Discrete Mathematics 26 (1979) 235–241

THE GRAPHS WITH ONLY SELF-DUAL SIGNINGS*

Frank HARARY and Helene J. KOMMEL
University of Michigan, Ann Arbor, MI 48109, USA.

Received 4 May 1977
Revised 14 December 1978

Given a graph G, it is possible to attach positive and negative signs to its lines only, to its points only, or to both. The resulting structures are called respectively signed graphs, marked graphs and nets. The dual of each such structure is obtained by changing every sign in it. We determine all graphs G for which every suitable marked graph on G is self-dual (the M-dual graphs), and also the corresponding graphs G for signed graphs (S-dual) and for nets (N-dual.

A graph G is M-dual if and only if G or $\bar{G}$ is one of the graphs K_{2m}, $2K_m$, mK_2, $K_m \times K_2$ or $2C_4$. The S-dual graphs are C_6, $2C_3$, $2C_4$, $2K_{1,n}$, $2nK_2$, $K_{1,2n}$, $nK_{1,2}$, $K_{2,n}$, $\bar{K}_n$ and all graphs obtained from these by the addition of isolated points. Finally, the only N-dual graph other than $\bar{K}_{2n}$ is $2K_2$.

1. Duality

Let G be a graph (V, E) with p points and q lines. All graphical notation and concepts not defined here can be found in the book [1]. From a graph $G = (V, E)$, a *signed graph* S is obtained by signing each element of E positive or negative; a *marked graph* M by signing V; and a *net* N by signing both V and E. The dual structures S^*, M^*, N^* result when every sign is changed in S, M, N, respectively. We say that S is *self-dual* if S^* and S are isomorphic signed graphs; then M or N is defined as *self-dual* similarly. Obviously if M is self-dual, then $p = 2m$ is even, if S is self-dual, then $q = 2s$ is even, and if N is self-dual, then both p and q are even.

A (p, q) graph $G = (V, E)$ is called *S-dual* if for every signing of E using s positive and s negative signs, the resulting signed graph S is self-dual. We say similarly that G is *M-dual* if for every signing of V using m positive and m negative signs, the marked graph M is self-dual. The concept of an *N-dual* graph G is analogous.

2. The M-dual graphs

Fortunately and coincidentally, these graphs have been obtained already by Kelly and Merriell [4] in another context. A graph G is *bisectable* if $p = 2m$ is

* Research supported in part by grant SOC76-16824 from the National Science Foundation.

even and for each subset U of m points, the induced subgraphs $\langle U\rangle$ and $\langle V-U\rangle$ are isomorphic.

Theorem KM (Kelly and Merriell). *The only bisectable graphs are K_{2m}, $2K_m$, mK_2, $K_m \times K_2$, $2C_4$ and their complements.*

For a subset $U \subset V(G)$ of points of a graph G, define the subgraph $\langle\langle U\rangle\rangle$ *produced by* U, to consist of all lines incident with at least one point of U. Then for all $U \subset V(G)$, $\langle\langle U\rangle\rangle \cup \langle\langle V-U\rangle\rangle = G$, whereas it is not in general true that $\langle U\rangle \cup \langle V-U\rangle = G$.

Our first result shows that the concepts of a bisectable graph and an M-dual graph are equivalent.

Theorem 1. *A graph G is M-dual if and only if G or $\bar{G}$ is one of the graphs K_{2m}, $2K_m$, mK_2, $K_m \times K_2$, $2C_4$.*

Proof. Certainly if G is M-dual, so is its complement $\bar{G}$. Evidently G is M-dual if and only if, for all $U \subset V(G)$ with $|U| = m$, $\langle\langle U\rangle\rangle \cong \langle\langle V-U\rangle\rangle$. Let $|U| = m$ and write $+$ for the usual join operation [1, p. 21]. For $U \subset V(K_{2m})$, clearly

$$\langle\langle U\rangle\rangle \cong \langle\langle V-U\rangle\rangle \cong K_m + \bar{K}_m.$$

For $U \subset V(2K_m)$, with r points of U in one component and $m-r$ in the other,

$$\langle\langle U\rangle\rangle \cong \langle\langle V-U\rangle\rangle \cong (K_r + \bar{K}_{m-r}) \cup (K_{m-r} + \bar{K}_r).$$

For $U \subset V(mK_2)$, with points of U appearing in r components, $\langle\langle U\rangle\rangle \cong \langle\langle V-U\rangle\rangle \cong rK_2$. The graph $K_m \times K_2$ may be regarded as $2K_m$ plus a 1-factor. The graph $2K_m$ is M-dual, and the addition of a 1-factor does not alter this. Hence $K_m \times K_2$ is M-dual. The graph $2C_4$ is immediately seen to be M-dual.

