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EXTENSION OF EIGENFUNCTION–EXPANSION SOLUTIONS OF A FOKKER–PLANCK EQUATION—I. FIRST ORDER SYSTEM

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Abstract—The work is concerned with eigenfunction–expansion solutions to the forward Fokker–Planck equation associated with a specific, non-linear, first-order system subject to white noise excitation. Using a digital computer, a substantial number of new terms in the expansions have been generated. With this new information, inverted Domb–Sykes plots revealed a pattern in the coefficients for certain ranges of values of the parameters. Through this pattern, Dingle's theory of terminants was used to recast the series into a more favorable computational form.

1. INTRODUCTION

Random excitation of non-linear systems is a challenging field of perennial interest (see Caughey's review [1]) and several approaches to obtaining solutions to associated Fokker–Planck equations have been developed. The present work is concerned with eigenfunction–expansion procedures, as described by, for example, Atkinson [2]. Specifically, the work of Payne [3, 4] on a first-order, weakly non-linear system is considerably extended. This is achieved by means of tools developed elsewhere for the analysis and improvement of perturbation series (see Van Dyke [5]). Also, the theory of terminants developed by Dingle [6] is harnessed in the work.

The spirit of the approach is to obtain a sufficient number of computer generated terms that a pattern, if it exists, emerges. This pattern is then used to recast the expansion into a more favorable computational form. Using this approach, a substantial body of new information on the steady-state, mean square response of a specific first-order system to a white noise excitation is presented.

2. EIGENFUNCTION–EXPANSION SOLUTIONS

Following Payne [3, 4], the system considered is

$$\frac{dx}{dt}+x+\varepsilon x^3=n(t) \tag{1}$$

where ε is a small parameter and $n(t)$ is a white noise process with the properties: (i) $n(t_i)$, $i=1, 2\ldots$, are mutually independent and (ii) $n(t)$ has a Gaussian probability distribution with $E[n(t)]=0$, $E[n(t)n(s)]=2D\delta(t-s)$, E denoting expected value, δ being the delta function, and D a constant which measures the white noise intensity. The response x is modeled as a Markov process and the forward Fokker–Planck equation associated with equation (1) is

$$\frac{\partial p}{\partial t}=\frac{\partial}{\partial x}[(x+\varepsilon x^3)p]+D\frac{\partial^2 p}{\partial x^2} \qquad t>0 \tag{2}$$

where $p=p(x, t|x_0; \varepsilon)$ is the transition probability density and satisfies

$$\int_{-\infty}^{\infty} p(x, t|x_0; \varepsilon)\,dx=1 \tag{3}$$

$$\lim_{t \to 0} \int_{-\infty}^{\infty} p(x, t|x_0; \varepsilon) h(x)\, \mathrm{d}x = h(x_0) \tag{4}$$

where $h(x)$ is an arbitrary continuous function. Also, on integrating (2) with respect to x and using (3), it follows that

$$\lim_{x \to \pm\infty} \left[D \frac{\partial p}{\partial x} + (x + \varepsilon x^3) p \right] = 0. \tag{5}$$

Introducing the variable $\zeta = x/\sqrt{2D}$, the expression

$$p(\zeta, t|\zeta_0; \varepsilon) = p_s(\zeta; \varepsilon) \sum_{n=0}^{\infty} v_n(\zeta_0; \varepsilon) v_n(\zeta; \varepsilon) \exp[-\lambda_n(\varepsilon)] t \tag{6}$$

where the subscript s stands for 'steady-state', is a solution to (2) provided

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} - 2\zeta \frac{\mathrm{d}}{\mathrm{d}\zeta} + 2\lambda_n(\varepsilon) \right] v_n(\zeta; \varepsilon) = 4\varepsilon D \zeta^3 \frac{\mathrm{d}}{\mathrm{d}\zeta} v_n(\zeta; \varepsilon). \tag{7}$$

It can readily be shown that the steady state solution is

$$p_s(\zeta; \varepsilon) = [\exp(-\zeta^2 - \varepsilon D \zeta^4)] / \left[\int_{-\infty}^{\infty} \exp(-\eta^2 - \varepsilon D \eta^4)\, \mathrm{d}\eta \right]. \tag{8}$$

Also, by definition

$$v_0(\zeta_0; \varepsilon) = 1,\ v_0(\zeta; \varepsilon) = 1,\ \lambda_0(\varepsilon) = 0. \tag{9}$$

Payne has shown that (2) and (7) constitute an eigenvalue problem. Moreover, he gave conditions on p, assumed to be true here, under which the spectrum would be discrete. Thus, (6) constitutes an eigenfunction–expansion solution. The eigenfunctions have the orthogonality property

$$\int_{-\infty}^{\infty} p_s(\zeta; \varepsilon) v_n(\zeta; \varepsilon) v_m(\zeta; \varepsilon)\, \mathrm{d}\zeta = \delta_{nm} \tag{10}$$

where δ_{nm} denotes the Kronecker delta.

