Subspaces of some nuclear sequence spaces

by M. Alpseymen, M. S. Ramanujan and T. Terzioglu

M. Alpseymen and M. S. Ramanujan: Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109.
T. Terzioglu: Department of Mathematics, Middle East Tech. Univ., Ankara, Turkey

Communicated by Prof. H. Freudenthal at the meeting of September 30, 1978

In this paper we investigate some special nuclear Fréchet spaces in regard to their infinite dimensional closed subspaces and, in theme, continue the earlier work of [7]; in its general technique this is similar to Zahariuta's [10] proof of the result that no power series space of infinite type can contain a subspace isomorphic to a power series space of finite type; Zahariuta proved this result by showing that all linear continuous maps from a power series space of finite type into one of infinite type are compact.

Following Zahariuta we shall say that for locally convex spaces (l.c.s.) X and Y, $(X, Y) \in R$ if all continuous linear maps of X into Y are also compact. Crone and Robinson [3], De Grande-De Kimpe [4], De Grande-De Kimpe and Robinson [5] and Zahariuta [10] have obtained necessary and/or sufficient conditions for $(X, Y) \in R$ to hold for various pairs X, Y chosen from among power series spaces (finite or infinite type) and the spaces $L_f(b, r)$. In this paper we build into this pairing the smooth sequence spaces of infinite type; we first prove a sufficient condition for the relation $(L_f(b, \infty), \lambda(A)) \in R$ to hold when $\lambda(A)$ is a Schwartz space and obtain a partial converse of this. This result is an analogue of a result of Zahariuta's [10]. Then we consider the problem of $(X, Y) \in R$ where Y is either an infinite type power series space [or an $L_f(b, \infty)$ space] and show that if Y is stable and X is a nuclear Fréchet space with a (d_3) [or a (d_f)]-basis then $(X, Y) \notin R$. This result is actually a consequence of the following more general result proved in Theorem 3 of this paper: if X is a nuclear Fréchet space with a (d_3) [or (d_f)] basis and if $\Lambda_\infty(\alpha)$ [or $L_f(b, \infty)$] is nuclear and stable, then X has a complemented subspace which is isomorphic to a subspace of $\Lambda_\infty(\alpha)$ [or $L_f(b, \infty)$]. This last mentioned result is derived by using a result of the first author [1].

DEFINITIONS, NOTATIONS, PRELIMINARY RESULTS

1) A (monotone) *Köthe set* A is a collection

$$A = \{a^k : k = 1, 2, \ldots\}$$

of sequences of positive numbers such that

(i) $a_n^k \leqslant a_n^{k+1}, \ k, n \in \mathbb{N}$,

(ii) $\forall k \ \forall l \ \exists j \ni a^k = O(a^j)$ and $a^l = O(a^j)$.

The *Köthe space* $\lambda(A)$ is defined to be the sequence space of scalar sequences

$$\lambda(A) = \{t = (t_n) : \|t\|_k = \sum_{n=1}^{\infty} |t_n| a_n^k < \infty \text{ for all } k \in \mathbb{N}\}$$

and is topologized by the seminorms $\|\cdot\|_k$, $k = 1, 2, \ldots$.

2) *Grothendieck-Pietsch criterion.* A Köthe space $\lambda(A)$ is nuclear if and only if

$$\forall k \ \exists l \ni (a_n^k / a_n) \in l_1.$$

3) A Köthe set $A = \{a^k\}$ is said to be *normalized* if $a_n^1 = 1$ for all n. It can be shown that every Köthe space is isomorphic to a Köthe space $\lambda(B)$ where B is normalized.

4) A Köthe set $A = \{a^k\}$ is said to satisfy the *normalized* (d_3) *condition* if

$$\forall k \ \exists l \ni (a^k)^2 = O(a^l).$$

5) A Köthe space $\lambda(A)$ is called a G_∞*-space* if

(i) $1 \leqslant a_n^k \leqslant a_{n+1}^k, \quad k, n \in \mathbb{N}$,

(ii) the Köthe set A satisfies the normalized (d_3) condition.

6) From the Grothendieck-Pietsch criterion it follows that a G_∞-space $\lambda(A)$ is nuclear if and only if for some $k \in \mathbb{N}$, $(1/a_n^k) \in l_1$.

A G_∞-space $\lambda(A)$ is a Schwartz space if and only if for some $k \in \mathbb{N}$, $(1/a_n^k) \in c_0$ (see [8].

