymbol{K}}, \boldsymbol{K}) = \int A(\bar{\boldsymbol{K}}, \bar{\boldsymbol{R}}) \mathscr{K}(\bar{\boldsymbol{R}}, \boldsymbol{R}) A^*(\boldsymbol{K}, \boldsymbol{R}) \mathrm{d}\bar{\boldsymbol{R}} \mathrm{d}\boldsymbol{R}, \tag{3}$$

where the function $A(\boldsymbol{K}, \boldsymbol{R})$ which generates the transform is a short-hand notation for

$$A(\boldsymbol{K}, \boldsymbol{R}) = \prod_{i=1}^{n} \prod_{\alpha = x, y, z} A(K_{i_\alpha}, X_{i_\alpha}) \tag{4a}$$

[see also eqs. (2.3)–(2.9) of ref. [6]]; and the one-dimensional factor is defined by

$$A(K_x, X) = \pi^{-\frac{1}{4}} \exp\{-\tfrac{1}{2}K_x^2 - \tfrac{1}{2}X^2 + \sqrt{2}K_x X\}. \tag{4b}$$

In the present application the generating function property of A is of prime importance:

$$A(K_x, X) = \sum_{n=0}^{\infty} \Psi_n^*(X) \frac{K_x^n}{\sqrt{n!}}. \tag{5}$$

Here, $\Psi_n(X)$ is a one-dimensional harmonic oscillator function, and $K_x^n/\sqrt{n!}$ is the normalized oscillator function in Bargmann space. The Bargmann space function,

$$P(\boldsymbol{K})^{(Q0)}_{n_x n_y n_z} = \frac{K_x^{n_x}}{\sqrt{n_x!}} \frac{K_y^{n_y}}{\sqrt{n_y!}} \frac{K_z^{n_z}}{\sqrt{n_z!}}, \tag{6a}$$

has SU(3) irreducible tensor character (Q0) with subgroup labels here given in a Cartesian oscillator basis. On the other hand,

$$P(\boldsymbol{K}^*)^{(0Q)}_{n_x n_y n_z} = \frac{K_x^{*n_x}}{\sqrt{n_x!}} \frac{K_y^{*n_y}}{\sqrt{n_y!}} \frac{K_z^{*n_z}}{\sqrt{n_z!}} \tag{6b}$$

has SU(3) irreducible character (0Q). An SU(3) coupled Bargmann space polynomial can be defined by

$$[P(\boldsymbol{K}_1)^{(Q_1 0)} \times P(\boldsymbol{K}_2)^{(Q_2 0)}]^{(\lambda\mu)}_{\alpha} = \sum_{\alpha_1 \alpha_2} \langle (Q_1 0)\alpha_1 (Q_2 0)\alpha_2 | (\lambda\mu)\alpha \rangle P(\boldsymbol{K}_1)^{(Q_1 0)}_{\alpha_1} P(\boldsymbol{K}_2)^{(Q_2 0)}_{\alpha_2}, \tag{7}$$

where the subgroup labels α in the U(3) coupling coefficient can be chosen in any convenient fashion; e.g. $\alpha = n_x n_y n_z$ in a Cartesian oscillator basis, or $\alpha = \kappa LM$ in an angular momentum basis. With $\boldsymbol{K}_1 = \boldsymbol{K}_2$ a renormalization factor is needed to construct the normalized Bargmann space polynomial

$$[P(\boldsymbol{K})^{(Q_1 0)} \times P(\boldsymbol{K})^{(Q_2 0)}]_\alpha^{(\lambda\mu)} = \delta_{\lambda,(Q=Q_1+Q_2)}\delta_{\mu 0}\left[\frac{Q!}{Q_1!Q_2!}\right]^{\frac{1}{2}} P(\boldsymbol{K})_\alpha^{(Q0)}. \tag{8}$$

The product of two A-functions can then be written as

$$A(\boldsymbol{K}_1, \boldsymbol{R}_1)A(\boldsymbol{K}_2, \boldsymbol{R}_2) = \sum_{Q_1 Q_2} \sum_{(\lambda_{12}\mu_{12})} \sum_{\alpha_{12}} \times [P(\boldsymbol{K}_1)^{(Q_1 0)} \times P(\boldsymbol{K}_2)^{(Q_2 0)}]_{\alpha_{12}}^{(\lambda_{12}\mu_{12})} [\chi^*(\boldsymbol{R}_1)^{(Q_1 0)} \times \chi^*(\boldsymbol{R}_2)^{(Q_2 0)}]_{\alpha_{12}}^{(\lambda_{12}\mu_{12})}. \tag{9}$$

By successive SU(3) coupling this can be generalized to

$$\prod_{i=1}^{n} A(\boldsymbol{K}_i, \boldsymbol{R}_i) = \sum_{Q_1 \ldots Q_n} \sum_{(\lambda_{12}\mu_{12})\ldots(\lambda\mu)} \sum_{\alpha} [P(\boldsymbol{K}_1, \ldots, \boldsymbol{K}_{n-1})^{(\lambda_c\mu_c)} \times P(\boldsymbol{K}_n)^{(Q_n 0)}]_\alpha^{(\lambda\mu)}$$
$$\times [[\ldots [\chi^*(\boldsymbol{R}_1)^{(Q_1 0)} \times \chi^*(\boldsymbol{R}_2)^{(Q_2 0)}]^{(\lambda_{12}\mu_{12})} \ldots \chi^*(\boldsymbol{R}_{n-1})^{(Q_{n-1}, 0)}]^{(\lambda_c\mu_c)} \times \chi^*(\boldsymbol{R}_n)^{(Q_n 0)}]_\alpha^{(\lambda\mu)}. \tag{10}$$

For the components with $(\lambda_c\mu_c)$, for which $Q_1, \ldots, Q_{n-1}$ are restricted to lowest Pauli-allowed values, the order and the detail of the SU(3) coupling are immaterial. The Bargmann space functions in the variables $\boldsymbol{K}_1, \ldots, \boldsymbol{K}_{n-1}$ for the internal degrees of freedom of the heavy cluster fragment can thus be characterized by the quantum numbers $(\lambda_c\mu_c)$ alone. By expressing all functions in eq. (10) in terms of SU(3) coupled irreducible tensors it is possible to avoid SU(3) Wigner coefficients and express physically meaningful results largely in terms of SU(3) recoupling coefficients. For the norm kernel, [or any kernel $\mathscr{K}(\bar{\boldsymbol{R}}, \boldsymbol{R})$ which is an SU(3) scalar], in particular, the simple property

$$\int \mathrm{d}\bar{\boldsymbol{R}}\mathrm{d}\boldsymbol{R}[\ldots [\chi^*(\bar{\boldsymbol{R}}_1)^{(\bar{Q}_1 0)} \times \chi^*(\bar{\boldsymbol{R}}_2)^{(\bar{Q}_2 0)}]^{(\bar{\lambda}_{12}\bar{\mu}_{12})} \ldots \chi^*(\bar{\boldsymbol{R}}_n)^{(\bar{Q}_n 0)}]_{\bar{\alpha}}^{(\bar{\lambda}\bar{\mu})} \mathscr{K}(\bar{\boldsymbol{R}}, \boldsymbol{R})$$
$$\times [\ldots [\chi(\boldsymbol{R}_1)^{(Q_1 0)} \times \chi(\boldsymbol{R}_2)^{(Q_2 0)}]^{(\lambda_{12}\mu_{12})} \ldots \chi(\boldsymbol{R}_n)^{(Q_n 0)}]_\alpha^{(\lambda\mu)} = \delta_{(\bar{\lambda}\bar{\mu})(\lambda\mu)}\delta_{\bar{\alpha}\alpha} I_{(\bar{\lambda}\bar{\mu})}, \tag{11}$$

eliminates all dependence on subgroup labels, since the numbers $I_{(\bar{\lambda}\bar{\mu})}$ must also be independent of $\bar{\alpha}$. For the kernel of the unit operator the integrals $I_{(\bar{\lambda}\bar{\mu})}$ are related to the norms, N, of the normalized cluster functions of the type (1),

$$N((\lambda_c\mu_c)(Q_n 0)(\bar{\lambda}\bar{\mu}))\mathscr{A}[\varphi_{\alpha_{n+1}} \Phi^{(\lambda_c\mu_c)} \chi(\boldsymbol{R}_n)^{(Q_n 0)}]_{\kappa LM}^{(\bar{\lambda}\bar{\mu})}, \tag{12}$$

by

$$I_{(\bar{\lambda}\bar{\mu})} = 1/[N((\lambda_c\mu_c)(Q_n 0)(\bar{\lambda}\bar{\mu}))]^2. \tag{13}$$