When $\varepsilon = 0$, (1) is linear and then (7) reduces to Hermite's equation, the solution to which is

$$v_{n0}(\zeta) = H_n(\zeta)/\sqrt{2^n n!} \tag{11}$$

$$\lambda_{n0} = n \tag{12}$$

where the Hermite polynomial is defined by

$$H_n(\zeta) = n! \sum_{k=0}^{k \le n/2} \frac{(-1)^k}{k!(n-2k)!} (2\zeta)^{n-2k}. \tag{13}$$

In (11) and (12) the additional subscript 0 designates the $\varepsilon = 0$ case. Equations (8) and (10) now combine to give

$$\int_{-\infty}^{\infty} p_s(\zeta) v_{n0}(\zeta) v_{m0}(\zeta)\, \mathrm{d}\zeta = \delta_{nm} \tag{14}$$

where

$$\sqrt{\pi} p_s(\zeta) = \mathrm{e}^{-\zeta^2}. \tag{15}$$

When $\varepsilon \neq 0$, but small, one may think of the process as perturbing the eigenfunctions v_{n0} and eigenvalues λ_{n0}. Within this conceptual framework the following expansions are set forth (see Courant and Hilbert [7]):

$$v_n(\zeta; \varepsilon) = \sum_{j=0}^{\infty} v_{nj}(\zeta) \varepsilon^j \tag{16}$$

$$\lambda_n(\varepsilon) = \sum_{j=0}^{\infty} \lambda_{nj} \varepsilon^j. \tag{17}$$

Substituting Eqs. (16) and (17) into Eq. (7), and setting the coefficients of the various powers of ε separately to zero yields

$$Lv_{nj}+2\sum_{i,k=0}^{\infty}\delta(j-i-k)\lambda_{ni}v_{nk}=4D\zeta^3v'_{n(j-1)}\qquad j=0,1,2,\ldots \tag{18}$$

where the operator L is given by

$$L=\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}-2\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta} \tag{19}$$

and the prime denotes differentiation w.r.t. ζ. Here, and throughout, any quantity with a negative subscript is to be understood as having a value of zero.

Still another expansion is introduced at this stage, namely

$$v_{ni}(\zeta)=\sum_{j=0}^{\infty}a_{nij}v_{j0}(\zeta)\qquad i=0,1,2,\ldots \tag{20}$$

where the a_{nij} are constants to be determined and the v_{j0} are the scaled Hermite polynomials given by equation (11). Using their orthogonality property, (14), the a_{nij} may be written

$$a_{nij}=(v_{ni},v_{j0}) \tag{21}$$

where an inner product notation has been introduced, namely

$$(u,v)=\int_{-\infty}^{\infty}p_s(\zeta)u(\zeta)v(\zeta)\,\mathrm{d}\zeta. \tag{22}$$

Recursion relations will now be developed. Multiplying (18) by $p_s(\zeta)v_{l0}(\zeta)$ and integrating gives

$$(v_{l0},Lv_{nj})+2\sum_{i,k=0}^{\infty}\delta(j-i-k)\lambda_{ni}(v_{l0},v_{nk})=4D(v_{l0},\zeta^3v'_{n(j-1)})\qquad j=0,1,2,\ldots \tag{23}$$

L is a self adjoint operator, i.e. $(Lu,v)=(u,Lv)$. Using this together with equation (7) with $\varepsilon=0$ and (21), (23) may be written

$$2\sum_{i,k=0}^{\infty}\delta(j-i-k)\lambda_{ni}a_{nkl}-2\lambda_{l0}a_{njl}=4D(v_{l0},\zeta^3v'_{n(j-1)})\qquad j=0,1,2,\ldots. \tag{24}$$

To obtain recursion relations for the λ's, consider $l=n$. Using a property of the delta function, (24) may be written

$$\sum_{i=1}^{j}\lambda_{ni}a_{n(j-i)n}=2D(v_{n0},\zeta^3v'_{n(j-i)})\qquad j=0,1,2,\ldots. \tag{25}$$