Conversely let G be M-dual, with $U \subset V(G)$ and $|U| = m$. Then the graphs produced by U and by $V-U$ are isomorphic so certainly the graphs they induce are isomorphic and G is bisectable. By Theorem KM, G or $\bar{G}$ is one of the graphs listed in the statement. □

3. The S-dual graphs.

We consider now those graphs with $q = 2s$ lines which are always self-dual with respect to line signings. For a subset $F \subset E(G)$ of lines of G, let $\langle F\rangle$ denote the subgraph induced by F.

Theorem 2. *The only S-dual graphs are C_6, $2C_3$, $2C_4$, $2K_{1,n}$, $2nK_2$, $K_{1,2n}$, $nK_{1,2}$, $K_{2,n}$, $\bar{K}_n$ and all graphs obtained from these by the addition of isolated points.*

Proof. The graphs $\overline{K_n}$ are trivially S-dual and obviously a graph obtained by adding isolated points to an S-dual graph is again S-dual. For any $F \subset E(G), \langle F \rangle \cup \langle E-F \rangle = G$, so that G is S-dual if and only if, for all $F \subset E(G)$ with $|F| = s, \langle F \rangle \cong \langle E-F \rangle$. It is easily verified that the graphs listed in the statement are S-dual.

It is convenient to develop the sufficiency argument by means of several lemmas. The lemmas serve to find all S-dual graphs with no trivial components.

Remark. The lemmas are proved by assuming that an S-dual graph G is not one of the graphs of the theorem. We then identify a nonempty subset F of lines of G such that $\langle F \rangle$ has some structural property not held by $\langle E-F \rangle$. Then one of these subgraphs has at most s lines which may all be given the same sign. Any completion in G of such a signing is not self-dual, a contradiction. As this reasoning is used repeatedly, the proofs of the lemmas end when such an exceptional subset of lines is constructed.

Lemma 2a. *Let q_d be the number of lines of G which are incident with a point of degree d. If G is S-dual, then $q_d = q$ for all $d > 0$ for which a point of degree d exists.*

Proof. Let G be S-dual. Assume there is an integer $d > 0$ such that a point of degree d exists but $q_d < q$. Then there is a line of G not incident with a point of degree d. Let E_d be the set of all lines incident with a point of degree d, do that $|E_d| = q_d$. By hypothesis, neither E_d nor $E - E_d$ is empty. By the Remark, this is a contradiction. Hence $q_d = q$ for all degrees $d > 0$ occurring in G. □

Lemma 2b. *If G is S-dual with no trivial components, then G is regular or is bipartite with all points of the same subset having the same degree.*

Proof. By Lemma 2a, every line of G must be incident with a point of degree d for every degree occurring in G. Thus at most two different degrees can occur in G. Either G is regular or points of different degrees d_1 and d_2 occur. In this case G is bipartite since by Lemma 2a, no line of G can join two points of the same degree. □

The next three lemmas serve to find all S-dual regular graphs.

Lemma 2c. *If G is S-dual and not a forest, then each component of G is a block.*

Proof. If G is S-dual then certainly all components of G are isomorphic. For if not, then for some component H of G, all the components isomorphic to H together have at most s lines, a contradiction by the Remark. Thus if G is also not a forest, then no component of G is a tree. Let H be a component of G. Then H

contains a cycle and no line of this cycle is incident with a point of degree 1, so by Lemma 2a, there can be no point of degree 1 in H.

Assume H is not a block, and let E^c be the set of all lines of H incident with a cutpoint. There must be some block B of H containing only one cutpoint v. Since no point of B is of degree 1 in B, the removal of all cutpoints from H removes only v from B, hence must leave some lines between points of B. These lines are incident with no cutpoint, hence $E(H)-E^c$ is not empty. The Remark completes the proof. □

Lemma 2d. *If G is S-dual and regular of degree d, then $d \leq 2$.*

Proof. If G is regular of degree d then $s = pd/4$. First, if $d \geq 4$ then $s \geq p$. Choose one point v of G and sign the d lines incident with v positive. It requires at most $p-d-2$ additional positive lines to cover the $p-d-1$ points not adjacent to v. Then we have used at most $p-2<s$ positive signs to make all lines incident with v positive while leaving no point with all incident lines negative.

The Remark applies here to give a contradiction. Hence $d \leq 3$.