7) Let $\alpha = (\alpha_n)$ be a non-decreasing sequence of positive numbers. The *infinite type power series space* (p.s.s.) $\Lambda_\infty(\alpha)$, generated by α, is the G_∞-space $\lambda(A)$ where $a_n^k = k^{\alpha_n}$ (or, equivalently, $a_n^k = \exp(k \alpha_n)$).

8) Let f be an odd, increasing, logarithmically convex function (i.e., the function ϕ defined by $\phi(x) = \ln f(\exp x)$ is convex); $b = (b_n)$, $b_n \uparrow \infty$. Then the space $L_f(b, \infty)$ is defined to be the G_∞-space $\lambda(A)$ where $a_n^k = \exp f(k\, b_n)$.

An odd, increasing, logarithmically convex function is called a *Dragilev* *function*. For a Dragilev function f, the limit

$$\tau(a) = \lim_{x\to\infty} \frac{f(ax)}{f(x)}$$

exists for all $a > 1$. Moreover, there are two possibilities: either
(i) $\forall a > 1,\ \tau(a) = +\infty$ or
(ii) $\forall a > 1,\ 1 < \tau(a) < +\infty$ and $\tau(a) \uparrow +\infty$ as $a \uparrow +\infty$.

In the first case f is said to be *rapidly increasing*, and in the second case f is said to be *slowly increasing*. It is known that a space $L_f(b, \infty)$ is isomorphic to a p.s.s. $\Lambda_\infty(\alpha)$ if and only if f is slowly increasing.

9) A l.c.s. X is called *stable* if $X \times X \cong X$. For G_∞- and power series spaces we have the following criteria:
(i) a G_∞-space $\lambda(A)$ is stable if and only if

$$\forall k\ \exists l \ni \sup_n (a_{2n}^k / a_n^l) < \infty,$$

(ii) a p.s.s. $\Lambda_\infty(\alpha)$ is stable if and only if

$$\sup_n (\alpha_{2n}/\alpha_n) < \infty.$$

10) Let X be a l.c.s. For two absolutely convex zero neighborhoods V and U with $V \prec U$ (i.e., $V \subset rU$ for some $r > 0$), the n-th Kolmogorov diameter of V with respect to U is defined as

$$d_n(V, U) = \inf\{r > 0 : V \subset rU + L\} : L \text{ is a linear subspace of } X \text{ with } \dim L \leqslant n\}.$$

The *diametral dimension* $\Delta(X)$ of X is then defined to be the set of all scalar sequences (t_n) such that

$$\forall U\ \exists V \ni V \prec U \text{ and } \lim_{n\to\infty} t_n d_n(V, U) = 0.$$

RESULTS

We shall start with the discussion of sufficient conditions for

$$(L_f(b, \infty), \lambda(A)) \in R.$$

Throughout this section we shall assume that f is a Dragilev function which is either rapidly increasing or slowly increasing, in which case we take it, without loss of generality, to be the identity function.

THEOREM 1. Suppose A is a normalized Köthe set and $\lambda(A)$ is a Schwartz space. Assume that for each k there exists an l such that

$$\lim_{n\to\infty} \frac{f^{-1} \log a_n^k}{f^{-1} \log a_n^l} = 0.$$

Then for each $(\beta_n) \uparrow \infty$, $(L_f(\beta, \infty), \lambda(A)) \in R$.

One can supply a proof of this modelled on the proof of a theorem of Zahariuta's ([10]; Theorem 4) which this generalizes.

REMARK: Later in this paper we shall actually show that the sufficient condition appearing in the theorem also implies that the normalized Köthe set A satisfies the normalized (d_3) condition.

COROLLARY. Under the conditions on A and f in Theorem 1, $\lambda(A)$ does not contain a subspace isomorphic to $L_f(\beta, \infty)$ for any $\beta=(\beta_n)\uparrow\infty$.

Examples where the above results apply are either known or easily constructed.

(a) Taking $f=$identity in the corollary, we obtain Corollary 10 in [7].

(b) Take rapidly increasing Dragilev functions f and g and take $\lambda(A)=L_g(\gamma, \infty)$; then we obtain a result of Zahariuta [10].

(c) Start with a nuclear G_∞-system $A=\{a^k\}$ such that for each k there exists an l such that

$$\frac{\log a_n^k}{\log a_n^l} \to 0,$$

and let f be a Dragilev function. Define (b^k) by setting $b_n^k = \exp f(\log a_n^k)$. Then $\lambda(B)$ has no subspace isomorphic to $L_f(\beta, \infty)$.

