In this paper we investigate some special nuclear Fréchet spaces in regard to their infinite dimensional closed subspaces and, in theme, continue the earlier work of [7]; in its general technique this is similar to Zahariuta's [10] proof of the result that no power series space of infinite type can contain a subspace isomorphic to a power series space of finite type; Zahariuta proved this result by showing that all linear continuous maps from a power series space of finite type into one of infinite type are compact.

Following Zahariuta we shall say that for locally convex spaces (l.c.s.) $X$ and $Y$, $(X, Y) \in R$ if all continuous linear maps of $X$ into $Y$ are also compact. Crone and Robinson [3], De Grande-De Kimpe [4], De Grande-De Kimpe and Robinson [5] and Zahariuta [10] have obtained necessary and/or sufficient conditions for $(X, Y) \in R$ to hold for various pairs $X$, $Y$ chosen from among power series spaces (finite or infinite type) and the spaces $L_\tau(b, \tau)$. In this paper we build into this pairing the smooth sequence spaces of infinite type; we first prove a sufficient condition for the relation $(L_\tau(b, \infty), \lambda(A)) \in R$ to hold when $\lambda(A)$ is a Schwartz space and obtain a partial converse of this. This result is an analogue of a result of Zahariuta's [10]. Then we consider the problem of $(X, Y) \in R$ where $Y$ is either an infinite type power series space [or an $L_\tau(b, \infty)$ space] and show that if $Y$ is stable and $X$ is a nuclear Fréchet space with a $(d_8)$ [or a $(d_\tau)$]-basis then $(X, Y) \not\in R$. This result is actually a consequence.
of the following more general result proved in Theorem 3 of this paper: if \( X \) is a nuclear Fréchet space with a \((d_3)\) [or \((d_5)\)] basis and if \( A_{\infty}(\alpha) \) [or \( L_f(b, \infty) \)] is nuclear and stable, then \( X \) has a complemented subspace which is isomorphic to a subspace of \( A_{\infty}(\alpha) \) [or \( L_f(b, \infty) \)]. This last mentioned result is derived by using a result of the first author [1].

DEFINITIONS, NOTATIONS, PRELIMINARY RESULTS

1) A (monotone) Köthe set \( A \) is a collection
\[
A = \{a^k : k = 1, 2, \ldots\}
\]
of sequences of positive numbers such that
(i) \( a^k_n < a^{k+1}_n \), \( k, n \in \mathbb{N} \),
(ii) \( \forall k \forall l \exists j \exists a^k = O(a^l) \) and \( a^l = O(a^k) \).

The Köthe space \( \lambda(A) \) is defined to be the sequence space of scalar sequences
\[
\lambda(A) = \{t = (t_n) : \|t\|_k = \sum_{n=1}^{\infty} |t_n|a^k_n < \infty \text{ for all } k \in \mathbb{N} \}
\]
and is topologized by the seminorms \( \|\cdot\|_k \), \( k = 1, 2, \ldots \).

2) Grothendieck-Pietsch criterion. A Köthe space \( \lambda(A) \) is nuclear if and only if
\[
\forall k \exists \exists (a^k_n/a^n) \in l_1.
\]

3) A Köthe set \( A = \{a^k\} \) is said to be normalized if \( a^k_1 = 1 \) for all \( n \).

It can be shown that every Köthe space is isomorphic to a Köthe space \( \lambda(B) \) where \( B \) is normalized.

4) A Köthe set \( A = \{a^k\} \) is said to satisfy the normalized \((d_3)\) condition if
\[
\forall k \exists \exists (a^k_1/a^n) \in l_1.
\]

5) A Köthe space \( \lambda(A) \) is called a \( G_\infty \)-space if
(i) \( 1 < a^k_n < a^{k+1}_n \), \( k, n \in \mathbb{N} \),
(ii) the Köthe set \( A \) satisfies the normalized \((d_3)\) condition.

6) From the Grothendieck-Pietsch criterion it follows that a \( G_\infty \)-space \( \lambda(A) \) is nuclear if and only if for some \( k \in \mathbb{N} \), \( (1/a^k) \in l_1 \).

A \( G_\infty \)-space \( \lambda(A) \) is a Schwartz space if and only if for some \( k \in \mathbb{N} \), \( (1/a^k) \in c_0 \) (see [8]).

