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THE DECOMPOSITIONS OF LINE GRAPHS, MIDDLE GRAPHS AND TOTAL GRAPHS OF COMPLETE GRAPHS INTO FORESTS

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We construct decompositions of $L(K_n)$, $M(K_n)$ and $T(K_n)$ into the minimum number of line-disjoint spanning forests by applying the usual criterion for a graph to be eulerian. This gives a realization of the arboricity of each of these three graphs.

1. Preliminaries

In this paper a graph is considered as finite, undirected, with single lines and no loops.

The arboricity of a graph G, denoted by $\Upsilon(G)$, is the least number of line-disjoint spanning forests into which G can be partitioned. $\{x\}$ is the least integer not less than x, and $\Delta(G)$ is the maximum degree among the points of G. $V(G)$, $E(G)$, and $|S|$ denote the point set of G, the line set of G, and the number of elements of a set S, respectively. $L(G)$, $M(G)$ and $T(G)$ denote the line graph of G, the middle graph of G (see [3]), and the total graph of G respectively. Other definitions not presented here may be found in [4]. Later on, the following two Theorems will be applied.

Theorem A (C.St.J.A. Nash-Williams, [5, 6]). *Let G be a nontrivial (p, q)-graph and let q_k be the maximum number of lines in any subgraph of G having k points, then*

$$\Upsilon(G) = \max_{1<k\leq p} \{q_k/(k-1)\}.$$

Theorem B (L.W. Beineke, [2]). *For the complete graph K_n,*

$$\Upsilon(K_n) = \{n/2\}.$$

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As a special case of Theorem A, we derive the following result which gives the arboricity for a regular graph.

Theorem 1.1. *If G is n-regular* ($n \geq 1$), *then we have*

$$\Upsilon(G) = \{(n+1)/2\}.$$

Proof. If H is a (p', q')-subgraph of G, then

$$q'/(p'-1) \leq \Delta(H)p'/2(p'-1) = \tfrac{1}{2}(\Delta(H) + \Delta(H)/(p'-1)) \leq \tfrac{1}{2}(n+1)$$

since $\Delta(H) \leq \min(n, p'-1)$. But $q/(p-1) = np/2(p-1) > \frac{1}{2}n$ so, by Theorem A, $\Upsilon(G) = \{(n+1)/2\}$.

Since $L(K_n)$, $T(K_n)$, $L(K_{m,n})$ are $2(n-2)$-regular, $2(n-1)$-regular, $(m+n-2)$-regular, respectively, we obtain the following Corollary immediately from Theorem 1.

Corollary 1.1.

$$\Upsilon(L(K_n)) = n-1 \quad (n \geq 2), \qquad \Upsilon(T(K_n)) = n$$

and

$$\Upsilon(L(K_{m,n})) = \{(m+n-1)/2\}.$$

2. A decomposition of $L(K_n)$ into line-disjoint spanning forests

The arboricity of a graph G is nothing but the minimum number of colors with which the lines of G can be colored, such that the coloring satisfies the following condition (*).

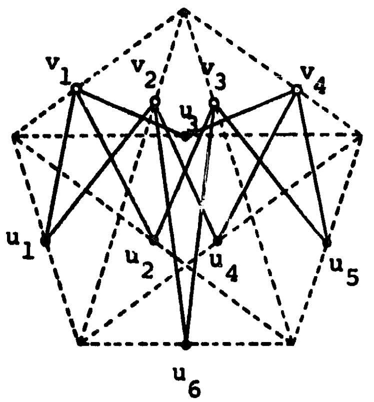
(*) Each subgraph of G induced by a set of monochromatic lines is a forest.

In what follows, we refer to such a line-coloring as a *forest-coloring of the graph G*.

By Corollary 1.1, $L(K_n)$ has an $(n-1)$-forest-coloring; we will now *construct* such a coloring. This is accomplished by induction on n. Now $L(K_3) \simeq K_3$ obviously has a 2-forest-coloring so let $n > 3$ and assume that $L(K_{n-1})$ has an $(n-2)$-forest-coloring.

