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THE EXPONENT OF UNil

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INTRODUCTION

CAPPELL[3] HAS introduced obstruction groups for his splitting theorem—$\mathrm{UNil}_k^h(R; M_1, M_2)$ and $\mathrm{UNil}_{2k}^s(R; M_1, M_2)$. (Here R is a ring, M_1 and M_2 are R-bimodules, and k is an integer.) He showed they are 2-primary in the geometrically interesting cases. In these cases, we prove the exponent of $\mathrm{UNil}_{2k}^h(\ ;\ ,\ )$ divides 4. (See Theorem 1.3.) Our techniques probably give the same result for $\mathrm{UNil}_{2k+1}^h(\ ;\ ,\ )$ and $\mathrm{UNil}_{2k}^s(\ ;\ ,\ )$; we don't attempt this to avoid obscuring our argument with technical details. It occured to the author, after completing this paper, that a sufficiently general localization theorem in L-theory would probably yield reasoning as in [9] and [5], a direct proof that 8 annihilates $\mathrm{UNil}_{2k}^h(\ ;\ ,\ )$ (for the same cases as above). Ranicki[10] has recently constructed such a localization theorem.

We obtain some additional information about $L_3(Z_2 * Z_2)$. (See Theorem 4.1.)

§1. MAIN RESULT

Let R be a ring with 1 and involution $r \to \bar{r}$, and M a R-bimodule with involution also denoted by $x \to \bar{x}$ (see e.g. [3]). Let $\mathscr{F} = (P, \lambda, \mu)$ be a $(-1)^k$ Hermitian form over M and $f: V \times V \to \mathbf{Z}$ a symmetric (integral valued) bilinear form on a finitely generated, free, abelian group V. Define $f\mathscr{F} = (V \otimes P, \lambda', \mu')$ to be a new $(-1)^k$ Hermitian form over M. We explain the terms occurring in $f\mathscr{F}$. First, $V \otimes P$ is tensor product with respect to Z; $V \otimes P$ inherits a right R-module structure from P; clearly, $V \otimes P$ is a free, finitely generated R-module. Next, the bilinear pairing λ' is determined by the equation

$$(1) \qquad \lambda'(v \otimes x, w \otimes y) = f(v, w)\lambda(x, y)$$

for v, $w \in V$ and x, $y \in P$. Finally, the quadratic map μ' is determined by

$$(2) \qquad \mu'(v \otimes x) = f(v, v)\mu(x)$$

for $v \in V$ and $x \in P$.

We collect together some notation. Let $P^* = \mathrm{Hom}_R(P, R)$ and $\lambda^*: P \to P^* \otimes_R M$ be the adjoint of λ; i.e. the composite of the map $P \to \mathrm{Hom}_R(P, M)$ defined by $x \to \lambda(x, \ )$ with the inverse of the canonical isomorphism

$$P^* \otimes_R M \to \mathrm{Hom}_R(P, M).$$

Similarly, let $V^* = \mathrm{Hom}(V, \mathrm{Z})$ and define f^*: $V \to V^*$ by $f^*(m) = f(m, \ )$. The following diagram commutes

$$(3) \qquad \begin{tikzcd} V \otimes P \arrow[r, "(\lambda')^*"] \arrow[dr, "f^* \otimes \lambda^*"'] & (V \otimes P)^* \otimes_R M \\ & V^* \otimes (P^* \otimes_R M) \arrow[u] \end{tikzcd}$$

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where the vertical map is the canonical isomorphism. Recall f is non-singular if f^* is an isomorphism. When f is non-singular, define

$$f^{-1}: V^* \times V^* \to \mathbf{Z}$$

by requiring $(f^{-1})^* = (f^*)^{-1}$.

Let D_{2n} denote the dihedral group of order $2n$. Fix generators α and γ for D_{2n} with $\alpha^2 = 1 = \gamma^n$ and $\alpha\gamma\alpha^{-1} = \gamma^{-1}$; define $\beta = \gamma\alpha$. (Note $\beta^2 = 1$.) Let $\mathscr{L} = (V, f)$ be a $\mathbf{Z}D_{2n}$-lattice; i.e. V is a finitely generated, $\mathbf{Z}$-free, D_{2n}-module and $f: V \times V \to \mathbf{Z}$ is a symmetric, D_{2n}-invariant, non-singular form. Define associated, symmetric, non-singular forms f_1, f_2: $V \times V \to \mathbf{Z}$ by

$$f_1(v, w) = f(\alpha v, w), \qquad f_2(v, w) = f(\beta v, w) \tag{4}$$

for $v, w \in V$. Notice that f_1^* is the composite of f^* and multiplication by α; f_2^* the composite of f^* with multiplication by β. Set $\mathscr{L}^{-1} = (V^*, f^{-1})$, then $\mathscr{L}^{-1}$ is also a $\mathbf{Z}D_{2n}$-lattice.