The result in Theorem 1 can be used to construct for each given Dragilev function f a suitable G_∞-space $\lambda(B)$ such that $\lambda(B)$ is isomorphic to $L_f(\beta, \infty)\times\lambda(B_0)$, for a suitable $\beta_n\uparrow\infty$ and such that $\lambda(B_0)$ is a G_∞-space which has no subspace isomorphic to $L_f(\gamma, \infty)$ for any γ. One such construction is given in [7].

Now we shall obtain a partial converse of Theorem 1.

THEOREM 2. Suppose f is a Dragilev function and A is a normalized Köthe set and that $\lambda(A)$ is nuclear. Suppose

(a) for each $k, m \in \mathbb{N}$, $\exists l=l(k, m)\in\mathbb{N}$ such that

$$\frac{f^{-1}\log a_n^l}{f^{-1}\log a_n^k} \geqslant m \text{ for all } n \text{ (or for } n>n_0(k, m)),$$

(b) $$\frac{f^{-1}\log a_n^{k+1}}{f^{-1}\log a_n^k} \leqslant \frac{f^{-1}\log a_{n+1}^{k+1}}{f^{-1}\log a_{n+1}^k} \quad \forall k, n,$$

(c) $\lambda(A)$ is not isomorphic to $L_f(\beta, \infty)$ for any β.

Then for each k there exists an l such that

$$\lim_{n\to\infty} \frac{f^{-1}\log a_n^l}{f^{-1}\log a_n^k} = \infty.$$

PROOF. Write $b_n^k = f^{-1}\log a_n^k$, $k, n\in\mathbb{N}$. If possible the conclusion in the theorem be false; then there exists a k (fixed) so that $\lim_n b_n^k=\infty$ and for each l, there is a subsequence $(m_n^l)=(m_n)$ of positive integers and $M_l>0$ such that $b_{m_n}^l/b_{m_n}^k \leqslant M_l$, $n=1, 2, \ldots$. But then, because of (b) above $b_n^l/b_n^k \leqslant M_l$, $n=1, 2, \ldots$, and hence $\log a_n^l \leqslant f(M_l\, \alpha_n)$, $n=1, 2, \ldots$, where $\alpha_n = f^{-1} \log a_n^k = b_n^k$ (k already fixed). Also by (a) above, for each m there is an l such that $f(m\, \alpha_n) \leqslant \log a_n^l$, $n=1, 2, \ldots$. This shows that $\lambda(A)$ is set theoretically equal and topologically isomorphic to $\lambda(B)$ where $b_n^k = \exp f(k\, \alpha_n)$.

Let now π be a permutation of $\mathbf{N}$ such that $(\beta_n)=(\alpha_{\pi(n)})$ is increasing. Then by the nuclearity of $\lambda(A)$ and the fact that $\lambda(A) \cong \lambda(B)$ we have $\lambda(A) = L_f(\beta, \infty)$, and this gives a contradiction to (c).

REMARK. The strange looking condition (a) is only an analogue of the normalized (d_3) condition in the presence of f; if $f=$identity then (a) gives exactly the normalized (d_3) condition. So we shall now prove

PROPOSITION. Suppose f is a Dragilev function and $A=\{a^k\}$ satisfies condition (a) in Theorem 2. Then A satisfies the normalized (d_3) condition.

PROOF. Pick c so that $\tau(c)>2$. For this c and a given k pick, using (a), a^l so that $f^{-1} \log a_n^l \geqslant c\, f^{-1} \log a_n^k$. Then

$$\exp \left[f\left(\frac{1}{c} f^{-1} \log a_n^l \right) \right] \geqslant a_n^k,$$

since f increases. But

$$\frac{f(f^{-1} \log a_n^l)}{f\left(\frac{1}{c} f^{-1} \log a_n^l\right)} \to \tau(c) \text{ as } n \to \infty;$$

so for $n \geqslant n_0$,

$$a_n^l = \exp\, [f(f^{-1} \log a_n)] \geqslant \exp \left[2\, f\left(\frac{1}{c} f^{-1} \log a_n^l \right) \right] \geqslant (a_n^k)^2.$$

REMARKS

1) The above proof could also be adopted to show that

$$\forall k\, \exists l \ni \lim_n \frac{f^{-1} \log a_n^l}{f^{-1} \log a_n^k} = \infty \Rightarrow \forall k\, \exists l' \ni \lim_n \frac{\log a_n^{l'}}{\log a_n^k} = \infty.$$

Thus if A is as in Theorem 1 and satisfies the condition there for *some* Dragilev function f then it satisfies the same condition for the identity function and so $\lambda(A)$ contains no subspace isomorphic to a p.s.s. (of infinite type).