7) Let \( \alpha = (\alpha_n) \) be a non-decreasing sequence of positive numbers. The infinite type power series space (p.s.s.) \( A_{\infty}(\alpha) \), generated by \( \alpha \), is the \( G_\infty \)-space \( \lambda(A) \) where \( a^k_n = k^\alpha_n \) (or, equivalently, \( a^k_n = \exp(k \alpha_n) \)).

8) Let \( f \) be an odd, increasing, logarithmically convex function (i.e., the function \( \phi \) defined by \( \phi(x) = \ln f(\exp x) \) is convex); \( b = (b_n) \), \( b_n \uparrow \infty \). Then the space \( L_f(b, \infty) \) is defined to be the \( G_\infty \)-space \( \lambda(A) \) where \( a^k_n = \exp f(k b_n) \).

An odd, increasing, logarithmically convex function is called a Dragilev
function. For a Dragilev function $f$, the limit

$$
\tau(a) = \lim_{x \to \infty} \frac{f(ax)}{f(x)}
$$

exists for all $a > 1$. Moreover, there are two possibilities: either

(i) $\forall a > 1$, $\tau(a) = +\infty$ or

(ii) $\forall a > 1$, $1 < \tau(a) < +\infty$ and $\tau(a) \uparrow +\infty$ as $a \uparrow +\infty$.

In the first case $f$ is said to be rapidly increasing, and in the second case $f$ is said to be slowly increasing. It is known that a space $L_f(b, \infty)$ is isomorphic to a p.s.s. $A_\infty(a)$ if and only if $f$ is slowly increasing.

9) A l.c.s. $X$ is called stable if $X \times X \cong X$. For $G_\infty$ and power series spaces we have the following criteria:

(i) a $G_\infty$-space $\lambda(A)$ is stable if and only if

$$
\forall k \exists s \sup_n (a_{kn}^n/d_n^k) < \infty,
$$

(ii) a p.s.s. $A_\infty(\alpha)$ is stable if and only if

$$
\sup_n (a_{2\alpha}/\alpha_n) < \infty.
$$

10) Let $X$ be a l.c.s. For two absolutely convex zero neighborhoods $V$ and $U$ with $V \subset U$ (i.e., $V \subset rU$ for some $r > 0$), the $n$-th Kolmogorov diameter of $V$ with respect to $U$ is defined as

$$
d_n(V, U) = \inf \{r > 0 : V \subset rU + L\} : L \text{ is a linear subspace of } X \text{ with } \dim L < n\}
$$

The diametral dimension $\Delta(X)$ of $X$ is then defined to be the set of all scalar sequences $(t_n)$ such that

$$
\forall U \forall V \exists V < U \text{ and } \lim_{n \to \infty} t_n d_n(V, U) = 0.
$$

RESULTS

We shall start with the discussion of sufficient conditions for

$$(L_f(b, \infty), \lambda(A)) \in R.$$ 

Throughout this section we shall assume that $f$ is a Dragilev function which is either rapidly increasing or slowly increasing, in which case we take it, without loss of generality, to be the identity function.

**Theorem 1.** Suppose $A$ is a normalized Köthe set and $\lambda(A)$ is a Schwartz space. Assume that for each $k$ there exists an $l$ such that

$$
\lim_{s \to \infty} \frac{f^{-1} \log a_k^x}{f^{-1} \log a_k^y} = 0.
$$

Then for each $(\beta_n) \uparrow \infty$, $(L_f(\beta, \infty), \lambda(A)) \in R$. 

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One can supply a proof of this modelled on the proof of a theorem of Zahariuta's ([10]: Theorem 4) which this generalizes.

REMARK: Later in this paper we shall actually show that the sufficient condition appearing in the theorem also implies that the normalized Köthe set $A$ satisfies the normalized $(d_3)$ condition.

COROLLARY. Under the conditions on $A$ and $f$ in Theorem 1, $\lambda(A)$ does not contain a subspace isomorphic to $L_f(\beta, \infty)$ for any $\beta = (\beta_n) \uparrow \infty$.

Examples where the above results apply are either known or easily constructed.

(a) Taking $f = $ identity in the corollary, we obtain Corollary 10 in [7].

(b) Take rapidly increasing Dragilev functions $f$ and $g$ and take $\lambda(A) = L_g(\gamma, \infty)$; then we obtain a result of Zahariuta [10].