If $d=3$, since q is even, we have $p=4n$ and $q=6n$ for some n. Since K_4 is not S-dual, n is at least 2. By Lemma 2c, each component of g is a block. It is well known that a cubic block has no bridge. Petersen [6] proved that every cubic bridgeless graph has a 1-factor, i.e., a spanning subgraph of the form mK_2. By signing each line of such a subgraph in G positive, every point of G is incident with a positive line. Choose one point v of G and sign the other two lines incident with v positive also. This uses $2n+2 \leq 3n = s$ positive signs. Again the Remark applies, since we have all lines incident with v positive and no point with all incident lines negative. Hence $d \leq 2$. □

Lemma 2e. *If G is S-dual and regular of degree 2, then G is C_6, C_4, $2C_3$, or $2C_4$.*

Proof. By hypothesis, G is a union of cycles. As in Lemma 2c, the components of G are isomorphic so G is of the form kC_n. Further, $k \leq 2$ for if $k>2$, make one C_n component all positive. This leaves $nk/2-n=n(k-2)/2>k-2$ positive signs, enough to use one on each of the other $k-1$ cycles, leaving no cycle all negative.

If G consists of odd cycles C_{2m+1}, then G must be $2C_{2m+1}$ and in fact, $G=2C_3$. For if $m>1$, we may sign one cycle with a P_{m+1} subgraph positive and a P_{m+2} subgraph negative and sign the other with a $P_{m+1} \cup P_2$ subgraph positive and a $P_m \cup P_2$ subgraph negative. This is not self-dual.

If G consists of even cycles C_{2m}, then $m \leq 3$. For if $m>3$ we may choose in each component m lines which induce the subgraph $P_2 \cup P_m$ while the remaining m lines induce $P_3 \cup P_{m-1}$. Again this is not self-dual. If one component is C_6, then there is only one component. For if there are two components then we may sign one with a P_5 subgraph positive and a P_3 subgraph negative, and the other with a $2P_2$ subgraph positive and a $2P_3$ subgraph negative. Hence $G=2C_3$, C_6, C_4, or $2C_4$. □

Evidently, if G is S-dual and regular of degree 1, then $G = 2nK_2$. If G is S-dual and regular with $d = 0$, then $G = \bar{K}_n$ trivially.

We now find all S-dual graphs which are not regular, looking first for connected graphs.

Lemma 2f. *If G is S-dual and a tree, then $G = K_{1,2n}$.*

Proof. Since G is a tree, there exist points of degree 1. By Lemma 2a, either $G = K_2$ or each line of G is incident with exactly one point of degree 1. Since G is S-dual, $G \neq K_2$. Thus the q lines of G insure the presence of q distinct points of degree 1 in G. Since the tree G has $q+1$ points, $G = K_{1,q}$. □

By Lemma 2b, S-dual graphs which are not regular are bipartite. Let $V(G) = U_1 \cup U_2$ with n_i points of degree d_i in U_i, $i = 1, 2$, and $d_1 \neq d_2$.

Lemma 2g. *Let G be S-dual and not regular. If $n_1 > 2$, then $d_1 \leq 2$.*

Proof. Consider $n_1 > 2$. If $d_1 > 2$, choose a point u from U_1 and sign the d_1 lines incident with u positive. As $s = n_1 d_1/2$, this leaves $n_1 d_1/2 - d_1 = d_1(n_1 - 2)/2 > n_1 - 2$ positive signs. Thus there are enough plus signs available to place one of them on some line incident with each of the $n_1 - 1$ other points of U_1. Since $d_1 \neq d_2$, this is not self-dual. Hence $d_1 \leq 2$. □

Lemma 2h. *If G is S-dual, connected, not regular, and not a tree, then $G = K_{2,n}$, $n > 2$.*

Proof. By Lemma 2a, if G is not a tree there are no points of degree 1. If G is not regular we may assume $1 < d_1 < d_2$ and since $n_1 d_1 = n_2 d_2$ and $d_1 \leq n_2$, we have $n_1 > n_2 > 1$. Hence, $n_1 > 2$ so $d_1 = 2$ by Lemma 2g. If also $n_2 > 2$ then $d_2 = 2$; but $d_1 \neq d_2$ so $n_2 = 2$. Then $d_1 = n_2$, so that G is a complete bigraph; hence $G = K_{2,n}$ when $n = n_1 > 2$. □

Finally we look for disconnected nonregular S-dual graphs.

Lemma 2i. *If G is S-dual, not regular, and disconnected with no trivial components, then $G = nK_{1,2}$* or *$G = 2K_{1,n}$.*

Proof. Since G is disconnected and $d_1 \neq d_2$, we may assume $1 < n_1 < n_2$, so $d_1 > d_2$. We have $n_2 > 2$ so $d_2 \leq 2$ by Lemma 2g. In one case, $n_1 > 2$ also. Then $d_1 \leq 2$ so $d_1 = 2$, $d_2 = 1$. Since $n_1 d_1 = n_2 d_2 = q$, we have $n_2 = 2n_1$. Evidently $G = nK_{1,2}$ when $n = n_1$.