Equations (12) and (25) give the recursion relations

$$\lambda_{n0}=n \tag{26}$$

$$\lambda_{n1}=2D(v_{n0},\zeta^3v'_{n1}) \tag{27}$$

$$\lambda_{nj}=2D(v_{n0},\zeta^3v'_{nj})-\sum_{i=1}^{j-1}\lambda_{ni}a_{n(j-i)n}\qquad j\geq 2. \tag{28}$$

Consider now $l\neq n$. Equations (12) and (24) give

$$(n-l)a_{njl}+\sum_{i=1}^{\infty}\lambda_{ni}a_{n(j-i)l}=2D(v_{l0},\zeta^3v'_{n(j-1)})\qquad j=0,1,2,\ldots \tag{29}$$

from which it follows that

$$a_{n0l}=\delta_{nl}$$

$$a_{njl}=\frac{1}{n-l}\left[2D(v_{l0},\zeta^3v'_{n(j-1)})-\sum_{i=1}^{j}\lambda_{ni}a_{n(j-i)l}\right] \tag{30}$$

$$n\neq l,\ j=1,2,\ldots.$$

The recursion relation for a_{nin} must now be calculated. Expanding exp $(-\varepsilon D\zeta^4)$ and exp $(-\varepsilon D\eta^4)$ in (8) in Maclaurin series can be shown to lead to

$$p_s(\zeta;\varepsilon)=\frac{e^{-\zeta^2}}{\sqrt{\pi}}\left[\sum_{n=0}^{\infty}\frac{(-D\zeta^4)^n}{n!}\varepsilon^n\right]\Big/\left[\sum_{n=0}^{\infty}\frac{(-D\varepsilon)^n}{n!}\int_{-\infty}^{\infty}\eta^{4n}e^{-\zeta^2}d\eta\right]$$
$$=\frac{e^{-\zeta^2}}{\sqrt{\pi}}\left[\sum_{n=0}^{\infty}\frac{(-D\zeta^4)^n}{n!}\varepsilon^n\right]\Big/\left[\sum_{n=0}^{\infty}\frac{(-D)^n[1.3.5.\cdots(4n-1)]}{2^{2n}n!}\varepsilon^n\right]. \tag{31}$$

When the indicated division of series is performed, (31) can be written

$$p_s(\zeta;\varepsilon)=\frac{e^{-\zeta^2}}{\sqrt{\pi}}\sum_{n,m=0}^{\infty}\frac{(-D\zeta^4)^n}{n!}b_m\varepsilon^{n+m} \tag{32}$$

where

$$b_0=1,\ b_m=-\sum_{j=1}^{m}\frac{(-D)^j[1.3.5.\cdots(4j-1)]}{2^{2j}j!}b_{m-j}\qquad m\geqslant 1. \tag{33}$$

Equations (10), (16), (20) and (32) give, on interchanging summation and integration

$$\sum_{i,j,k,l=0}^{\infty}\frac{(-D)^{i+j}b_j}{i!}\sum_{r,s=0}^{\infty}a_{nkr}a_{mls}(\zeta^{4i}v_{r0},v_{s0})\varepsilon^{i+j+k+l}=\delta_{nm}. \tag{34}$$

Let $m=n$ and group according to powers of ε, denoting the coefficient of ε^N by c_N. To satisfy (34) with $n=m$, it must follow that $c_0=1$, which can readily be shown to be true, and $c_N=0$, $N>0$. This latter condition together with (34), requires that

$$\sum_{i,j,k,l=0}^{N}\delta(N-i-j-k-l)\frac{(-D)^{i+j}}{i!}b_j\sum_{r,s=0}^{\infty}a_{nkr}a_{nls}(\zeta^{4i}v_{r0},v_{s0})=0\qquad N>0. \tag{35}$$

By systematically isolating terms of the form a_{nkn} and a_{nln} in (35), the recursion relation for a_{nNn} can be obtained as