By the use of eq. (10) for both $A(\bar{\boldsymbol{K}}, \bar{\boldsymbol{R}})$ and $A^*(\boldsymbol{K}, \boldsymbol{R})$, combined with eqs. (11) and (13), the BS transform of the norm kernel can be expressed by

$$H(\bar{\boldsymbol{K}},\boldsymbol{K}) = \sum_{\substack{(\lambda_c\mu_c)Q_n\\(\bar{\lambda}\bar{\mu})}} \sum_{\bar{\alpha}} 1/[N((\lambda_c\mu_c)(Q_n 0)(\bar{\lambda}\bar{\mu}))]^2$$

$$\times [P(\bar{\boldsymbol{K}}_1 \ldots, \bar{\boldsymbol{K}}_{n-1})^{(\lambda_c\mu_c)} \times P(\bar{\boldsymbol{K}}_n)^{(Q_n 0)}]^{(\bar{\lambda}\bar{\mu})}_{\bar{\alpha}} [P(\boldsymbol{K}_1^*, \ldots, \boldsymbol{K}_{n-1}^*)^{(\mu_c\lambda_c)} \times P(\boldsymbol{K}_n^*)^{(0Q_n)}]^{(\bar{\mu}\bar{\lambda})}_{\bar{\alpha}}$$

$$= \sum_{\substack{(\lambda_c\mu_c)Q_n\\(\bar{\lambda}\bar{\mu})}} (1/[N((\lambda_c\mu_c)(Q_n 0)(\bar{\lambda}\bar{\mu}))]^2)[\dim(\bar{\lambda}\bar{\mu})]^{\frac{1}{2}}$$

$$\times [[P(\bar{\boldsymbol{K}}_1, \ldots, \bar{\boldsymbol{K}}_{n-1})^{(\lambda_c\mu_c)} \times P(\bar{\boldsymbol{K}}_n)^{(Q_n 0)}]^{(\bar{\lambda}\bar{\mu})} \times [P(\boldsymbol{K}_1^*, \ldots, \boldsymbol{K}_{n-1}^*)^{(\mu_c\lambda_c)} \times P(\boldsymbol{K}_n^*)^{(0Q_n)}]^{(\bar{\mu}\bar{\lambda})}]^{(00)}, \tag{14}$$

where $\dim(\lambda\mu) = \frac{1}{2}(\lambda+1)(\mu+1)(\lambda+\mu+2)$ is the dimension of $(\lambda\mu)$.

By using the orthonormality of the $P(\boldsymbol{K})^{(\lambda_c\mu_c)}$ in $\boldsymbol{K}$-space it would, in principle, be possible to integrate over the complex variables $\bar{\boldsymbol{K}}_1 \ldots \bar{\boldsymbol{K}}_{n-1}, \boldsymbol{K}_1^* \ldots \boldsymbol{K}_{n-1}^*$ with the Bargmann $\boldsymbol{K}$-space measure [6]) to select a single, specific value of $(\lambda_c\mu_c)$. In practice, it is easier to expand the full $H(\bar{\boldsymbol{K}},\boldsymbol{K})$ in terms of the SU(3) coupled BS space polynomials of eq. (14) and determine the norms N by selecting the coefficients for the desired values of $(\lambda_c\mu_c)$ of the heavy fragment wave functions.

Since it is our aim to give an analytic expression for the function $H(\bar{\boldsymbol{K}},\boldsymbol{K})$ from which the norms N, for fixed $(\lambda_c\mu_c)$, can be evaluated for arbitrary values of $(\bar{\lambda}\bar{\mu})$ and Q_n, including Q_n values corresponding to arbitrary, high excitations of the α-heavy fragment relative motion, we shall make use of general properties of the BS transform of the norm kernel. These enable us to express $H(\bar{\boldsymbol{K}},\boldsymbol{K})$ in terms of a few exponentials in the variable $(\bar{\boldsymbol{K}}_n \cdot \boldsymbol{K}_n^*)$ and a few SU(3) coupled Bargmann-space polynomials built from $P(\boldsymbol{K}_n)^{(m0)}$, $P(\boldsymbol{K}_n^*)^{(0m)}$ of relatively low degree ($m \leqq 4$ for $\alpha+(A-4)$ cluster systems with $A-4 \leqq 16$); as well as the $P(\boldsymbol{K}_1, \ldots, \boldsymbol{K}_{n-1})^{(\lambda_c\mu_c)}$ which are the BS space transforms of the heavy fragment internal wave functions. Specifically, we shall show that $H(\bar{\boldsymbol{K}},\boldsymbol{K})$ can be given by

$$H(\bar{\boldsymbol{K}},\boldsymbol{K}) = \sum_{m=0}^{m_{\max}} \sum_{(\lambda\lambda)=(00)}^{(mm)} \sum_{\rho=1}^{d} \sum_{l=0}^{4} C_{m,l,(\lambda\lambda)\rho} \left\{ \sum_{p=0}^{4} D_l(p) \exp\left[\frac{4(A-4)-pA}{4(A-4)}(\bar{\boldsymbol{K}}_n \cdot \boldsymbol{K}_n^*)\right]\right\}$$

$$\times [[P(\bar{\boldsymbol{K}}_1, \ldots, \bar{\boldsymbol{K}}_{n-1})^{(\lambda_c\mu_c)} \times P(\boldsymbol{K}_1^*, \ldots, \boldsymbol{K}_{n-1}^*)^{(\mu_c\lambda_c)}]^{(\lambda\lambda)\rho} \times [P(\bar{\boldsymbol{K}}_n)^{(m0)} \times P(\boldsymbol{K}_n^*)^{(0m)}]^{(\lambda\lambda)}]^{(00)}. \tag{15}$$

The p-sum in the curly bracket arises from the antisymmetrizer, $\mathscr{A}$, and leads to a linear combination of five exponentials each multiplied by its own polynomial in the BS space SU(3) coupled irreducible tensors. The $p = 0$ term arises from the identity, etc., the $p = 4$ term from the permutation which exchanges all four nucleons of the α-particle with four nucleons in the heavy fragment. The coefficients $D_l(p)$ are given in table 1. The simplicity of these numbers is related to the fact that they are a group theoretical construct. Their values can be derived from properties of the permutation group. Eq. (15) is particularly simple when the heavy fragment in the $\alpha+(A-4)$ cluster system is a p-shell nucleus with $A-4 \leqq 16$. In this case, $m_{\max} = 4$, and the

TABLE 1

Values of $D_l(p)$

l \ p	0	1	2	3	4
0	1	−4	6	−4	1
1	0	−4	12	−12	4
2	0	0	6	−12	6
3	0	0	0	−4	4
4	0	0	0	0	1

coefficients, C, have the simple form $C_{m,l,(\lambda\lambda)\rho} = \delta_{m,l} C_{m,(\lambda\lambda),\rho}$; that is, the label l becomes redundant in this case, and the number of possible m, $(\lambda\lambda)$ values is 15. With λ_c and μ_c both non-zero the representation $(\lambda\lambda)$ may occur with a d-fold multiplicity in the Kronecker product $(\lambda_c\mu_c)\times(\mu_c\lambda_c)$, with $d > 1$; so that coupled SU(3) tensors and their coefficients must both be characterized by a multiplicity label ρ ($\rho = 1, 2, \ldots, d$). In many simple α+heavy cluster systems, however, either $\lambda_c = 0$ or $\mu_c = 0$ so that $d = 1$, and the multiplicity label ρ can be dropped. On the other hand, with an $A = 10$ heavy fragment, e.g., $(\lambda_c\mu_c) = (22)$ for highest space symmetry; and $d = 1, 2, 3, 2, 1$ for $(\lambda\lambda) = (00), (11), (22), (33), (44)$, respectively. Even in this case the number of possible m, $(\lambda\lambda)$, ρ values is a manageable number, *viz.* 27. For an $\alpha+(A-4)$ cluster system for which the heavy fragment is an sd shell nucleus $(A-4 > 16)$, $m_{\max} \leqq 8$; but the number of m, l combinations with non-zero coefficients is at most 15.