The construction depends on the parity of n, but since the two cases are similar we only give the details for n even, say $n = 2m$ ($m \geq 2$).

First we note that $L(K_n)$ consists of two point-disjoint subgraphs $A \simeq K_{n-1}$ and $B \simeq L(K_{n-1})$, and a bipartite subgraph H, whose bipartition is $V(A)$ and $V(B)$. The subgraph A is the one induced in $L(K_n)$ by the lines incident at one of the points of K_n and B is the subgraph induced in $L(K_n)$ on the remaining lines of K_n.



$V(A)=\{v_1, v_2, v_3, v_4\}$,
$V(B)=\{u_1, u_2, u_3, u_4, u_5, u_6\}$.
(The solid lines induce a bipartite graph H.)

Fig. 1. $L(K_5)$.

This is illustrated in Fig. 1.

By the induction assumption, $B \simeq L(K_{n-1})$ can be forest-colored with $2m-2$ distinct colors $c_1, c_2, \ldots, c_{2m-2}$. And by Theorem B, $A \simeq K_{n-1}$ has a forest-coloring in the $\{(n-1)/2\} = \{(2m-1)/2\} = m$ colors $c_1, c_2, \ldots, c_m$. In fact, we can forest-color A with the colors $c_1, c_2, \ldots c_m$ so that condition $(**)$ holds.

$(**)$ The lines assigned color c_m are not incident with $m-1$ specified points of A.

For, if we let $V(A) = \{v_1, v_2, \ldots, v_{2m-1}\}$, then by Theorem B, $A - \{v_{2m-1}\}$ can be decomposed into $m-1$ spanning paths $f'_1, f'_2, \ldots, f'_{m-1}$ such that v_i is an endpoint of f'_i, $i = 1, 2, \ldots, m-1$. Let

$$f_i = f'_i \cup \{v_i v_{2m-1}\}, \quad i = 1, 2, \ldots, m-1,$$

and

$$f_m = \Big(\bigcup_{v_i \in V'_m} \{v_i v_{2m-1}\}\Big) \cup \{v_1, v_2, \ldots, v_{m-1}\},$$

where $V'_m = \{v_m, \ldots, v_{2m-2}\}$; i.e., K_{2m-1} is decomposed into $m-1$ spanning paths and one nonpath subgraph consisting of half the lines incident at one point and having $m-1$ isolated points.

To complete the forest coloring of $L(K_n)$ it remains to properly color the bigraph H. But the degree of each point of A in H is $2m-2$ and the degree of each point of B in H is 2, so *H is Eulerian.*

We choose any one of the Eulerian trials in H, and alternately color its lines with a new color c_{2m-1} (Fig. 2). Consequently, exactly half the lines of H are colored with the color c_{2m-1}, i.e., $\frac{1}{2}(2m-2)(2m-1) = (m-1)(2m-1)$ lines of H are assigned the color c_{2m-1}. There now remain $(m-2)(2m-1)$ lines of H to be colored.

By $(**)$, none of the lines incident with v_i, $(i = 1, \ldots, m-1)$ are colored with the color c_m in A. Hence, we can color one of the uncolored lines of H incident

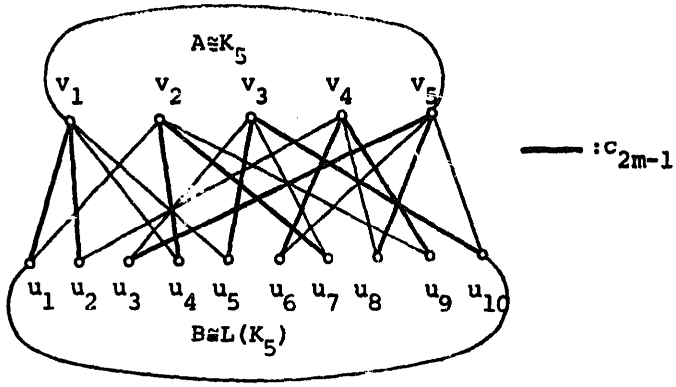


Fig. 2. $L(K_6)$.

with v_i $(i = 1, \ldots, m-1)$ with the color c_m. Now we color an arbitrary uncolored line of H incident with each of the remaining $(2m-1)-(m-1)=m$ points of A with the colors $c_1, \ldots\ldots, c_m$ successively (Fig. 3).