$$\begin{aligned}a_{nNn}=&-\tfrac{1}{2}\sum_{i,j,k,l=0}^{N}\delta(N-i-j-k-l)\frac{(-D)^{i+j}}{i!}b_j\Bigg[\sum_{r,s=0}^{n-1}a_{nkr}a_{nls}(\zeta^{4i}v_{r0},v_{s0})\\&+\sum_{r=0}^{n-1}\sum_{s=n+1}^{\infty}(a_{nkr}a_{nls}+a_{nks}a_{nlr})(\zeta^{4i}v_{r0},v_{s0})+\sum_{r,s=n+1}^{\infty}a_{nkr}a_{nls}(\zeta^{4i}v_{r0},v_{s0})\Bigg]\\&-\tfrac{1}{2}\sum_{i,j=0}^{N}\sum_{k,l=0}^{N-1}\delta(N-i-j-k-l)\frac{(-D)^{i+j}}{i!}b_j\Bigg[\sum_{r=0}^{n-1}(a_{nkr}a_{nln}+a_{nkn}a_{nlr})(\zeta^{4i}v_{r0},v_{n0})\\&+a_{nkn}a_{nln}(\zeta^{4i}v_{n0},v_{n0})+\sum_{r=n+1}^{\infty}(a_{nkn}a_{nlr}+a_{nkr}a_{nln})(\zeta^{4i}v_{r0},v_{n0})\Bigg]\qquad N>0.\end{aligned} \tag{36}$$

Finally, the requirements of (9), together with (16), (17) and (20), can be shown to yield

$$\lambda_{0j}=0\qquad \text{all } j \tag{37}$$

$$a_{0ij}=0\qquad i>0. \tag{38}$$

The recursion relations for the eigenvalues and the eigenfunctions can be employed once the various inner product expressions have been evaluated, a task postponed until later. First an expression for the mean square response of the system will be developed. The autocorrelation function of the response, which is assumed to be a stationary Markov process, is given by, where τ denotes a time lag

$$R_{xx}(\tau)=2DE[\zeta(t),\zeta(t+\tau)]=2D\iint_{-\infty}^{\infty}\zeta_{t+\tau}\zeta_t p(\zeta_{t+\tau},t+\tau|\zeta_t,t)p(\zeta_t,t|\zeta_0)\,d\zeta_t\,d\zeta_{t+\tau}$$

or, noting that a stationary process is independent of a shift in the time origin

$$R_{xx}(\tau)=2D\lim_{t\to\infty}\iint_{-\infty}^{\infty}\zeta_{t+\tau}\zeta_t p(\zeta_{t+\tau},t+\tau|\zeta_t,t)p(\zeta_t,t|\zeta_0)\,d\zeta_t\,d\zeta_{t+\tau}. \tag{39}$$

Using (6) and taking the limit inside the integral, (39) gives on rearranging

$$R_{xx}(\tau)=2D\sum_{n=0}^{\infty}\alpha_n^2(\varepsilon)\exp\left[-\lambda_n(\varepsilon)\tau\right] \tag{40}$$

where

$$\alpha_n(\varepsilon)=\int_{-\infty}^{\infty}\zeta p_s(\zeta;\varepsilon)v_n(\zeta;\varepsilon)\mathrm{d}\zeta. \tag{41}$$

By means of (6), (16) and (20), Eq. (41) can be written

$$\alpha_n(\varepsilon)=\sum_{i,j,k,l=0}^{\infty}a_{nkl}\frac{(-D)^i b_j}{i!}\varepsilon^{i+j+k}\int_{-\infty}^{\infty}p_s(\zeta)\zeta^{4i+1}v_{l0}(\zeta)\mathrm{d}\zeta. \tag{42}$$

Noting that $v_{10}(\zeta)=\sqrt{2}\zeta$ and using (22), (42) gives

$$\alpha_n(\varepsilon)=\sum_{i,j,k,l=0}^{\infty}a_{nkl}\frac{(-D)^i}{\sqrt{2}i!}b_j(\zeta^{4i}v_{10},v_{l0})\varepsilon^{i+j+k}. \tag{43}$$

Setting $\tau=0$ and employing (40) and (43), the mean square response is obtained as

$$E[x^2]=\sum_{m=0}^{\infty}a_m\varepsilon^m \tag{44}$$

where

$$a_m=2D\sum_{n=0}^{\infty}\sigma_{nm} \tag{45}$$

$$\sigma_{nm}=\tfrac{1}{2}\left[\sum_{i,j,k,p,q,r=0}^{m}\delta(m-i-j-k-p-q-r)\frac{b_jb_q}{i!p!}(-D)^{i+p}\right]$$
$$\times\left[\sum_{l,s=0}^{\infty}a_{nkl}a_{nrs}(\zeta^{4i}v_{10},v_{l0})(\zeta^{4p}v_{10},v_{s0})\right]. \tag{46}$$