The form of $H(\bar{\boldsymbol{K}}, \boldsymbol{K})$, eq. (15), can be established from the general structure of the BS transform of the norm for an $(n+1)$-cluster system made up of $n+1$ fragments with 0s internal wave functions [n α-clusters and a single k-particle cluster (with $k = 1, 2, 3, 4$) in our special case]. If all oscillator wave functions in such an $(n+1)$-cluster system have equal size, the BS transform of the norm has the simple Gaussian form

$$H(\bar{\boldsymbol{K}}, \boldsymbol{K}) = \sum_{\beta} a_\beta \exp\{\sum_{i,j=1}^{n} \sigma_{ij}(\beta)(\bar{\boldsymbol{K}}_i \cdot \boldsymbol{K}_j^*)\} \tag{16}$$

[see eqs. (4.4)–(4.6) of ref. [6])].

By using a double coset (DC) decomposition for the antisymmetrizer, $\mathscr{A}$, in the cluster function the β-sum in eq. (16) can be restricted to one over DC generators with simple weighting coefficients a_β. For large n this can still be a very large number of terms. For the restricted cluster function made up of a single α-cluster and a heavy $A = [4(n-1)+k]$ particle fragment, frozen into the lowest Pauli-allowed state, $(\lambda_c\mu_c)$, the number of DC generators is reduced to five, and the equivalent β-sum contains at most five terms. However, the general structure of the BS transform for the unrestricted n α-cluster + k-particle cluster function can be used to derive the form of the BS transform for the restricted heavy fragment plus α-particle cluster function, by expanding the exponentials of eq. (16) and limiting the powers of $\bar{\boldsymbol{K}}_1, \ldots \bar{\boldsymbol{K}}_{n-1}, \boldsymbol{K}_1^*, \ldots \boldsymbol{K}_{n-1}^*$ to their minimum Pauli-allowed values for the internal function of the heavy fragment. Since the scalar products $\bar{\boldsymbol{K}}_i \cdot \boldsymbol{K}_j^*$ are also SU(3) scalars, the polynomials in the $\boldsymbol{K}$-space variables resulting from this expansion must be built from SU(3) irreducible tensors coupled to resultant $(\lambda\mu) = (00)$. All expansions make use of the basic relation

$$\frac{(\bar{\boldsymbol{K}}_i \cdot \boldsymbol{K}_j^*)^n}{n!} = [P(\bar{\boldsymbol{K}}_i)^{(n0)} \times P(\boldsymbol{K}_j^*)^{(0n)}]^{(00)}[\dim(n0)]^{\frac{1}{2}}. \tag{17}$$

To carry out the full expansion of eq. (16) it is convenient to split the exponentials in $H(\bar{\boldsymbol{K}}, \boldsymbol{K})$ into three factors:

(i) The first component is to include only those terms in the sum with both $i, j \leqq n-1$; i.e., it can carry only oscillator excitations of internal degrees of freedom of the heavy ($A = [4(n-1)+k]$ particle) fragment. It must thus be built from tensors of the form

$$[P(\bar{\boldsymbol{K}}_1, \ldots, \bar{\boldsymbol{K}}_{n-1})^{(\lambda'\mu')} \times P(\boldsymbol{K}_1^*, \ldots, \boldsymbol{K}_{n-1}^*)^{(\mu'\lambda')}]^{(00)}, \tag{18}$$

where $(\lambda'\mu')$ corresponds to a U(3) representation $[f_1' f_2' f_3'] = [\lambda' + \mu' + f_3', \mu' + f_3', f_3']$ which carries a number of oscillator quanta $f_1' + f_2' + f_3' = Q_1' + Q_2' + \ldots + Q_{n-1}' \leqq Q_0$. Here, Q_0 is defined as the minimum Pauli-allowed Q-value for the $A = [4(n-1)+k]$-particle fragment; e.g., $Q_0 = 8$ for $n = 4$, $k = 0$; and $Q_0 = 11$ for $n = 4$, $k = 3$. The expansion of this factor of the exponentials is thus limited to relatively low powers.

(ii) The second component is made up of the cross terms

$$\exp\{\sum_{i=1}^{n-1} \sigma_{in}(\bar{\boldsymbol{K}}_i \cdot \boldsymbol{K}_n^*) + \sum_{j=1}^{n-1} \sigma_{nj}(\bar{\boldsymbol{K}}_n \cdot \boldsymbol{K}_j^*)\}. \tag{19}$$

The number of oscillator quanta which can be contributed to Q_n by the $\boldsymbol{K}_n^*$ term of this factor, to be denoted by the integer m, must be equal to the number of oscillator quanta contributed by the $\bar{\boldsymbol{K}}_1, \bar{\boldsymbol{K}}_2, \ldots, \bar{\boldsymbol{K}}_{n-1}$ terms of this factor, so that

$$m = Q_0 - (Q_1' + Q_2' + \ldots + Q_{n-1}').$$

Similarly, the number of oscillator quanta contributed to Q_n by $\bar{\boldsymbol{K}}_n$, to be denoted by m', must be equal to the number of oscillator quanta carried by $\boldsymbol{K}_1^*, \boldsymbol{K}_2^*, \ldots, \boldsymbol{K}_{n-1}^*$, so that $m' = Q_0 - (Q_1' + Q_2' + \ldots + Q_{n-1}')$; and consequently $m' = m$. This second factor in the exponential thus contributes terms of the form

$$\begin{aligned}
&[[P(\bar{\boldsymbol{K}}_1, \ldots, \bar{\boldsymbol{K}}_{n-1})^{(m0)} \times P(\boldsymbol{K}_n^*)^{(0m)}]^{(00)}[P(\bar{\boldsymbol{K}}_n)^{(m0)} \times P(\boldsymbol{K}_1^*, \ldots, \boldsymbol{K}_{n-1}^*)^{(0m)}]^{(00)}]^{(00)} \\
&\quad = \sum_{(\lambda\lambda)=(00)}^{(mm)} \frac{[\dim(\lambda\lambda)]^{\frac{1}{2}}}{\dim(m0)} [[P(\bar{\boldsymbol{K}}_1, \ldots, \bar{\boldsymbol{K}}_{n-1})^{(m0)} \times P(\boldsymbol{K}_1^*, \ldots, \boldsymbol{K}_{n-1}^*)^{(0m)}]^{(\lambda\lambda)} \\
&\qquad \times [P(\bar{\boldsymbol{K}}_n)^{(m0)} \times P(\boldsymbol{K}_n^*)^{(0m)}]^{(\lambda\lambda)}]^{(00)}
\end{aligned} \tag{20}$$

to the expansion. The second line of eq. (20) is obtained from the first by a trivial SU(3) recoupling transformation.

(iii) The third factor of eq. (16) is left in its exponential form

$$\exp\{\sigma_{nn}(\bar{\boldsymbol{K}}_n \cdot \boldsymbol{K}_n^*)\}. \tag{21}$$

Only the σ_{nn} for the five double coset generators of the $\alpha+(A = [4(n-1)+k])$ two component cluster function are needed. These can be read from the p-sum of eq. (15) (the $p = 2$ term, e.g., corresponds to a permutation which exchanges two particles from the α-cluster component with two particles from the heavy fragment).