In the other case, $n_1 = 2$. Since G is disconnected, $d_2 < n_1$. Hence $d_2 = 1$, n_2 is even, and $d_1 = n_2/2$. Thus $G = 2K_{1,n}$ when $n = n_2/2$. □

This completes the proof of the theorem. □□

Without reference to line signing, we may define a graph G to be E-bisectable if $q=2s$ is even and for every subset $F \subset E(G)$ with $|F|=s$, the subgraphs induced by F and by $E-F$ are isomorphic. The concepts of an E-bisectable graph and an S-dual graph are equivalent.

4. The N-dual graph

We turn now to N-dual graphs. Certainly every N-dual graph is both S-dual and M-dual. Thus we need only search among the graphs of Theorems 1 and 2 although a direct determination of all N-dual graphs is not difficult.

Theorem 3. *The only N-dual graphs are* $2K_2$ *and* $\bar{K}_{2n}$.

Proof. The graphs $\bar{K}_{2n}$ are trivially N-dual. There are only three nonisomorphic signings of $2K_2$ as a net and each is readily verified to be N-dual. They are shown in Fig. 1.

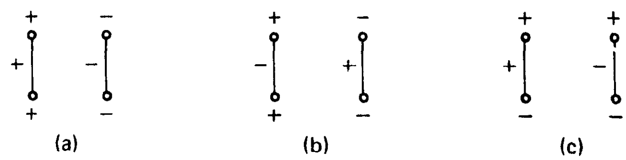

Fig. 1. The three nets on $2K_2$ with equidistributed signs.

By Theorems 1 and 2, the only graphs G which are both M-dual and S-dual are $\bar{K}_{2n}$, $2nK_2$, C_6, $2C_3$, and $2C_4$. None of the latter three graphs is N-dual as illustrated in Fig. 2. Hence G is $\bar{K}_{2n}$ or G is of the form $2nK_2$.

Assume $n>1$ and sign the lines arbitrarily. As there are at least four K_2 components, we may construct the following net. Let one positive line have both its points positive, let one positive line have both points negative, and let all remaining lines have one point positive and one negative. This net is not self-dual, a contradiction. Hence $n=1$ and $G=2K_2$. □

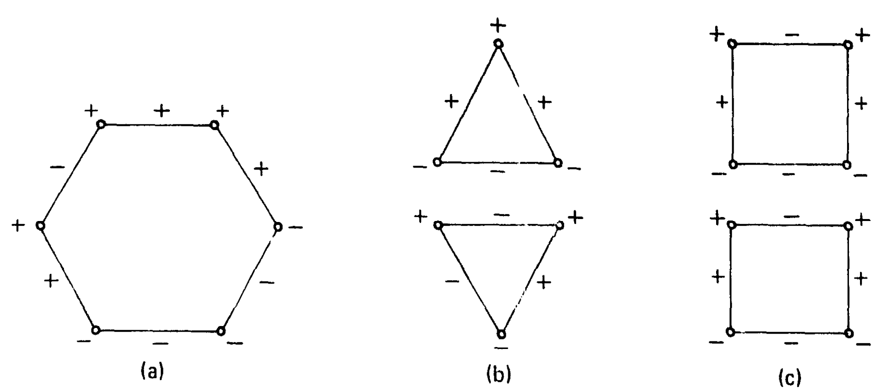

Fig. 2. Non-self-dual nets on C_6, $2C_3$, $2C_4$.

5. Related results

Kelly and Merriell [5] found all the bisectable digraphs and, hence, all M-dual digraphs. Self-dual marked graphs, signed graphs, and nets are counted in [2]. The converse D' of a digraph D is obtained when the orientation of every arc in D is reversed. A theorem analogous to those obtained above was derived in [3] for the dual operation of taking the converse.

Theorem (Harary, Palmer and Smith). *The only connected graphs for which every orientation is self-converse are* K_1, K_2, C_3, C_4, *and* C_5.

References

[1] F. Harary, Graph Theory (Addison-Wesley, Reading, MA, 1969).
[2] F. Harary, E.M. Palmer, R.W. Robinson, and A.J. Schwenk, Enumeration of graphs with signed points and lines, J. Graph Theory 1 (1977) 295–308.
[3] F. Harary, E.M. Palmer, and C.A.B. Smith, Which graphs have only self-converse orientations? Canad. Math. Bull. 10 (1967) 425–429.
[4] P.J. Kelly and D. Merriell, A class of graphs, Trans. Amer. Math. Soc. 96 (1960) 488–492.
[5] P.J. Kelly and D. Merriell, Directed graphs having all bisections, J. Combinatorial Theory 4 (1968) 4–18.
[6] J. Petersen, Die Theorie der regulären Graphen, Acta Math. 15 (1891) 1 '3–220.