The recursion relations involve inner products of the form $(v_{i0},\zeta^3v'_{jk})$ and (ζ^iv_{j0},v_{k0}) and these will now be evaluated. Noting that the Hermite polynomials satisfy

$$H'_0(\zeta)=0,\ H'_n(\zeta)=2nH_{n-1}(\zeta)$$

it follows from (11) and (20) that in fact all the inner products can be reduced to an evaluation of the general form (ζ^iv_{j0},v_{k0}), which can be written

$$(\zeta^iv_{j0},v_{k0})=\frac{1}{\sqrt{2^{j+k}j!k!}}\int_{-\infty}^{\infty}p_s(\zeta)\zeta^iH_j(\zeta)H_k(\zeta)\mathrm{d}\zeta. \tag{47}$$

Using (13) and (15), (47) can be shown to yield

$$(\zeta^iv_{j0},v_{k0})=\sqrt{2^{j+k}j!k!}\sum_{l=0}^{l\leq j/2}\sum_{m=0}^{m\leq k/2}\frac{E(i+j+k-2(l+m))}{(-4)^{l+m}(j-2l)!(k-2m)!l!m!} \tag{48}$$

where

$$E(N)=\begin{cases}0, & N\text{ odd}\\ 1, & N=0\\ \dfrac{1.3.5\ldots(N-1)}{2^{N/2}} & N\text{ even.}\end{cases} \tag{49}$$

By means of (48), the recursion relations for the eigenvalues and eigenfunctions, and the perturbation expansion for the mean square response can be calculated to any desired order using a digital computer.

3. COMPUTER EXTENSION OF PERTURBATION SOLUTIONS

The recursion relations to obtain the eigenvalues and the eigenfunction expansion coefficients and the formulas for the perturbation expansion of the mean square response were programmed on the Ford Motor Company Honeywell 6000 computer in the STRAN (Structured Fortran) computer language. Not surprisingly, the programming involved considerable effort and the authors would be pleased to supply interested readers with details on program listings and organization.

Payne [4] presented results for the present system out to the second order. Here the perturbations were carried out to the eighth order† for values of the white noise intensity D ranging between 0.0001 to 100 and this substantial body of new information is given in Table 1. Agreement with Payne to the second order should be noted.‡

Table 1. Mean square response perturbation coefficients for various values of the intensity D

	$D=0.0001$	$D=0.001$	$D=0.01$
a_0	0.1000000E-03	0.1000000E-02	0.1000000E-01
a_1	−0.1500450E-03	−0.1504500E-02	−0.1545000E-01
a_2	−0.4874324E-03	−0.4868214E-02	−0.14803938E-01
a_3	−0.4106030E-02	−0.4104062E-01	−0.4084888E 00
a_4	−0.5262706E-01	−0.5261046E-00	−0.5244580E 01
a_5	−0.8891938E-00	−0.8889810E-01	−0.8868638E 02
a_6	−0.1853723E 02	−0.1853363E 03	−0.1849780E 04
a_7	−0.4588644E 03	−0.4587894E 04	−0.4580418E 05
a_8	−0.1314790E 05	−0.1314604E 06	−0.1312752E 07

	$D=0.1$	$D=0.2$	$D=0.5$
a_0	0.1000000E 00	0.2000000E 00	0.5000000E 00
a_1	−0.1950000E 00	−0.4800000E 00	−0.1875000E 01
a_2	−0.3843750E 00	−0.4200000E 00	−0.3703125E 01
a_3	−0.3981656E 01	−0.8424000E 01	−0.4756640E 02
a_4	−0.5082874E 02	−0.9621540E 02	−0.1294970E 00
a_5	−0.8670164E 03	−0.1703229E 04	−0.6158097E 04
a_6	−0.1815437E 05	−0.3555803E 05	−0.5913441E 05
a_7	−0.4508292E 06	−0.8868724E 06	−0.2460330E 07
a_8	−0.1294787E 08	−0.2551653E 08	−0.5595717E 08

	$D=0.7$	$D=1$	$D=5$
a_0	0.7000000E 00	0.1000000E 01	0.5000000E 01
a_1	−0.3255000E 01	−0.6000000E 01	−0.1200000E 03
a_2	−0.1211438E 02	0.3750000E 02	0.4597501E 04
a_3	−0.1355898E 03	−0.4860000E 03	−0.2647349E 06
a_4	−0.8872697E 03	0.6505884E 04	0.2037707E 08
a_5	−0.2135682E 05	−0.1401909E 06	−0.1976074E 10
a_6	−0.1454986E 06	0.2938146E 07	0.2317035E 12
a_7	−0.7849864E 07	−0.8865712E 08	−0.3188347E 14
a_8	−0.2337453E 08	0.2493584E 10	0.4928034E 16