Finally, the SU(3) (00) tensors of eqs. (18) and (20) are combined with as yet unknown coefficients $c_{m,l,(\lambda'\mu')}$ to make the mth degree terms of the polynomials in $\bar{\boldsymbol{K}}_n$ and $\boldsymbol{K}_n^*$ needed for the full expansion of the BS transform $H(\bar{\boldsymbol{K}}, \boldsymbol{K})$ of eq. (15):

$$\begin{aligned}\mathscr{P}_{m,l} \equiv \sum_{(\lambda'\mu')} c_{m,l,(\lambda'\mu')} \Bigg\{ \sum_{(\lambda\lambda)=(00)}^{(mm)} \frac{[\dim(\lambda\lambda)]^{\frac{1}{2}}}{\dim(m0)} & \\ \times [[P(\bar{\boldsymbol{K}}_1, \ldots, \bar{\boldsymbol{K}}_{n-1})^{(\lambda'\mu')} \times P(\boldsymbol{K}_1^*, \ldots, \boldsymbol{K}_{n-1}^*)^{(\mu'\lambda')}]^{(00)} & \\ \times [[P(\bar{\boldsymbol{K}}_1, \ldots, \bar{\boldsymbol{K}}_{n-1})^{(m0)} \times P(\boldsymbol{K}_1^*, \ldots, \boldsymbol{K}_{n-1}^*)^{(0m)}]^{(\lambda\lambda)} & \\ \times [P(\bar{\boldsymbol{K}}_n)^{(m0)} \times P(\boldsymbol{K}_n^*)^{(0m)}]^{(\lambda\lambda)}]^{(00)}]^{(00)} \Bigg\}. &\end{aligned} \tag{22}$$

In principle, the $c_{m,l,(\lambda'\mu')}$ can be evaluated from the σ_{ij} and the DC expansion of the full $(n+1)$-cluster problem. In practice, it is easier to calculate them by indirect means from the norms of the cluster functions, eqs. (1), with $Q_1, Q_2, \ldots, Q_n$ all set equal to their minimum Pauli-allowed values. Such an antisymmetrized cluster function is equivalent to a simple (valence) shell model wave function; and its norm can be calculated by shell model techniques [11,14]). [For both p and sd shell nuclei, the four-particle c.f.p. needed are available [17,18]). For sd shell nuclei the norms for a few very simple core excited states [11]) may also be needed.]

To make this calculation it is necessary to reorganize the terms in $\mathscr{P}_{m,l}$ by an SU(3) recoupling transformation

$$\begin{aligned}\mathscr{P}_{m,l} = \sum_{(\lambda'\mu')} c_{m,l,(\lambda'\mu')} \sum_{(\lambda\lambda)=(00)}^{(mm)} \frac{[\dim(\lambda\lambda)]^{\frac{1}{2}}}{\dim(m0)} \sum_{(\lambda_c\mu_c)\rho} \left[\frac{\dim(\lambda_c\mu_c)}{\dim(\lambda'\mu')\dim(m0)}\right]^{\frac{1}{2}} & \\ \times U((\lambda_c\mu_c)(\mu'\lambda')(\lambda\lambda)(0m); (m0) - -; (\mu_c\lambda_c) - \rho) & \\ \times [[P(\bar{\boldsymbol{K}}_1, \ldots, \bar{\boldsymbol{K}}_{n-1})^{(\lambda_c\mu_c)} \times P(\boldsymbol{K}_1^*, \ldots, \boldsymbol{K}_{n-1}^*)^{(\mu_c\lambda_c)}]^{(\lambda\lambda)\rho} & \\ \times [P(\bar{\boldsymbol{K}}_n)^{(m0)} \times P(\boldsymbol{K}_n^*)^{(0m)}]^{(\lambda\lambda)}]^{(00)}, &\end{aligned} \tag{23}$$

where the U-coefficients are SU(3) Racah coefficients in the notation of refs. [11,19]). These are readily available through the computer code of Draayer and Akiyama [19,20]).

For the most general $6(\lambda\mu)$ recoupling transformation the U-coefficient

$$U((\lambda_1\mu_1)(\lambda_2\mu_2)(\lambda\mu)(\lambda_3\mu_3);\quad (\lambda_{12}\mu_{12})\rho_{12}\rho_{12,3};\quad (\lambda_{23}\mu_{23})\rho_{23}\rho_{1,23})$$

is a unitary transformation matrix whose rows are specified by $(\lambda_{12}\mu_{12})\rho_{12}\rho_{12,3}$ and columns by $(\lambda_{23}\mu_{23})\rho_{23}\rho_{1,23}$; and in the most general case four multiplicity labels ρ are required. For the recoupling transformation needed for eq. (14), only one of the four SU(3) products has a multiplicity $d > 1$. Only the multiplicity label $\rho_{1,23}$ for the product $(\lambda_c\mu_c)\times(\mu_c\lambda_c)\to(\lambda\lambda)$ is needed and is abbreviated by ρ. Unnecessary multiplicity labels are replaced by a dash. Comparing with eq. (15), the coefficients $C_{m,l,(\lambda\lambda)\rho}$ and $c_{m,l,(\lambda'\mu')}$ are related by

$$C_{m,l,(\lambda\lambda)\rho} = \sum_{(\lambda'\mu')} c_{m,l,(\lambda'\mu')}\left[\frac{\dim(\lambda\lambda)\dim(\lambda_c\mu_c)}{\dim(\lambda'\mu')(\dim(m0))^3}\right]^{\frac{1}{2}} \times U((\lambda_c\mu_c)(\mu'\lambda')(\lambda\lambda)(0m);(m0)\text{ - -};(\mu_c\lambda_c)\text{ - }\rho). \tag{24}$$

3. The norm evaluation

The BS transform of the norm kernel is now established in suitable SU(3) irreducible tensor form with the evaluation of the coefficients $c_{m,l,(\lambda'\mu')}$. Both to calculate these coefficients from the norms for the minimum Pauli-allowed Q_n values, and to calculate the norms for higher excitations, it will be useful to expand the exponentials $\exp\{\sigma_{nn}(p)(\bar{\boldsymbol{K}}_n\cdot\boldsymbol{K}_n^*)\}$ by the use of eq. (17) and combine the (Q_n-m)th term in this expansion with the mth degree polynomials $P(\bar{\boldsymbol{K}}_n)^{(m0)}$ and $P(\boldsymbol{K}_n^*)^{(0m)}$ of eq. (23) to attain the full expansion of $H(\bar{\boldsymbol{K}},\boldsymbol{K})$ in the form of eq. (14). For this purpose an SU(3) recoupling transformation has to be made to take us from the

$$[[[(\lambda_c\mu_c)\times(\mu_c\lambda_c)](\lambda\lambda)\rho\times[(m0)\times(0m)](\lambda\lambda)](00)\times[(Q_n-m,0)\times(0,Q_n-m)](00)](00)$$

scheme to the needed $[[(\lambda_c\mu_c)\times(Q_n0)](\bar{\lambda}\bar{\mu})\times[(\mu_c\lambda_c)\times(0Q_n)](\bar{\mu}\bar{\lambda})](00)$ scheme. This recoupling transformation, together with an application of eq. (8), yields the BS transform $H(\bar{\boldsymbol{K}},\boldsymbol{K})$ in the form needed:

$$\begin{aligned}
H(\bar{\boldsymbol{K}},\boldsymbol{K}) = &\sum_{m=0}^{m_{\max}}\sum_{Q_n=m}^{\infty}\sum_{l=0}^{4}\left\{\sum_{p=0}^{4}D_l(p)\left[\frac{4(A-4)-pA}{4(A-4)}\right]^{Q_n-m}\right\}\binom{Q_n}{m}\\
&\times\sum_{(\lambda_c\mu_c)(\bar{\lambda}\bar{\mu})(\lambda'\mu')}(-)^{\lambda_c+\mu_c+Q_n-\bar{\lambda}-\bar{\mu}}c_{m,l,(\lambda'\mu')}\left[\frac{\dim(\lambda_c\mu_c)\dim(Q_n0)}{\dim(\lambda'\mu')(\dim(m0))^4}\right]^{\frac{1}{2}}\\
&\times\Big\{\sum_{(\lambda\lambda)=(00)}^{(mm)}\sum_{\rho=1}^{d}[\dim(\lambda\lambda)]^{\frac{1}{2}}\\
&\times U((\lambda_c\mu_c)(\mu'\lambda')(\lambda\lambda)(0m);(m0)\text{ - -};(\mu_c\lambda_c)\text{ - }\rho)\\
&\times U((Q_n0)(0,Q_n-m)(\lambda\lambda)(0m);(m0)\text{ - -};(0Q_n)\text{ - -})
\end{aligned}$$

$$\times U((Q_n0)(\lambda_c\mu_c)(Q_n0)(\mu_c\lambda_c);(\bar{\lambda}\bar{\mu})\text{ - -};(\lambda\lambda)\rho\text{ -})\}$$
$$\times[[P(\bar{\boldsymbol{K}}_1,\ldots,\bar{\boldsymbol{K}}_{n-1})^{(\lambda_c\mu_c)}\times P(\bar{\boldsymbol{K}}_n)^{(Q_n0)}]^{(\bar{\lambda}\bar{\mu})}$$
$$\times[P(\boldsymbol{K}_1^*,\ldots,\boldsymbol{K}_{n-1}^*)^{(\mu_c\lambda_c)}\times P(\boldsymbol{K}_n^*)^{(0Q_n)}]^{(\bar{\mu}\bar{\lambda})}]^{(00)}. \tag{25}$$