2) That A is a Köthe set satisfying the normalized (d_3) condition does not imply (a) of Theorem 2 for every Dragilev function f is seen by taking $A=(e^{k\alpha_n})$ and $f(x)=e^x$.

3) Take $a_n^k = \exp(k^{\alpha_n})$, where $\alpha = (\alpha_n)$ is a nuclear exponent sequence (i.e., $\Lambda_\infty(\alpha)$ is nuclear) and $f(x) = e^x$. Then conditions (a) and (b) of Theorem 2 are satisfied, but (c) is not true since

$$\frac{f^{-1}\log a_n^l}{f^{-1}\log a_n^k} = \frac{\log l}{\log k}.$$

Thus in this case $\lambda(A)$ is isomorphic to $L_f(\beta, \infty)$ but has no subspace isomorphic to a p.s.s.

This remark also shows that condition (a) of Theorem 2 which is a weaker assumption than the condition in Theorem 1 is not sufficient to obtain the conclusion of Theorem 1.

Now we shall reverse the roles of the spaces considered in Theorem 1 and consider the problem "$(\lambda(A), L_f(\beta, \infty)) \in R$". The following definitions and results are relevant to our discussion.

Let X be a nuclear Fréchet space with a basis (x_n) and a continuous norm. The basis (x_n) is called a (d_3)-*basis* (see Dubinsky [6]) if there exists on X a fundamental sequence $(\|\cdot\|_k)$ of norms so that

$$\forall k\, \exists \varepsilon_k > 0 \ni \varepsilon_k \frac{\|x_n\|_{k+1}}{\|x_n\|_k} \leqslant \frac{\|x_n\|_{k+2}}{\|x_n\|_{k+1}}, \quad n \in \mathbf{N}.$$

Given a Dragilev function f, the basis (x_n) on X is called a (d_f)-*basis* (see Alpseymen [1]) if there exist $M > 1$ and a fundamental sequence of norms $(\|\cdot\|_k)$ on X so that

$$Mf^{-1}\log\frac{\|x_n\|_{k+1}}{\|x_n\|_k} \leqslant f^{-1}\log\frac{\|x_n\|_{k+2}}{\|x_n\|_{k+1}}, \quad k, n \in \mathbf{N}.$$

Putting f = identity, the (d_f)-condition reduces to an equivalent form of the (d_3)-condition (see [6]).

The following result of the first author [1] is needed in our proofs.

LEMMA. Let f be either a rapidly increasing Dragilev function or the identity function. Assume the space $L_f(b, \infty)$ is nuclear and stable. X be a Fréchet space with a basis (x_n). Then X is isomorphic to a subspace of $L_f(b, \infty)$ if and only if

(a) the basis (x_n) is a (d_f)-basis and

(b) $\Delta(L_f(b, \infty)) \subset \Delta(X)$.

For the case f = identity and $b_n = \log(n+1)$ (so that $L_f(b, \infty) = (s)$, the space of rapidly decreasing sequences) the above result is due to Dubinsky [6] and to Vogt [9].

We are now in a position to state an interesting consequence of the above result and derive useful corollaries.

THEOREM 3. Let f be a Dragilev function which is rapidly increasing or be the identity function and let $L_f(b, \infty)$ be stable and nuclear. Let

X be a nuclear Fréchet space with a (d_f)-basis (x_n). Then X has a complemented subspace isomorphic to a subspace of $L_f(b, \infty)$.

PROOF. Since (x_n) is a (d_f)-basis, there exist a fundamental sequence $(\|\cdot\|_k)$ of norms on X and $M>1$ such that

$$Mf^{-1}\log\frac{\|x_n\|_{k+1}}{\|x_n\|_k}\leqslant f^{-1}\log\frac{\|x_n\|_{k+2}}{\|x_n\|_{k+1}},\quad k, n\in \mathbf{N}.$$

It follows that $(\|x_n\|_{k+1})^2\leqslant\|x_n\|_1\|x_n\|_{2k+1}$. Define

$$a_n^k=\frac{\|x_n\|_k}{\|x_n\|_1}.$$

Then $A=\{a^k\}$ is a normalized Köthe set and satisfies the normalized (d_3) condition since

(1) $\qquad (a_n^{k+1})^2\leqslant a_n^{2k+1}.$

Since X is nuclear we may assume that

$$\sum_n \frac{\|x_n\|_1}{\|x_n\|_2}<\infty \text{ and } \lim_n \frac{\|x_n\|_{k+1}}{\|x_n\|_1}=\lim_n a_n^{k+1}=\infty,\quad k\in\mathbf{N}.$$