Let M_1 and M_2 be R-bimodules with involution which are free as left R-modules, $\mathscr{C} = (\mathscr{F}_1; \mathscr{F}_2)$ a $(-1)^k$ UNil form over (M_1, M_2), where $\mathscr{F}_i = (P_i, \lambda_i, \mu_i)$ are $(-1)^k$ Hermitian forms over M_i $(i = 1, 2)$ with $P_2 = P_1^*$ (see e.g. [3]). Define a new $(-1)^k$ UNil form $\mathscr{L}\mathscr{C} = (\mathscr{F}_1', \mathscr{F}_2')$ by $\mathscr{F}_1' = f_1\mathscr{F}_1$ and $\mathscr{F}_2' = (f^{-1})_2\mathscr{F}_2$. (To be precise, $\mathscr{F}_2'$ is the pullback of $(f^{-1})_2\mathscr{F}_2$ to $(V \otimes P_1)^*$ via the canonical isomorphism $(V \otimes P_1)^* \to V^* \otimes P_1^*$.) Using (3), we see $\mathscr{L}\mathscr{C}$ satisfies the nilpotent condition in the definition of a $(-1)^k$ UNil form. (See [3].)

Recall $\mathscr{C}$ is a kernel if there exist free summands S_i of P_i $(i = 1, 2)$ with $S_2 \subset P_2 = P_1^*$ the annihilator of $S_1 \subset P_1$, and with $\lambda_i | S_i \times S_i$ and $\mu_i | S_i$ zero; we call the pair (S_1, S_2) a subkernel for $\mathscr{C}$.

LEMMA 1.1. *If either $\mathscr{C}$ is a kernel or $\mathscr{L}$ is a split lattice, then $\mathscr{L}\mathscr{C}$ is a kernel.*

Proof. First, assume $\mathscr{C} = (P_1, \lambda_1, \mu_1; P_2, \lambda_2, \mu_2)$ is a kernel with subkernel (S_1, S_2) and $\mathscr{L} = (V, f)$, then $(V \otimes S_1, V^* \otimes S_2)$ is a subkernel for $\mathscr{L}\mathscr{C}$.

Next, assume $\mathscr{L}$ is split (see [6], p. 294) and let W be a Lagrangian in V; i.e. W is a D_{2n}-submodule such that $W = W^\perp$ where

$$W^\perp = \{v \in V | f(v, w) = 0 \quad \text{for all } w \in W\}, \tag{5}$$

then $(W \otimes P_1, f^*W \otimes P_2)$ is a subkernel for $\mathscr{L}\mathscr{C}$.

COROLLARY 1.2. *The pairing $(\mathscr{L}, \mathscr{C}) \mapsto \mathscr{L}\mathscr{C}$ induces a unital $GW(D_{2n}, \mathbf{Z})$-module structure on* $\mathrm{UNil}_{2k}^h(R; M_1, M_2)$.

(See [6] for the definition of $GW(\ ,\ )$.)

In certain cases, Cappell constructs a map from UNil to the Wall surgery group. Namely, let $R \subset \Lambda_i$ $(i = 1, 2)$ be inclusions of rings with identity and involution. Assume that Λ_i has an R-bimodule with involution decomposition $\Lambda_i = R \oplus \hat{\Lambda}_i$, $\hat{\Lambda}_i$ a free left R-module. Let Λ denote the amalgamation ring $\Lambda_1 *_R \Lambda_2$, then there is a map

$$\rho: \mathrm{UNil}_{2k}^h(R; \hat{\Lambda}_1, \hat{\Lambda}_2) \to L_{2k}^h(\Lambda). \tag{6}$$