The coefficient of a specific $\boldsymbol{K}$-space tensor

$$[[P(\bar{\boldsymbol{K}}_1,\ldots,\bar{\boldsymbol{K}}_{n-1})^{(\lambda_c\mu_c)}\times P(\bar{\boldsymbol{K}}_n)^{(Q_n0)}]^{(\bar{\lambda}\bar{\mu})}\ldots]]^{(00)}$$

is $[\dim(\bar{\lambda}\bar{\mu})]^{\frac{1}{2}}/[N((\lambda_c\mu_c)(Q_n0)(\bar{\lambda}\bar{\mu}))]^2$ and can thus be used to calculate N.

The evaluation of the $c_{m,l,(\lambda'\mu')}$ proceeds along slightly different lines for nuclei with (i) $A \leqq 16$, (ii) $16 < A \leqq 20$, and (iii) $A > 20$. In the first two cases, when the heavy fragment is a p-shell nucleus, the index l is redundant: $c_{m,l,(\lambda'\mu')} = \delta_{ml}c_{m,(\lambda'\mu')}$. This is related to the fact that the removal of m oscillator quanta from the $(A-4)$-particle wave function $\Phi^{(\lambda_c\mu_c)}$ of the heavy fragment to make a wave function of symmetry $(\lambda'\mu')$ can yield a wave function $\Phi^{(\lambda'\mu')}$ with at most $(A-4-m)$ particles. The removal of m oscillator quanta requires the removal of m p-shell nucleons. Only exchange terms with $p \geqq m$ can thus contribute to such a term; and the non-zero values of $D_l(p)$ begin with $l = p$ (cf. table 1). In addition, in cases (i) and (ii) the SU(3) symmetry $(\lambda_c\mu_c)$ of the heavy (p-shell) fragment must correspond to a Young tableau with at most four columns, so that $m \leqq 4$ since the product $(\lambda'\mu')\times(m0)$ for $m > 4$ would have product functions with tableaux of more than four columns. With $A = 14$, e.g., with an $A = 10$ heavy fragment of highest possible space symmetry $(\lambda_c\mu_c) = (22)$, there are nine possible m, $(\lambda'\mu')$ values; determined by the number of ways in which m symmetrically coupled squares can be removed from the Young tableau for $(\lambda_c\mu_c) = (22)$:

$$m = 0,(\lambda'\mu') = (22);\qquad m = 1,(\lambda'\mu') = (31),(12);\qquad m = 2,(\lambda'\mu') = (02),(40),(21);$$
$$m = 3,(\lambda'\mu') = (11),(30);\qquad m = 4,(\lambda'\mu') = (20).$$

The nine $c_{m,(\lambda'\mu')}$ can be evaluated from the first Pauli-allowed term of the norm kernel, with $Q_n = 4$. In eq. (25) Pauli-forbidden terms with $Q_n \leqq 3$ are automatically equal to zero via the p-sum. This is guaranteed by the structure of the $D_l(p)$. With $Q_n = 4$ there are nine possible $(\bar{\lambda}\bar{\mu})$ values in this case since

$$(\lambda_c\mu_c)\times(Q_n0) = (22)\times(40) = (62)+(43)+(51)+(24)+(40)+(32)+(13)+(21)+(02).$$

Of these nine $(\bar{\lambda}\bar{\mu})$ values, however, only the single one, $(\bar{\lambda}\bar{\mu}) = (02)$, gives a Pauli-allowed wave function, the ordinary shell model ground state configuration wave function for $A = 14$: $|(0\mathrm{s})^4(0\mathrm{p})^{10}(\bar{\lambda}\bar{\mu}) = (02)\ ST = 01 \text{ or } 10\rangle$. The nine $c_{m,(\lambda'\mu')}$ can thus be evaluated from the coefficients of the $[(\lambda_c\mu_c)\times(Q_n0)](\bar{\lambda}\bar{\mu}) = [(22)\times(40)](\bar{\lambda}\bar{\mu})$ terms of eq. (25), where these coefficients have the value zero for the eight Pauli-forbidden $(\bar{\lambda}\bar{\mu})$ and the simple shell model value [11,14])

$$[\dim(\bar{\lambda}\bar{\mu})]^{\frac{1}{2}}/N^2 = \sqrt{6}[(3^2\times 5/2^6)(14/10)^4]$$

for the only allowed state with $(\bar{\lambda}\bar{\mu}) = (02)$.

The calculation for a nucleus of type (2) can be illustrated by the example $A = 19$ with a heavy fragment $(\lambda_c\mu_c)$ restricted to $(\lambda_c\mu_c) = (01)$, the only possible SU(3) wave function for the $A = 15$ fragment. In this case there are eight possible $m, (\lambda'\mu')$ values:

$$m = 0, (\lambda'\mu') = (01); \qquad m = 1, (\lambda'\mu') = (10), (02); \qquad m = 2, (\lambda'\mu') = (11), (03);$$

$$m = 3, (\lambda'\mu') = (12), (04); \qquad m = 4, (\lambda'\mu') = (13).$$

The first Pauli-allowed Q_n value is now $Q_n = 7$. The Pauli-forbidden terms in eq. (25) with $Q_n \leqq 3$ are again automatically equal to zero via the structure of the $D_l(p)$. The eight $c_{m,(\lambda'\mu')}$ are evaluated from the Pauli-forbidden terms with $4 \leqq Q_n \leqq 7$ and the single Pauli-allowed term with $Q_n = 7$. As required, these are eight in number corresponding to the $[(\lambda_c\mu_c) \times (Q_n 0)](\bar{\lambda}\bar{\mu}) = [(01) \times (Q_n 0)](\bar{\lambda}\bar{\mu})$ possibilities with

$$Q_n = 4, (\bar{\lambda}\bar{\mu}) = (41), (30); \qquad Q_n = 5, (\bar{\lambda}\bar{\mu}) = (51), (40);$$

$$Q_n = 6, (\bar{\lambda}\bar{\mu}) = (61), (50); \qquad Q_n = 7, (\bar{\lambda}\bar{\mu}) = (71), (60).$$

The coefficients in eq. (25) corresponding to the seven Pauli-forbidden terms are again equal to zero. The coefficient for the single Pauli-allowed term with $Q_n = 7, (\bar{\lambda}\bar{\mu}) = (60)$ has the value $[\dim(\bar{\lambda}\bar{\mu})]^{\frac{1}{2}}/N^2 = \sqrt{28}[(3^4 \times 5/2^{13})(19/15)^7]$ where N^2 for the shell model configuration $(0s)^4(0p)^{12}(sd)^3(60)$ has been calculated by the SU(3) recoupling techniques of refs. [11,14]).