At this stage, the number of uncolored lines incident with each point of A is $m-2$. Thus we use the $m-2$ colors $c_{m+1}, \ldots, c_{2m-2}$ (which were used only to color the forests of B) to color the remaining $m-2$ lines incident with v_i $(i = 1, \ldots, 2m-1)$.

Thus, if n is even, all the lines of $L(K_n)$ are colored in such a way that each subgraph of $L(K_n)$ induced be a set of monochromatic lines is a forest.

Remark. In the case that $n(=2m-1)$ is odd, the bipartite subgraph H of $L(K_n)$ is not Eulerian. We only briefly indicate how the coloring of H differs from that used when n is even.

Let the points of $V(A)$ be labelled $v_1, \ldots, v_{2m-1}$, and define the point set $V(C)$ as follows.

$$V(C) = \{u_i \mid u_i \in V(B),\ u_i \text{ is adjacent to the points } v_i \text{ and } v_{2m-i-1} \text{ of } V(A), \text{ where } i = 1, \ldots, m-1\}.$$

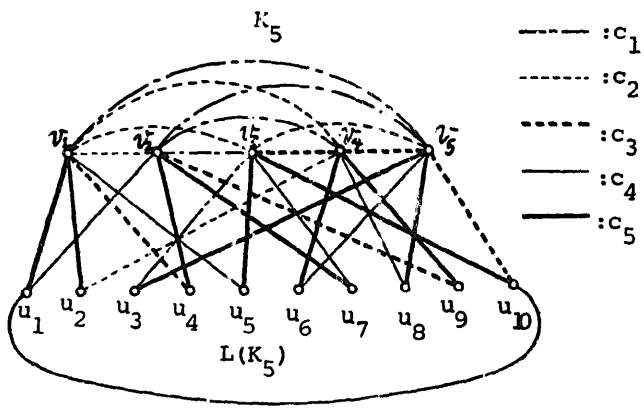


Fig. 3. $L(K_6)$.

First color all the lines which are incident with points of $V(C)$ with a new color, say c_{2m-2}. Then the remaining subgraph $H' = H - V(C)$ of H is *Eulerian*; we can apply the coloring used in the case when n is even.

3. Decompositions of $M(K_n)$ and $T(K_n)$ into line-disjoint spanning forests

It is easy to prove the following lemma.

Lemma 3.1. *Let H be a graph with arboricity n, and G be a graph obtained by adding a new point v and, combining arbitrary $p(p \leq n)$ points of H with v. Then G also has arboricity n.*

Theorem 3.1. *The middle graph $M(K_n)$ of a complete graph K_n has an $(n-1)$-forest-coloring.*

Proof. First we show that $\Upsilon(M(K_n)) = n-1$ and then how to construct such a coloring.

Since $L(K_n)$ is a subgraph of $M(K_n)$, we have $\Upsilon(M(K_n)) \geq n-1$.

On the other hand, since $M(K_n) \cong L(K_n \circ K_1)$ (see Theorem 1 in [3]), $M(K_n)$ is constructed by adding n new points v_i $(i = 1, \ldots, n)$ to $L(K_n)$, and joining each v_i with a certain $n-1$ points of $L(K_n)$.

Thus we see that $M(K_n)$ also has arboricity $n-1$ by Lemma 3.1.

We will now construct such a coloring, first color the forests of $L(K_n)$ with $(n-1)$ colors, say $c_1, \ldots, c_{n-1}$, as shown in the previous section. Then use all $n-1$ colors to color the lines incident with v_i $(i = 1, \ldots, n)$. Clearly this is a forest-coloring of $M(K_n)$.

Corollary 3.1. *The total graph $T(K_n)$ of a complete graph K_n has an n-forest-coloring.*

Proof. The forest-coloring of $T(K_n)$ is evident from the relation $T(K_n) \cong L(K_{n+1})$ (See [1]), and Theorem 1.1.

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