(See [3].) We now describe the situation of particular interest to us. Let H, G_1, G_2 be finitely presented groups with $H \subset G_i$ $(i = 1, 2)$ and $\omega_i: G_i \to \{\pm 1\}$ homomorphisms with $\omega_1|H = \omega_2|H$; these determine involutions on $\mathbf{Z}[H]$, $\mathbf{Z}[G_1]$, $\mathbf{Z}[G_2]$, $\mathbf{Z}[G]$ where $G = G_1 *_H G_2$. Let $\mathbf{Z}[\hat{G}_i]$ denote the $\mathbf{Z}[H]$ subbimodule with involution of $\mathbf{Z}[G_i]$ additively generated by $g \in G_i - H$. This fits into the above terminology with $R = \mathbf{Z}[H]$, $\Lambda_i = \mathbf{Z}[G_i]$, $\hat{\Lambda}_i = \mathbf{Z}[\hat{G}_i]$, and $\Lambda = \mathbf{Z}[G]$. But, in this specific situation, Cappell [3] shows the map ρ of (6) is a monomorphism. We use this fact in proving our main result.

THEOREM 1.3. *The exponent of* $\mathrm{UNil}^h_{2k}(\mathbf{Z}[H]; \mathbf{Z}[\hat{G}_1], \mathbf{Z}[\hat{G}_2])$ *divides* 4 (for all k).

To prove this, we first show that ρ factors through $\mathrm{UNil}^h_{2k}(\Lambda; \Lambda, \Lambda)$ which we abbreviate to $\mathrm{UNil}_{2k}(\Lambda)$. Let the $(-1)^k$ UNil form $\mathscr{C} = (P_1, \lambda_1, \mu_1; P_2, \lambda_2, \mu_2)$ represent an element in $\mathrm{UNil}^h_{2k}(R; \hat{\Lambda}_1, \hat{\Lambda}_2)$; associate to it the $(-1)^k$ UNil form over (Λ, Λ)

$$\hat{\mathscr{C}} = (P_1 \otimes_R \Lambda, \hat{\lambda}_1, \hat{\mu}_1; P_2 \otimes_R \Lambda, \hat{\lambda}_2, \hat{\mu}_2) \tag{7}$$

where $\hat{\lambda}_i$ and $\hat{\mu}_i$ $(i = 1, 2)$ are determined by

$$\begin{aligned} \hat{\lambda}_i(x \otimes s, y \otimes t) &= \bar{s}\lambda_i(x, y)t, \quad \text{and} \\ \hat{\mu}_i(x \otimes s) &= \bar{s}\mu_i(x)s \end{aligned} \tag{8}$$

for $x, y \in P_i$ and $s, t \in \Lambda$. The correspondence $\mathscr{C} \mapsto \hat{\mathscr{C}}$ induces a homomorphism

$$\hat{\rho}\colon \mathrm{UNil}^h_{2k}(R; \hat{\Lambda}_1, \hat{\Lambda}_2) \to \mathrm{UNil}_{2k}(\Lambda). \tag{9}$$

Cappell's procedure for defining ρ also gives a map

$$\rho'\colon \mathrm{UNil}_{2k}(\Lambda) \to L^h_{2k}(\Lambda).$$

Namely, ρ' is determined by associating to a $(-1)^k$ UNil form $(P_1, \lambda_1, \mu_1; P_2, \lambda_2, \mu_2)$ over (Λ, Λ) a $(-1)^k$ Hermitian form (P, λ, μ) over Λ with $P = P_1 \oplus P_2$ and

$$\begin{aligned} \lambda(x, y) &= \langle x, y \rangle && \text{for } x \in P_2 = P_1^*, \quad y \in P_1; \\ \lambda(x, y) &= \lambda_i(x, y) && \text{for } x, y \in P_i; \\ \mu(x) &= \mu_i(x) && \text{for } x \in P_i. \end{aligned} \tag{10}$$

Thus, we obtain the factorization.

LEMMA 1.4. *The map ρ factors as the composite of $\hat{\rho}$ with ρ'.*

Therefore, it suffices to show the exponent of image ρ' divides 4; for this, we need some more lemmas. Denote the identity of D_{2n} by e and the cyclic subgroups generated by α, β, γ, and e, respectively, by (α), (β), (γ), and (e); their inclusion maps into D_{2n} by i, j, k, and l, respectively.