For a nucleus of type (3) with $A > 20$ and a heavy fragment corresponding to an sd shell nucleus, higher m-values come into play, and both indices m and l are now needed. For $A = 23$, e.g., the $A = 19$ heavy fragment of highest possible SU(3) symmetry has $(\lambda_c\mu_c) = (60)$ with a shell model configuration $s^4[p^{12}(00)sd^3(60)](60)$. In this case the possible m, l, $(\lambda'\mu')$ values are 33 in number:

$$m, l = 0, 0: (\lambda'\mu') = (60); \qquad m, l = 1, 1: (\lambda'\mu') = (61), (50);$$

$$m, l = 2, 2: (\lambda'\mu') = (62), (51), (40); \qquad m, l = 2, 1: (\lambda'\mu') = (40);$$

$$m, l = 3, 3: (\lambda'\mu') = (63), (52), (41), (30); \qquad m, l = 3, 2: (\lambda'\mu') = (41), (30);$$

$$m, l = 4, 4: (\lambda'\mu') = (64), (53), (42), (31), (20);$$

$$m, l = 4, 3: (\lambda'\mu') = (42), (31), (20); \qquad m, l = 4, 2: (\lambda'\mu') = (20);$$

$$m, l = 5, 4: (\lambda'\mu') = (43), (32), (21), (10); \qquad m, l = 5, 3: (\lambda'\mu') = (21), (10);$$

$$m, l = 6, 4: (\lambda'\mu') = (22), (11), (00); \qquad m, l = 6, 3: (\lambda'\mu') = (00);$$

$$m, l = 7, 4: (\lambda'\mu') = (01).$$

These can be understood from the following examples: (i) For $m = 1$ the removal of one oscillator quantum from the $A = 19$ heavy fragment wave function, with

$(\lambda_c\mu_c) = (60)$ and $12+6 = 18$ oscillator quanta, can yield wave functions of symmetries $(\lambda'\mu') = (50)$ and (61) only. Both of these $(\lambda'\mu')$ can be made only from shell model configurations $s^r[p^{11}(01)sd^3(60)](\lambda'\mu')$, with $r \leqq 4$, which therefore correspond to clusters with at most eighteen particles. To make functions of these $(\lambda'\mu')$ at least one nucleon must be removed from the $A = 19$ heavy fragment, and only exchange terms with $p \geqq 1$ can contribute to these terms. The $D_l(p)$ which begin with $p = 1$ are given by $l = 1$; see table 1. The four non-zero $D_1(p)$ give the relative fractional parentage contributions of the $s^r p^{11} sd^3$ configurations with $r = 4, 3, 2, 1$, corresponding to $p = 1, 2, 3, 4$. [A detailed group-theoretical interpretation of the $D_l(p)$ in terms of the concept of fractional parentage will be given in a subsequent publication which will deal with more complicated cluster systems.] (ii) For $m = 4$, the removal of four oscillator quanta from the $A = 19$ function with $(\lambda_c\mu_c) = (60)$ yields functions of symmetries $(\lambda'\mu') = (64), (53), (42), (31), (20)$. Of these, functions with $(\lambda'\mu') = (64)$ and (53) can be constructed only from the shell model configuration $s^4[p^8(04)sd^3(60)](\lambda'\mu')$ with fifteen particles. At least four nucleons must be removed from the $A = 19$ heavy fragment to make functions with these $(\lambda'\mu')$. Only the single exchange term with $p = 4$ can contribute in this removal process so that only the $l = 4$ term, [with only $D_4(p = 4) \neq 0$], can contribute to coefficients with these $(\lambda'\mu')$. Functions with $(\lambda'\mu') = (42)$ and (31) can be constructed both from the shell model configuration $s^4[p^8(04)sd^3(60)](\lambda'\mu')$ and from configurations $s^r[p^{10}(02)sd^2(40)](\lambda'\mu')$ with $r = 4$ or 3. The first one corresponds to an $l = 4$ term, as before. The second one can get contributions from exchange terms with both $p = 3$ and 4 and corresponds to $l = 3$. Finally, the function with $(\lambda'\mu') = (20)$ can be constructed from the above two shell model configurations leading to $l = 4$ and $l = 3$ contributions. In addition, it can also be constructed from the configurations $s^r[p^{12}(00)sd^1(20)](20)$ with $r = 4, 3, 2$ which can get contributions from exchange terms with $p = 2, 3, 4$. These are made possible by including terms with the index $l = 2$. The $\boldsymbol{K}$-space polynomials with $m = 4$, $(\lambda'\mu') = (20)$ are thus fully determined by three coefficients $c_{m,l,(\lambda'\mu')}$ with $l = 4, 3, 2$.

In all, there are 33 different $c_{m,l,(\lambda'\mu')}$. The first Pauli-allowed Q_n value is now $Q_n = 8$, but states with $4 \leqq Q_n \leqq 8$ are now only 32 in number, corresponding to the $[(\lambda_c\mu_c) \times (Q_n 0)](\bar{\lambda}\bar{\mu}) = [(60) \times (Q_n 0)](\bar{\lambda}\bar{\mu})$ possibilities with

$$Q_n = 4, (\bar{\lambda}\bar{\mu}) = (10{,}0), (81), (62), (43), (24);$$

$$Q_n = 5, (\bar{\lambda}\bar{\mu}) = (11{,}0), (91), (72), (53), (34), (15);$$

$$Q_n = 6, (\bar{\lambda}\bar{\mu}) = (12{,}0)\, (10{,}1), (82), (63), (44), (25), (06);$$

$$Q_n = 7, (\bar{\lambda}\bar{\mu}) = (13{,}0), (11{,}1), (92), (73), (54), (35), (16);$$

$$Q_n = 8, (\bar{\lambda}\bar{\mu}) = (14{,}0), (12{,}1), (10{,}2), (\underline{83}), (\underline{64}), (\underline{45}), (\underline{26}).$$

Of the $Q_n = 8$ states the last four, which are underlined above, are now Pauli-allowed. The norms for these states can be found in table 1 of ref. [11]). Norms for a few simple additional states with $Q_n \geqq 9$ can also be calculated by the SU(3) recoupling techniques of ref. [11]). [All Pauli-allowed states with $Q_n = 9$ are included in table 1 of ref. [11]).] It is thus easy to choose a simple set of 33 equations with known norms from which the 33 coefficients $c_{m,l,(\lambda'\mu')}$ can be calculated.

Once the $c_{m,l,(\lambda'\mu')}$ have been calculated, the norms for $\alpha-(A-4)$ heavy fragment relative motion of arbitrary excitation, Q_n, follow at once from the BS transform of the norm kernel, eq. (25), by choosing the Q_n, $(\bar{\lambda}\bar{\mu})$, $(\lambda_c\mu_c)$. It is convenient to absorb some trivial factors into the definition of the coefficients, and in place of $c_{m,l,(\lambda'\mu')}$ introduce

$$y_{m,l,(\lambda'\mu')} = \frac{c_{m,l,(\lambda'\mu')}}{[\dim(\lambda'\mu')]^{\frac{1}{2}}} \frac{1}{m!\dim(m0)} \left[\frac{4(A-4)}{A}\right]^m. \tag{26}$$

The norm is then given by

$$\begin{aligned}
&\frac{(-)^{\lambda_c+\mu_c+Q_n-\bar{\lambda}-\bar{\mu}}}{[N((\lambda_c\mu_c)(Q_n0)(\bar{\lambda}\bar{\mu}))]^2}\left[\frac{\dim(\bar{\lambda}\bar{\mu})}{\dim(\lambda_c\mu_c)\dim(Q_n0)}\right]^{\frac{1}{2}} \\
&\qquad = \sum_{ml(\lambda'\mu')} \mathscr{D}^{Q_n}_{m,l} \frac{y_{m,l,(\lambda'\mu')}}{\dim(m0)} \frac{Q_n!}{(Q_n-m)!}\left[\frac{A}{4(A-4)}\right]^m \\
&\qquad \times \Big\{ \sum_{(\lambda\lambda)=(00)}^{(mm)} \sum_{\rho=1}^{d} [\dim(\lambda\lambda)]^{\frac{1}{2}} \\
&\qquad \times U((\lambda_c\mu_c)(\mu'\lambda')(\lambda\lambda)(0m); (m0)\text{ - -}; (\mu_c\lambda_c)\text{ - }\rho) \\
&\qquad \times U((Q_n0)(0,Q_n-m)(\lambda\lambda)(0m); (m0)\text{ - -}; (0Q_n)\text{ - -}) \\
&\qquad \times U((Q_n0)(\lambda_c\mu_c)(Q_n0)(\mu_c\lambda_c); (\bar{\lambda}\bar{\mu})\text{ - -}; (\lambda\lambda)\rho\text{ -})\Big\},
\end{aligned} \tag{27}$$

where

$$\mathscr{D}^{Q_n}_{m,l} = \sum_{p=0}^{4} D_l(p)\left[\frac{4(A-4)-pA}{4(A-4)}\right]^{Q_n-m} \tag{28}$$