(c) Start with a nuclear $G_\infty$-system $A = \{a^k\}$ such that for each $k$ there exists an $l$ such that
\[
\frac{\log a_{nk}^k}{\log a_{nk}^l} \rightarrow 0,
\]
and let $f$ be a Dragilev function. Define $(b^k)$ by setting $b_{nk}^k = \exp f(\log a_{nk}^k)$. Then $\lambda(B)$ has no subspace isomorphic to $L_f(\beta, \infty)$.

The result in Theorem 1 can be used to construct for each given Dragilev function $f$ a suitable $G_\infty$-space $\lambda(B)$ such that $\lambda(B)$ is isomorphic to $L_f(\beta_0, \infty) \times \lambda(B_0)$, for a suitable $\beta_0 \uparrow \infty$ and such that $\lambda(B_0)$ is a $G_\infty$-space which has no subspace isomorphic to $L_f(\gamma, \infty)$ for any $\gamma$. One such construction is given in [7].

Now we shall obtain a partial converse of Theorem 1.

THEOREM 2. Suppose $f$ is a Dragilev function and $A$ is a normalized Köthe set and that $\lambda(A)$ is nuclear. Suppose

(a) for each $k$, $m \in \Omega$, $\mathcal{H} = l(k, m) \in \Omega$ such that
\[
\frac{f^{-1}\log a_{nk}^k}{f^{-1}\log a_{nk}^l} > m \text{ for all } n \text{ (or for } n > n_0(k, m)),
\]

(b) $\frac{f^{-1}\log a_{nk}^{k+1}}{f^{-1}\log a_{nk}^k} < \frac{f^{-1}\log a_{nk+1}^{k+1}}{f^{-1}\log a_{nk+1}^k} \quad \forall k, n,$

(c) $\lambda(A)$ is not isomorphic to $L_f(\beta, \infty)$ for any $\beta$.

Then for each $k$ there exists an $l$ such that
\[
\lim_{n \to \infty} \frac{f^{-1}\log a_{nk}^k}{f^{-1}\log a_{nk}^l} = \infty.
\]

PROOF. Write $b_{nk}^k = f^{-1}\log a_{nk}^k$, $k, n \in \Omega$. If possible the conclusion in the theorem be false; then there exists a $k$ (fixed) so that $\lim_n b_{nk}^k = \infty$ and
for each $i$, there is a subsequence $(m_n^i) = (m_n)$ of positive integers and $M_1 > 0$ such that $b_{m_n^i}/b_{m_n^i}^k < M_1$, $n=1, 2, \ldots$. But then, because of (b) above $b_n/b_n < M_1$, $n=1, 2, \ldots$, and hence $\log a_n^i < f(A_n^i)$, $n=1, 2, \ldots$, where $A_n^i = f^{-1}\log a_n^i = b_n^k$ ($k$ already fixed). Also by (a) above, for each $m$ there is an $n$ such that $f(m \xi_n) < \log a_n^m$, $n=1, 2, \ldots$. This shows that $\lambda(A)$ is set theoretically equal and topologically isomorphic to $\lambda(B)$ where $b_n^k = \exp f(k \lambda_n)$.

Let now $\pi$ be a permutation of $\Omega$ such that $(\beta_n) = (\alpha_n \xi_0)$ is increasing. Then by the nuclearity of $\lambda(A)$ and the fact that $\lambda(A) \cong \lambda(B)$ we have $\lambda(A) = \Lambda_f(\beta, \infty)$, and this gives a contradiction to (c).

**REMARK.** The strange looking condition (a) is only an analogue of the normalized $(d_3)$ condition in the presence of $f$; if $f$ = identity then (a) gives exactly the normalized $(d_3)$ condition. So we shall now prove

**PROPOSITION.** Suppose $f$ is a Dragilev function and $A = \{a_k\}$ satisfies condition (a) in Theorem 2. Then $A$ satisfies the normalized $(d_3)$ condition.