LEMMA 1.5. *For each $r \in GW((\gamma), \mathbf{Z})$ and $x \in \mathrm{UNil}_{2k}(\Lambda)$, $k_*(r)x = 0$.*

Proof. Let $\mathscr{L} = (V, f)$ represent r and $\mathscr{C} = (P_1, \lambda_1, \mu_1; P_2, \lambda_2, \mu_2)$ represent x, then k_*r is represented by the $\mathbf{Z}D_{2n}$-lattice (W, g) where $W = V \oplus V$,

$$g = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \tag{11}$$

and α, β act (relative to this decomposition) via the matrices

$$\begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \gamma \\ \gamma^{-1} & 0 \end{pmatrix}, \tag{12}$$

respectively. Then, $V_1 \otimes P_1$ is a subkernel for $\mathscr{L}\mathscr{C}$ where V_1 is the first component of W.

PROPOSITION 1.6. *For each $x \in \mathrm{UNil}_{2k}(\Lambda)$, there exists an integer N_x such that for all $n > N_x$ and every $r \in GW((\alpha), \mathbf{Z})$ and $s \in GW((\beta), \mathbf{Z})$,*

$$\rho'(i_*(r)x) = 0 = \rho'(j_*(s)x).$$

PROPOSITION 1.7. *When n is a power of 2,*

$$i_*(2) + j_*(2) + k_*(2) - l_*(2) = 4$$

is an equation in $GW(D_{2n}, \mathbf{Z})$.

We postpone the prooofs of these propositions to §2 and §3 and complete the proof of Theorem 1.3. As already observed, it suffices to show $4\rho'(x) = \rho'(4x) = 0$ for all $x \in \mathrm{UNil}_{2k}(\Lambda)$. Let n be a power of 2; $n > N_x$. By Proposition 1.7,

$$i_*(2)x + j_*(2)x + k_*(2)x - l_*(2)x = 4x, \tag{13}$$

but Lemma 1.5 shows $k_*(2)x = 0 = l_*(2)x$. (Note that l factors through k.) Applying ρ' to (13), we obtain

$$\rho'(i_*(2)x) + \rho'(j_*(2)x) = \rho'(4x).$$

The result now follows from Proposition 1.6.

Remark 1.8. Proposition 1.6 was geometrically motivated by Browder's paper [1] and Lemma 1.5 by the Browder–Levine paper [2].

§2. PROOF OF PROPOSITION 1.6.

The proof of Proposition 1.6 divides into a few slightly different cases; we prove only one of these (Proposition 1.6′) and leave the others to the reader.

PROPOSITION 1.6′. *For each $x \in \mathrm{UNil}_{2k}(\Lambda)$, there exists an integer N_x such that for all even integers $n > N_x$ and every $r \in GW((\alpha), \mathbf{Z})$, $\rho'(i_*(r)x) = 0$.*

Proof. Let $\mathscr{L} = (V, f)$ represent r and $\mathscr{C} = (P, \lambda_1, \mu_1; P^*, \lambda_2, \mu_2)$ represent x. For any *even* integer $n = 2m$, $\rho'(i_*(r)x)$ is represented by a $(-1)^k$ Hermitian form (Q, λ, μ) with

$$Q = P_1 \oplus P_2 \oplus \cdots \oplus P_n \oplus P_1^* \oplus \cdots \oplus P_n^* \tag{14}$$

where $P_i = V \otimes P$. The forms μ and λ have certain nice properties; first, $\mu | P_i^* = 0$ for all i and $\mu | P_i = 0$ for $i \neq m$ and n. Next, we discuss the properties of λ; define forms

$$\varphi: V \otimes P \times V \otimes P \to \Lambda, \text{ and}$$
$$(15) \qquad \psi: V^* \otimes P^* \times V^* \otimes P^* \to \Lambda$$

by the equations

$$(16) \qquad \varphi(v \otimes x, w \otimes y) = f(\alpha v, w)\lambda_1(x, y)$$

for $v, w \in V$ and $x, y \in P$, and

$$(17) \qquad \psi(v \otimes x, w \otimes y) = f^{-1}(\alpha v, w)\lambda_2(x, y)$$

for $v, w \in V^*$ and $x, y \in P^*$. Then, λ is described by the equations (where $x_i \in P_i^*$ and $y_j \in P_j$)

$$\lambda(x_i, y_j) = \begin{cases} 0 & \text{if } i \neq j \\ \langle x_i, y_i \rangle & \text{if } i = j \end{cases}$$

$$(18) \qquad \lambda(y_i, y_j) = \begin{cases} 0 & \text{if } i + j \neq n \\ \varphi(y_i, y_j) & \text{if } i + j = n, \text{ and} \end{cases}$$

$$\lambda(x_i, x_j) = \begin{cases} 0 & \text{if } i + j \neq n + 1 \\ \psi(x_i, x_j) & \text{if } i + j = n + 1. \end{cases}$$