The $D_l(p)$ are given in table 1. The $y_{m,l,(\lambda'\mu')}$ for all nuclei with $12 \leqq A \leqq 24$ and heavy fragment $(\lambda_c\mu_c)$ corresponding to highest possible SU(3) symmetry are very simple numbers and are collected in tables 2a, b and c. The expression in curly brackets in eq. (27) involving the sums over $(\lambda\lambda)$, ρ is akin to a 9-$(\lambda\mu)$ coefficient. The U-coefficients needed are readily available through the computer code of Akiyama and Draayer [20]). For $A \leqq 20$ the number of terms in the m, $(\lambda'\mu')$ sum is $\leqq 9$. Even for $A = 24$ the number of m, l, $(\lambda'\mu')$ values is only 35. For even heavier sd shell nuclei the technique can be made practical with very simple computer codes. The case $A = 20$ is particularly simple. In this case $(\lambda_c\mu_c) = (00)$, hence $(\lambda\lambda) = (00)$ only, and the expression in the curly brackets is replaced by the trivial factor 1; with $(\lambda'\mu') = (0m)$ the norm is given by a sum over five very simple terms. Numerical values for $A = 20$ norms have previously been given by Bandō [15]). For $A = 20$ a general algebraic expression for the norm, in very different form, has previously been given by Tomoda and Arima [13]. Insofar as the present method uses the generating function properties of $A(\boldsymbol{K}, \boldsymbol{R})$ the method is akin in philosophy to that employed by Suzuki [21] and Horiuchi [12]. By expressing the B-S transform of the norm kernel in full SU(3) irreducible tensor form, however, all sums over angular momentum or other subgroup labels are avoided altogether. The present method has the advantage that it automatically propagates information from the space of lowest Pauli-allowed excitations to arbitrarily high excitations of the relative motion functions of the cluster basis.

TABLE 2a

The coefficients $y_{m,(\lambda'\mu')}$ for the $[(A-4)+\alpha]$ cluster function norms

m	$A = 12$		$A = 13$		$A = 14$		$A = 15$	
	$(\lambda'\mu')$	$y_{m,(\lambda'\mu')}$	$(\lambda'\mu')$	$y_{m,(\lambda'\mu')}$	$(\lambda'\mu')$	$y_{m,(\lambda'\mu')}$	$(\lambda'\mu')$	$y_{m,(\lambda'\mu')}$
0	(40)	1	(31)	1	(22)	1	(13)	1
1	(30)	1	(40)	$\frac{5}{16}$	(31)	$\frac{2}{3}$	(22)	$\frac{9}{8}$
			(21)	$\frac{15}{16}$	(12)	$\frac{5}{6}$	(03)	$\frac{5}{8}$
2	(20)	1	(11)	$\frac{5}{6}$	(02)	$\frac{5}{9}$	(31)	1
			(30)	$\frac{5}{12}$	(40)	$\frac{5}{18}$	(12)	$\frac{5}{4}$
					(21)	$\frac{5}{6}$		
3	(10)	1	(01)	$\frac{5}{8}$	(11)	$\frac{10}{9}$	(40)	$\frac{5}{8}$
			(20)	$\frac{5}{8}$	(30)	$\frac{5}{9}$	(21)	$\frac{15}{8}$
4	(00)	1	(10)	$\frac{5}{4}$	(20)	$\frac{5}{3}$	(30)	$\frac{5}{2}$

TABLE 2b

The coefficients $y_{m,(\lambda'\mu')}$ for the $[(A-4)+\alpha]$ cluster function norms

m	$A = 16$		$A = 17$		$A = 18$		$A = 19$		$A = 20$	
	$(\lambda'\mu')$	$y_{m,(\lambda'\mu')}$	$(\lambda'\mu')$	$y_{m,(\lambda'\mu')}$	$(\lambda'\mu')$	$y_{m,(\lambda'\mu')}$	$(\lambda'\mu')$	$y_{m,(\lambda'\mu')}$	$(\lambda'\mu')$	$y_{m,(\lambda'\mu')}$
0	(04)	1	(03)	1	(02)	1	(01)	1	(00)	1
1	(13)	2	(12)	$\frac{15}{8}$	(11)	$\frac{5}{3}$	(10)	$\frac{5}{4}$	(01)	3
			(04)	$\frac{3}{8}$	(03)	$\frac{5}{6}$	(02)	$\frac{3}{2}$		
2	(22)	3	(21)	$\frac{5}{2}$	(20)	$\frac{5}{3}$	(11)	$\frac{10}{3}$	(02)	6
			(13)	1	(12)	$\frac{25}{12}$	(03)	$\frac{5}{3}$		
					(04)	$\frac{5}{12}$				
3	(31)	4	(30)	$\frac{5}{2}$	(21)	$\frac{25}{6}$	(12)	$\frac{25}{4}$	(03)	10
			(22)	$\frac{9}{4}$	(13)	$\frac{5}{3}$	(04)	$\frac{5}{4}$		
4	(40)	5	(31)	6	(22)	$\frac{15}{2}$	(13)	10	(04)	15

TABLE 2c

The coefficients $y_{m,l,(\lambda'\mu')}$ for the $[(A-4)+\alpha]$ cluster function norms

m, l	$A = 21$		$A = 22$		$A = 23$		$A = 24$	
	$(\lambda'\mu')$	$y_{m,l,(\lambda'\mu')}$	$(\lambda'\mu')$	$y_{m,l,(\lambda'\mu')}$	$(\lambda'\mu')$	$y_{m,l,(\lambda'\mu')}$	$(\lambda'\mu')$	$y_{m,l,(\lambda'\mu')}$
0,0	(20)	1	(40)	1	(60)	1	(80)	1
1, 1	(21)	$\frac{5}{2}$	(41)	$\frac{7}{3}$	(61)	$\frac{9}{4}$	(81)	$\frac{11}{5}$
	(10)	$\frac{1}{2}$	(30)	$\frac{2}{3}$	(50)	$\frac{3}{4}$	(70)	$\frac{4}{5}$
2, 2	(22)	$\frac{9}{2}$	(42)	4	(62)	$\frac{15}{4}$	(82)	$\frac{18}{5}$
	(11)	$\frac{4}{3}$	(31)	$\frac{8}{5}$	(51)	$\frac{12}{7}$	(71)	$\frac{16}{9}$
	(00)	$\frac{1}{6}$	(20)	$\frac{2}{5}$	(40)	$\frac{15}{28}$	(60)	$\frac{28}{45}$
2, 1	(00)	$\frac{1}{8}$	(20)	$\frac{1}{4}$	(40)	$\frac{3}{8}$	(60)	$\frac{1}{2}$
3, 3	(23)	7	(43)	6	(63)	$\frac{11}{2}$	(83)	$\frac{26}{5}$
	(12)	$\frac{5}{2}$	(32)	$\frac{14}{5}$	(52)	$\frac{81}{28}$	(72)	$\frac{44}{15}$
	(01)	$\frac{1}{2}$	(21)	1	(41)	$\frac{5}{4}$	(61)	$\frac{7}{5}$
			(10)	$\frac{1}{5}$	(30)	$\frac{5}{14}$	(50)	$\frac{7}{15}$
3, 2	(01)	$\frac{5}{12}$	(21)	$\frac{7}{12}$	(41)	$\frac{3}{4}$	(61)	$\frac{11}{12}$
			(10)	$\frac{1}{4}$	(30)	$\frac{1}{2}$	(50)	$\frac{3}{4}$
4, 4	(24)	10	(44)	$\frac{25}{3}$	(64)	$\frac{15}{2}$	(84)	7
	(13)	4	(33)	$\frac{64}{15}$	(53)	$\frac{30}{7}$	(73)	$\frac{64}{15}$
	(02)	1	(22)	$\frac{9}{5}$	(42)	$\frac{15}{7}$	(62)	$\frac{7}{3}$
			(11)	$\frac{8}{15}$	(31)	$\frac{6}{7}$	(51)	$\frac{16}{15}$
			(00)	$\frac{20}{3}$	(20)	$\frac{3}{14}$	(40)	$\frac{1}{3}$
4, 3	(02)	$\frac{15}{16}$	(22)	$\frac{21}{20}$	(42)	$\frac{135}{112}$	(62)	$\frac{11}{8}$
			(11)	$\frac{7}{10}$	(31)	$\frac{81}{70}$	(51)	$\frac{11}{7}$
			(00)	$\frac{1}{8}$	(20)	$\frac{9}{20}$	(40)	$\frac{45}{56}$
4, 2			(00)	$\frac{1}{24}$	(20)	$\frac{1}{8}$	(40)	$\frac{1}{4}$
5, 4	(03)	$\frac{7}{4}$	(23)	$\frac{42}{25}$	(43)	$\frac{99}{56}$	(63)	$\frac{143}{75}$
			(12)	$\frac{7}{5}$	(32)	$\frac{81}{40}$	(52)	$\frac{891}{350}$
			(01)	$\frac{21}{50}$	(21)	$\frac{81}{70}$	(41)	$\frac{11}{6}$
					(10)	$\frac{3}{10}$	(30)	$\frac{5}{7}$
5, 3			(01)	$\frac{7}{40}$	(21)	$\frac{27}{80}$	(41)	$\frac{11}{20}$
					(10)	$\frac{3}{16}$	(30)	$\frac{1}{2}$
6, 4			(02)	$\frac{7}{15}$	(22)	$\frac{27}{40}$	(42)	$\frac{33}{35}$
					(11)	$\frac{3}{5}$	(31)	$\frac{44}{35}$
					(00)	$\frac{1}{8}$	(20)	$\frac{3}{5}$
6, 3					(00)	$\frac{1}{32}$	(20)	$\frac{1}{8}$
7, 4					(01)	$\frac{9}{56}$	(21)	$\frac{11}{28}$
							(10)	$\frac{1}{4}$
8, 4							(00)	$\frac{1}{16}$