**PROOF.** Pick $c$ so that $\tau(c) > 2$. For this $c$ and a given $k$ pick, using (a), $a_i$ so that $f^{-1}\log a_i > c f^{-1}\log a_n^k$. Then

$$\exp \left[ f \left( \frac{1}{c} f^{-1}\log a_i^k \right) \right] > a_n^k,$$

since $f$ increases. But

$$\frac{f(f^{-1}\log a_i^k)}{f^{-1}\log a_n^k} \to \tau(c) \text{ as } n \to \infty;$$

so for $n > n_0$,

$$a_i^k = \exp [f(f^{-1}\log a_i^k)] > \exp \left[ 2 \frac{f}{c} f^{-1}\log a_i^k \right] > (a_n^k)^2.$$

**REMARKS**

1) The above proof could also be adopted to show that

$$\forall k \mathcal{H} \ni \lim_{n} \frac{f^{-1}\log a_n^i}{f^{-1}\log a_n^k} = \infty \Rightarrow \forall k \mathcal{H} \ni \lim_{n} \frac{\log a_n^i}{\log a_n^k} = \infty.$$
3) Take $a_n^k = \exp(k^\alpha n)$, where $\alpha = (x_n)$ is a nuclear exponent sequence (i.e., $A_\alpha(x)$ is nuclear) and $f(x) = e^x$. Then conditions (a) and (b) of Theorem 2 are satisfied, but (c) is not true since

$$\frac{f^{-1} \log a_n^k}{f^{-1} \log a_n^k} = \frac{\log l}{\log k}.$$ 

Thus in this case $\lambda(A)$ is isomorphic to $L_f(\beta, \infty)$ but has no subspace isomorphic to a p.s.s.

This remark also shows that condition (a) of Theorem 2 which is a weaker assumption than the condition in Theorem 1 is not sufficient to obtain the conclusion of Theorem 1.

Now we shall reverse the roles of the spaces considered in Theorem 1 and consider the problem "$(\lambda(A), L_f(\beta, \infty)) \subseteq F$". The following definitions and results are relevant to our discussion.

Let $X$ be a nuclear Fréchet space with a basis $(x_n)$ and a continuous norm. The basis $(x_n)$ is called a $(d_\alpha)$-basis (see Dubinsky [6]) if there exists on $X$ a fundamental sequence $(\|x_n\|_n)$ of norms so that

$$Mf^{-1} \log \frac{\|x_n\|_{k+1}}{\|x_n\|_k} < f^{-1} \log \frac{\|x_n\|_{k+2}}{\|x_n\|_{k+1}}, \quad k, n \in \Omega.$$ 

Given a Dragilev function $f$, the basis $(x_n)$ on $X$ is called a $(d_f)$-basis (see Alpseyen [1]) if there exist $M > 1$ and a fundamental sequence of norms $(\| \cdot \|_k)$ on $X$ so that

$$Mf^{-1} \log \frac{\|x_n\|_{k+1}}{\|x_n\|_k} < f^{-1} \log \frac{\|x_n\|_{k+2}}{\|x_n\|_{k+1}}, \quad k, n \in \Omega.$$ 

Putting $f = \text{identity}$, the $(d_f)$-condition reduces to an equivalent form of the $(d_\alpha)$-condition (see [6]).

The following result of the first author [1] is needed in our proofs.

**Lemma.** Let $f$ be either a rapidly increasing Dragilev function or the identity function. Assume the space $L_f(b, \infty)$ is nuclear and stable. $X$ be a Fréchet space with a basis $(x_n)$. Then $X$ is isomorphic to a subspace of $L_f(\beta, \infty)$ if and only if

(a) the basis $(x_n)$ is a $(d_f)$-basis and

(b) $A(L_f(b, \infty)) \subseteq A(X)$.

For the case $f = \text{identity}$ and $b_n = \log(n+1)$ (so that $L_f(b, \infty) = (\mathbb{F})$, the space of rapidly decreasing sequences) the above result is due to Dubinsky [6] and to Vogt [9].

We are now in a position to state an interesting consequence of the above result and derive useful corollaries.

**Theorem 3.** Let $f$ be a Dragilev function which is rapidly increasing or be the identity function and let $L_f(b, \infty)$ be stable and nuclear. Let
X be a nuclear Fréchet space with a \((d_f)\)-basis \((x_n)\). Then X has a complemented subspace isomorphic to a subspace of \(L_f(b, \infty)\).