In matrix terminology, λ has the form

$$(19) \qquad \begin{pmatrix} A & \pm I \\ I & B \end{pmatrix}$$

where I is the identity matrix; B a "$n \times n$-matrix" with ψ along the skew diagonal and zero elsewhere; and A a "$n \times n$-matrix" with φ along the diagonal above the skew diagonal, also in the bottom, right corner and zero elsewhere.

Since $\mathscr{C}$ is a UNil form, $\lambda_1^*\lambda_2^*: P^* \to P^*$ is nilpotent; i.e. there is an integer N' such that $(\lambda_1^*\lambda_2^*)^p = 0$ for all $p \geq N'$, hence $h^p = 0$ for $p \geq N'$ where $h = \varphi^*\psi^*$. Now, if $m - 1 \geq N'$, we can construct a subkernel S for (Q, λ, μ); namely,

$$(20) \qquad S = P_1 \oplus \cdots \oplus P_{m-1} \oplus W \oplus P_{m+1}^* \oplus \cdots \oplus P_n^*$$

where it remains to describe W. To each $x \in (V \otimes P)^*$, associate $x' \in Q$ where the i-th component x_i' of x' is given by the formula

$$(21) \qquad x_i' = \begin{cases} 0 & \text{if either } i \leq m \text{ or } i > 3m \\ -\psi^* h^j(x) & \text{if } m < i \leq n, \text{ where } j = i - (m+1) \\ h^{3m-i}(x) & \text{if } n < i \leq 3m; \text{ i.e.,} \end{cases}$$

$$x' = (0, \ldots, -\psi^*(x), \ldots, -\psi^* h^{m-1}(x), h^{m-1}(x), \ldots, x, 0, \ldots);$$

let W be the submodule consisting of all x'. A straightforward calculation verifies that S is a subkernel.

§3. PROOF OF PROPOSITION 1.7.

Let $\mathbf{Q}$ denote the rational numbers, E_r the equation posited in Proposition 1.7 for $n = 2^r$, and $D^r = D_{2n}$. Since Dress ([6], Theorem 5) has shown that the map

$GW(D^r, \mathbf{Z}) \to GW(D^r, \mathbf{Q})$ is a monomorphism, it suffices to verify E_r in $GW(D^r, \mathbf{Q})$. We proceed by induction on r; the case D^1 (the Klein 4-group) can be checked directly and is left to the reader. When $r \geq 1$, Wall (see e.g. [12], p. 68) has observed that

$$\mathbf{Q}D^{r+1} \simeq \mathbf{Q}D^r \oplus M_2(\mathbf{Q}(\cos\theta)) \tag{22}$$

where $\theta = \pi/n$ and $M_2(\mathbf{Q}(\cos\theta))$ denotes the 2×2-matrix ring over the field $\mathbf{Q}(\cos\theta)$. In this decomposition, the map $\mathbf{Q}D^{r+1} \to \mathbf{Q}D^r$ is induced by the group homomorphism $D^{r+1} \to D^r$ which sends γ, α in D^{r+1} to γ, α, respectively, in D^r; the map $\mathbf{Q}D^{r+1} \to M_2(\mathbf{Q}(\cos\theta))$ is determined by sending

$$\begin{aligned} &\alpha \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and} \\ &\gamma \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \end{aligned} \tag{23}$$

Fröhlich and McEvert[8] have defined for a ring R with involution a group $\mathcal{M}(R)$ which reduces to the Wittring when R is a field with trivial involution, and for a finite group G, $\mathcal{M}(\mathbf{Q}G) = GW(G, \mathbf{Q})$. Applying $\mathcal{M}(\,)$ to (22), we obtain

$$GW(D^{r+1}, \mathbf{Q}) \simeq GW(D^r, \mathbf{Q}) \oplus \mathcal{M}(M_2(\mathbf{Q}(\cos\theta))); \tag{24}$$

therefore, to verify E_{r+1}, it suffices that it projects to a valid equation on each factor of (24). One shows, without much difficulty, that E_{r+1} projects to E_r on the first factor of (24).