Next we choose strictly increasing sequences $(i(k, n))_n$ of indices so that
(i) $(i(k+1, n))\subset(i(k, n))$,
(ii) $(a_{i(k,n)}^{(k+1)})_n$ is increasing and
(iii) $kb_n\leqslant f^{-1}\log(a_{i(k,n)}^{k+1})$.
Now let $j(n)$ denote the diagonal sequence $i(n, n)$. Then for $n\geqslant k$, $(j(n))\subset$ $\subset(i(k, n))$ and $j(n)\geqslant i(k, n)$ and $a_{i(k,n)}^{k+1}\leqslant a_{j(n)}^{k+1}$; therefore, again for $n\geqslant k$, we have

(2) $\qquad kb_n\leqslant f^{-1}\log a_{j(n)}^{k+1}$ or $\exp f(kb_n)\leqslant a_{j(n)}^{k+1}.$

Since $a_{i(k,n)}^{k+1}$ is increasing with n and since $\{j(n): n\geqslant k\}\subset\{i(k, n): n\geqslant k\}$ we have

(3) $\qquad a_{j(n)}^{k+1}$ is increasing with n, for $n\geqslant k$ and $k\in\mathbf{N}$.

Now since (1) is true also for $(a_{j(n)}^{k+1})$, by using, if necessary a small perturbation for the first $(k-1)$ elements of $(a_{j(n)}^{k+1})$ we may assume that $\lambda(a_{j(n)}^k)$ is a G_∞-space; this space is isomorphic to the complemented subspace Y of X generated by $(x_{j(n)})$. Also by [8],

$$\Delta(Y)=\Delta(\lambda(a_{j(n)}^k))=\{(t_n): \exists k, \exists C>0 \ni |t_n|\leqslant Ca_{j(n)}^k\}.$$

Then (2) shows $\Delta(L_f(b, \infty))\subset\Delta(Y)$. Also the basis $(x_{j(n)})$ of Y is a (d_f)-basis. Now it follows from the above lemma that Y is isomorphic to a subspace of $L_f(b, \infty)$.

NOTE. In the above theorem, taking $f=$ identity we obtain that each nuclear Fréchet space X with a (d_3)-basis has a complemented subspace isomorphic to a subspace of (a pre-assigned) stable, nuclear p.s.s. $\Lambda_\infty(\alpha)$. In this context the following result of Aytuna and Terzioglu [2] is of interest: a subspace X of $\Lambda_1(\alpha)$ with basis is either isomorphic to a subspace of $\Lambda_\infty(\alpha)$ or X has a complemented subspace which is isomorphic to a power series space of finite type.

COROLLARY TO THEOREM 3. If X is a nuclear Fréchet space with a (d_f)-basis (x_n) and $L_f(b, \infty)$ is nuclear and stable, then $(X, L_f(b, \infty)) \notin R$.

PROOF. By Theorem 3, we have a complemented subspace Y of X which is isomorphic to a subspace of $L_f(b, \infty)$. Let T denote the isomorphism map and P the projection of X onto Y. Then clearly TP is continuous but not compact.

REFERENCES

1. Alpseymen, M. – Basic sequences in some nuclear Köthe sequence spaces, Thesis, University of Michigan (1978).
2. Aytuna, A. and T. Terzioglu – On certain subspaces of a nuclear power series space of finite type (preprint).
3. Crone, L. and W. Robinson – Diagonal maps and diameters in Köthe spaces, Israel J. Math. **20**, 13–21 (1975).
4. DeGrande-DeKimpe, N. – L_f(a,r)-spaces between which all the operators are compact, I, Comment. Math. Univ. Carolinae **18**, 659–674 (1977).
5. DeGrande-DeKimpe, N. and W. Robinson – Compact maps and embeddings from an infinite type power series space to a finite type power series space, J. Reine Angew. Math. **293/294**, 52–61 (1977).
6. Dubinsky, E. – Basic sequences in (s), Studia Math. **59**, 283–293 (1977).
7. Ramanujan, M. S. and T. Terzioglu – Subspaces of smooth sequence spaces, to appear in Studia Math.
8. Terzioglu, T. – Die diametrale Dimension von lokalkonvexen Räumen, Collect. Math. **20**, 49–99 (1969)
9. Vogt, D. – Charakterisierung der Unterräume von s, Math. Z. **155**, 109–117 (1977).
10, Zahariuta, V. P. – On the isomorphism of the cartesian products of locally convex spaces, Studia Math. **46**, 201–221 (1973).