**Proof.** Since \((x_n)\) is a \((d_f)\)-basis, there exist a fundamental sequence \(\left(\|\cdot\|_k\right)\) of norms on X and \(M \geq 1\) such that

\[
Mf^{-1} \log \frac{\|x_n\|_{k+1}}{\|x_n\|_k} < \int f^{-1} \log \frac{\|x_n\|_{k+2}}{\|x_n\|_{k+1}}, \quad k, n \in \Omega.
\]

It follows that \(\left(\|x_n\|_{k+1}\right)^{3} \ll \|x_n\|_{1} \|x_n\|_{2k+1}\). Define

\[
a^k_n = \frac{\|x_n\|_{k}}{\|x_n\|_{1}}.\]

Then \(A = \{a^k\}\) is a normalized Köthe set and satisfies the normalized \((d_3)\) condition since

\[(1) \quad (a^{k+1}_n)^{3} \ll a^{2k+1}_n.\]

Since X is nuclear we may assume that

\[
\sum_n \frac{\|x_n\|_{1}}{\|x_n\|_{2}} < \infty \quad \text{and} \quad \lim_n \frac{\|x_n\|_{k+1}}{\|x_n\|_{1}} = \lim_n a^{k+1}_n = \infty, \quad k \in \Omega.
\]

Next we choose strictly increasing sequences \((i(k, n))\) of indices so that

(i) \(i(k+1, n)) \subset (i(k, n))\),

(ii) \((a^{k+1}_n)\) is increasing and

(iii) \(kb_n \leq f^{-1} \log (a^{k+1}_n)\).

Now let \(j(n)\) denote the diagonal sequence \((i(k, n))\). Then for \(n \geq k\), \((j(n)) \subset (i(k, n))\), \(j(n) \geq i(k, n)\) and \(a^{k+1}_{j(k, n)} < a^{k+1}_n\); therefore, again for \(n \geq k\), we have

\[(2) \quad kb_n \leq f^{-1} \log a^{k+1}_{j(k, n)} \text{ or } \exp f(kb_n) \leq a^{k+1}_{j(k, n)}.
\]

Since \(a^{k+1}_{j(k, n)}\) is increasing with \(n\) and since \(\{j(n) : n \geq k\} \subset \{i(k, n) : n \geq k\}\), we have

\[(3) \quad a^{k+1}_{j(k, n)} \text{ is increasing with } n, \text{ for } n \geq k \text{ and } k \in \Omega.
\]

Now since \((1)\) is true also for \((a^{k+1}_{j(k, n)})\), by using, if necessary a small perturbation for the first \((k-1)\) elements of \((a^{k+1}_{j(k, n)})\) we may assume that \(\lambda(a^k_{j(k)})\) is a \(G_{\infty}\)-space; this space is isomorphic to the complemented subspace \(Y\) of \(X\) generated by \((x_{j(k)})\). Also by [8],

\[A(Y) = A(\lambda(a^k_{j(k)})) = \{t_n) : \forall n, \forall C > 0 \exists |t_n| < C a^k_{j(k)}\}.
\]

Then \((2)\) shows \(A(L_f(b, \infty)) \subset A(Y)\). Also the basis \((x_{j(k)})\) of \(Y\) is a \((d_f)\)-basis. Now it follows from the above lemma that \(Y\) is isomorphic to a subspace of \(L_f(b, \infty)\).

**Note.** In the above theorem, taking \(f = \text{identity}\) we obtain that each nuclear Fréchet space \(X\) with a \((d_3)\)-basis has a complemented subspace.
isomorphic to a subspace of (a pre-assigned) stable, nuclear p.s.s. \( A_\infty(\alpha) \). In this context the following result of Aytuna and Terzioglu [2] is of interest: a subspace \( X \) of \( A_1(\alpha) \) with basis is either isomorphic to a subspace of \( A_\infty(\alpha) \) or \( X \) has a complemented subspace which is isomorphic to a power series space of finite type.

**Corollary to Theorem 3.** If \( X \) is a nuclear Fréchet space with a \((d_f)\)-basis \((x_n)\) and \( L_f(b, \infty) \) is nuclear and stable, then \((X, L_f(b, \infty)) \notin R\).

**Proof.** By Theorem 3, we have a complemented subspace \( Y \) of \( X \) which is isomorphic to a subspace of \( L_f(b, \infty) \). Let \( T \) denote the isomorphism map and \( P \) the projection of \( X \) onto \( Y \). Then clearly \( TP \) is continuous but not compact.

**References**