Next, observe that both 4 and $k_*(2)$ project to 0 in the second factor of (24). Now, $M_2(\mathbf{Q}(\cos\theta))$ is Morita equivalent (in the standard way) to $\mathbf{Q}(\cos\theta)$; via which, we identify $\mathcal{M}(M_2(\mathbf{Q}(\cos\theta)))$ to $\mathcal{M}(\mathbf{Q}(\cos\theta))$—the ordinary Wittring of the field $\mathbf{Q}(\cos\theta)$. After this identification, $l_*(2)$ clearly projects to $4 \in \mathcal{M}(\mathbf{Q}(\cos\theta))$; also, $i_*(2)$ goes to 2, while $j_*(2)$ projects to the element represented by the form $(1+\sin\theta) \perp (1+\sin\theta)$. Since 2 is the sum of two squares $(2 = 1^2 + 1^2)$, $(1+\sin\theta) \perp (1+\sin\theta)$ and $(2+2\sin\theta) \perp (2+2\sin\theta)$ represent the same element. But, $2+2\cos\theta$ is also the sum of two squares in $\mathbf{Q}(\cos\theta)$; namely,

$$2+2\sin\theta = (\cos\theta)^2 + (1+\sin\theta)^2. \tag{25}$$

(Note that $\sin\theta \in \mathbf{Q}(\cos\theta)$ since $\theta = \pi/2^r$.) Hence, $(2+2\sin\theta) \perp (2+2\sin\theta)$ and $1 \perp 1$ represent the same element in $\mathcal{M}(\mathbf{Q}(\cos\theta))$; namely, 2.

§4. EXAMPLE

Let D be the infinite dihedral group generated by α, γ subject to relations $\alpha^2 = 1$ and $\alpha\gamma\alpha^{-1} = \gamma^{-1}$, $D(n)$ the subgroup of index n generated by α and γ^n, and T_n the normal subgroup generated by γ^n. Note $D(n)$ is isomorphic to D and T_n is infinite cyclic; $T_n \subset D(n) \subset D$; denote these inclusions by i and j_n, respectively. Equip D with the trivial homomorphism $\omega: D \to \{\pm 1\}$ and let $\mathbf{Z}_2$ denote the cyclic group of order 2. Let $\beta_n = \gamma^n\alpha$ and $(\alpha), (\beta_n)$ denote the subgroups of $D(n)$ generated by these elements. (These subgroups are cyclic of order 2.) Wall ([11], p. 162) shows $L_3(\mathbf{Z}(\alpha)) = \mathbf{Z}_2 = L_3(\mathbf{Z}(\beta_n))$; identify the sum of their images in $L_3(\mathbf{Z}D(n))$ with $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.

THEOREM 4.1. *Either $L_3(\mathbf{Z}D)$ is $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ or it is not finitely generated*

We deduce this from two lemmas whose proofs are postponed to the end of this section. When $j: G \to H$ is an inclusion where G is a subgroup with finite index in H, recall there is a transfer map $j^*: L_*(\mathbf{Z}H) \to L_*(\mathbf{Z}G)$.

LEMMA 4.2. *To each $x \in L_3(\mathbf{Z}D)$ corresponds an integer N_x such that*

$$j_p^*(x) \in \mathbf{Z}_2 \oplus \mathbf{Z}_2$$

for all primes $p \geq N_x$.

LEMMA 4.3. *When p is an odd prime,*

$$j_p^* j_{p*}(x) = x + \frac{p-1}{2} i_* i^*(x)$$

for all $x \in L_3(\mathbf{Z}D(p))$.

Proof of Theorem 4.1. By [3],

$$L_3(\mathbf{Z}D(n)) = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathrm{UNil}_3(\mathbf{Z}) \tag{26}$$

where $\mathbf{Z}$ has the trivial involution. Our proof is by contradiction, hence assume $\mathrm{UNil}_3(\mathbf{Z})$ is non-zero but finitely generated. Since $\mathrm{UNil}_3(\mathbf{Z})$ is a quotient group (by definition) of $\mathrm{UNil}_4(\mathbf{Z}[H]; \mathbf{Z}[\hat{G}_1], \mathbf{Z}[\hat{G}_2])$ for appropriate choices of H, G_1, and G_2, its exponent divides 4 (Theorem 1.3); in particular, $L_3(\mathbf{Z}D(n))$ is a finite group annihilated by 4. It is well known there are arbitrarily large primes of the form $8m + 1$, hence there is a prime p such that

$$\begin{gathered} j_p^*: L_3(\mathbf{Z}D) \to \mathbf{Z}_2 \oplus \mathbf{Z}_2 \underset{\neq}{\subset} L_3(\mathbf{Z}D(p)), \text{ and} \\ (j_p)^*(j_p)_* = \text{identity}: L_3(\mathbf{Z}D(p)) \to L_3(\mathbf{Z}D(p)). \end{gathered} \tag{27}$$

(Use Lemmas 4.2 and 4.3.) But, (27) is self-contradictory.