4. Concluding remarks

By combining SU(3) recoupling techniques with the use of BS integral transforms, a very general yet simple expression has been gained for the norm of a cluster wave function for a two component cluster system consisting of an α-cluster and a heavy fragment. This simple example has been chosen for purposes of illustration. The combined use of BS transforms and SU(3) recoupling techniques can also simplify the evaluation of matrix elements of more challenging operators or more complicated cluster systems. The BS transforms of more complicated kernels will again be constructed from linear combinations of exponentials and polynomials in the BS space variables $\bar{K}_1, \ldots, \bar{K}_n, K_1^*, \ldots, K_n^*$. Besides the SU(3) scalar variables $\bar{K}_n \cdot K_n^*$, the exponentials will also contain variables such as $\bar{K}_n \cdot \bar{K}_n$ and $K_n^* \cdot K_n^*$ which are SU(3) irreducible tensors of ranks (20) and (02). For an α+heavy fragment two-component cluster system with different harmonic oscillator constants for the α-particle and heavy fragment, e.g., the construction of the BS transform of the norm kernel can be carried out by steps which parallel the construction given here. The main difference comes from the fact that the exponentials $\exp\{\sigma_{nn}(\bar{K}_n \cdot K_n^*)\}$ will have to be replaced by products of the form

$$\exp\{\rho_{nn}(\bar{K}_n \cdot \bar{K}_n)\} \exp\{\sigma_{nn}(\bar{K}_n \cdot K_n^*)\} \exp\{\tau_{nn}(K_n^* \cdot K_n^*)\}.$$

An expansion of such products, combined with SU(3) recoupling techniques of the type illustrated here, will now lead to BS transforms of the norm in which the BS transforms of the α+heavy fragment wave functions of the bra and ket sides of a matrix element, *viz.*

$$[P(\bar{K}_1, \ldots, \bar{K}_{n-1})^{(\lambda_c\mu_c)} \times P(\bar{K}_n)^{(\bar{Q}_n 0)}]_{\bar{\alpha}}^{(\bar{\lambda}\bar{\mu})} \quad \text{and} \quad [P(K_1^*, \ldots, K_{n-1}^*)^{(\mu_c\lambda_c)} \times P(K_n^*)^{(0Q_n)}]_{\alpha}^{(\mu\lambda)}$$

can now be coupled to SU(3) irreducible tensors with resultant $(\lambda_0\mu_0) \neq (00)$; cf. eq. (14). The SU(3) recoupling techniques can furnish an even more powerful tool in the construction of the BS transforms of such kernels. Some details of this technique have been given elsewhere [22]) for the simple $^{12}C = \alpha+\alpha+\alpha$ cluster system for which the kernel of the interaction is considered. The combination of SU(3) recoupling and BS transform techniques can also lead to a ready evaluation of matrix elements in an angular momentum coupled basis for cluster systems involving more than two fragments. By merging the power of the integral transform techniques with readily available SU(3) technology, the detailed microscopic treatment of relatively complicated cluster structures, with several fragments including p-shell nuclei, e.g., may now be within striking distance.

It is a pleasure to acknowledge valuable discussions with T. H. Seligman.

References

1) K. Wildermuth and W. McClure, Cluster representations of nuclei, Springer Tracts in Modern Physics, vol. 41 (Springer, Berlin, 1966);
K. Wildermuth and Y. C. Tang, A unified theory of the nuclei (Vieweg, Braunschweig, 1975)
2) H. H. Hackenbroich, Symp. on present status and novel developments in the many-body problem, Rome (Editrice Compositori, Bologna, 1973)
3) B. Wiedmann, Diplomarbeit, Koeln, 1972;
H. H. Hackenbroich, private communication
4) P. Kramer, M. Moshinsky and T. H. Seligman, in Group theory and its applications, vol. 3, ed. E. M. Loebl (Academic Press, New York, 1975)
5) W. Zahn, Burg monographs in science, vol. 2 (Burg Verlag, Basel, 1975);
H. H. Hackenbroich, T. H. Seligman and W. Zahn, Helv. Phys. Acta **50** (1977) 723
6) T. H. Seligman and W. Zahn, J. of Phys. **G2** (1976) 79
7) W. Suenkel and K. Wildermuth, Phys. Lett. **41B** (1972) 439
8) H. Friedrich, Nucl. Phys. **A224** (1974) 537
9) P. Kramer and D. Schenzle, Rev. Mex. Fiz. **22** (1973) 25
10) T. H. Seligman and W. Zahn, Helv. Phys. Acta **49** (1976) 217
11) K. T. Hecht, Nucl. Phys. **A283** (1977) 223
12) H. Horiuchi, Prog Theor. Phys. **51** (1974) 745
13) T. Tomoda and A. Arima, Nucl. Phys. **A303** (1978) 217
14) M. Ichimura, A. Arima, E. C. Halbert and T. Terasawa, Nucl. Phys. **A204** (1973) 225
15) H. Bandō, Nucl. Phys. **A261** (1976) 269
16) P. Kramer and T. H. Seligman, Nucl. Phys. **A136** (1969) 545; **A186** (1972) 49
17) K. T. Hecht and D. Braunschweig, Nucl. Phys. **A244** (1975) 365
18) D. Braunschweig, Comp. Phys. Commun. **14** (1978) 109
19) J. P. Draayer and Y. Akiyama, J. Math. Phys. **14** (1973) 1904
20) Y. Akiyama and J. P. Draayer, Comp. Phys. Commun. **5** (1973) 405
21) Y. Suzuki, Prog. Theor. Phys. **50** (1973) 341
22) K. T. Hecht and W. Zahn, in Proc. 3rd Int. Conf. on clustering aspects of nucleus structure and nuclear reactions, Winnipeg, Manitoba, 1978; in Proc. Seventh Int. Coll. on group-theoretical methods in physics, Austin, Texas, 1978