It remains to discuss Lemmas 4.2 and 4.3. The first can be proven geometrically. Let N be a 10-dimensional, connected, orientable manifold containing a simply connected (connected), codimension-1 sub-manifold M which separates N into two components A and B with cyclic fundamental groups of order 2 and universal covers diffeomorphic to $M \times [0, 1]$. (Such spaces are easily constructed.) Note that $\pi_1 N \simeq D$ and its universal cover is diffeomorphic to $M \times \mathbf{R}$. By Wall ([11], p. 66), each $x \in L_3(\mathbf{Z}D)$ determines a surgery problem

$$\begin{gathered} f: W \to N \times [0, 1], \text{ with} \\ f_-: \partial_- W \to N \times 0 \quad \text{the identity map} \end{gathered} \tag{28}$$

and having obstruction x. Associated to $D(p) \subset D$, we have p-sheeted covers $\hat{N}$, $\hat{W}$ and an induced surgery problem

$$\hat{f}: \hat{W} \to \hat{N} \times [0, 1] \tag{29}$$

with obstruction $j_p^*(x)$. Now, M lifts to $\hat{N}$ and

$$\hat{f}_+ : \partial_+ \hat{W} \to \hat{N} \times 1 \tag{30}$$

splits along M for all p sufficiently large by Browder's result[1]. Making $\hat{f}$ transverse to the rest of $M \times [0, 1]$ and completing surgery on this membrane, we see that $j_p^*(x)$ is the sum of elements coming from $L_3(\mathbf{Z}(\alpha))$ and $L_3(\mathbf{Z}(\beta_p))$.

Finally, Lemma 4.3 would be an immediate consequence of the Mackey subgroup property. Dress ([6], p. 302) shows that L-theory satisfies such a property for finite groups and subgroups. It's probably true for arbitrary groups and subgroups of finite index. In any event, a simple direct argument, similar to that used to prove ([7], Lemma 2.7), can be given for Lemma 4.3; the details are left to the reader.

Remark 4.4. Our proof of Theorem 4.1 was motivated by Cappell's paper[4] where he showed that $L_2(ZD)$ is not finitely generated.

REFERENCES

1. W. BROWDER: Structures on $M \times \mathbf{R}$, *Proc. Camb. Phil. Soc.* **61** (1965), 337–345.
2. W. BROWDER and J. LEVINE: Fibering manifolds over a circle, *Comm. Math. Helv.* **40** (1966), 153–160.
3. S. E. CAPPELL: Unitary nilpotent groups and Hermitan K-theory. I, *Bull. Amer. Math. Soc.* **80** (1974), 1117–1122.
4. S. E. CAPPELL: On connected sums of manifolds, *Topology* **13** (1974), 395–400.
5. F. X. CONNOLLY: Linking numbers and surgery, *Topology* **12** (1973), 389–410.
6. A. DRESS: Induction and structure theorems for orthogonal representations of finite groups, *Ann. of Math.* **102** (1975), 291–325.
7. F. T. FARRELL and W. C. HSIANG: Rational L-groups of Bierberbach groups, *Comm. Math. Helv.* **52** (1977), 89–109.
8. A. FRÖHLICH and A. M. MCEVETT: Forms over rings with involution, *J. Alg.* **12** (1969), 79–104.
9. D. S. PASSMANN and T. PETRIE: Surgery with coefficients in a field, *Ann. of Math.* **95** (1972), 385–405.
10. A. RANICKI: The algebraic theory of surgery, preprint.
11. C. T. C. WALL: *Surgery on Compact Manifolds*. Academic Press, New York (1970).
12. C. T. C. WALL: Classification of Hermitian forms. VI Group rings, *Ann. of Math.* **103** (1976), 1–80.

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