	$D=10$	$D=100$
a_0	0.1000000E 02	0.1000000E 03
a_1	−0.4650000E 03	−0.4515000E 05
a_2	0.3625126E 05	0.3569202E 08
a_3	−0.4186026E 07	−0.4139694E 11
a_4	0.6461296E 09	0.6407094E 14
a_5	−0.1255596E 12	−0.1247123E 18
a_6	0.2947952E 14	0.2931510E 21
a_7	−0.8118050E 16	−0.8079800E 24
a_8	0.2510452E 19	0.2499916E 28

†After this order, computer costs became prohibitively high.

‡The authors confirmed with Payne in a private communication that there is a typographical error in Reference [4]. The term $\lambda n_1 a_{nn}$, in his notation, should be subtracted from 48.

These results, though valuable in themselves, can be made considerably more useful by means of the theory of terminants advanced by Dingle [6], and recently used by Buchanan [8] in a study on improvement of series representations. A basic aim of Dingle's work is the analysis and improvement of divergent asymptotic series, and inspection of Table 1 clearly shows that the series at hand are divergent. It should be noted that the sign pattern of the series changes from a single-sign series for small values of the intensity D to an alternating-sign series for large values of D, with the exception of the range $0.2 < D < 1.0$. These values represent a transition region and, presumably, with more terms of the series available a stable pattern would emerge. They will not be pursued any further in this work. The overall sign pattern indicates that for small values of D, the expansion contains a Stoke's discontinuity, but that this does not occur for large values of D.

Further insight can be gained by constructing inverted Domb–Sykes plots, that is, plots of a_{n-1}/a_n versus $1/n$, a feature impossible before this work since sufficient data was not available. Figure 1 shows such a plot for $D = 0.0001$, and is a typical result in that the

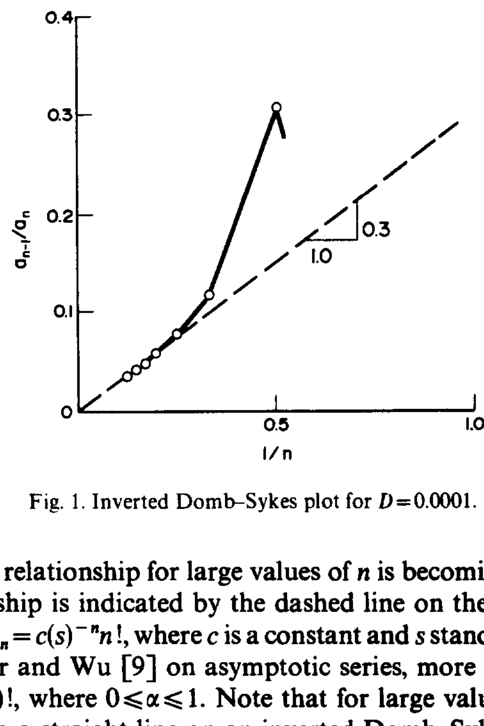

Fig. 1. Inverted Domb–Sykes plot for $D = 0.0001$.

establishing of a linear relationship for large values of n is becoming apparent. The best-fit slope for that relationship is indicated by the dashed line on the figure. This behavior is reproduced by taking $a_n = c(s)^{-n}n!$, where c is a constant and s stands for slope, but following related work of Bender and Wu [9] on asymptotic series, more flexibility is obtained by taking $a_n = c(s)^{-n}(n+\alpha)!$, where $0 \leqslant \alpha \leqslant 1$. Note that for large values of n this gives a very good approximation to a straight line on an inverted Domb–Sykes plot. The parameters c, s and α are to be determined using the higher order terms in the series. Adopting this procedure the general form of the series coefficients can be determined and is given in Table 2.

With this information the theory of terminants can now be employed. Consider the

Table 2. General form of the coefficients of the asymptotic series representation of the mean square response for various values of the intensity D

D	n-th order coefficient a_n
0.0001	$-0.720118\text{E-}05 \times (1/0.3)^n(n+\frac{1}{2})!$
0.001	$-0.719998\text{E-}04 \times (1/0.3)^n(n+\frac{1}{2})!$
0.01	$-0.718805\text{E-}03 \times (1/0.3)^n(n+\frac{1}{2})!$
0.1	$-0.707305\text{E-}02 \times (1/0.3)^n(n+\frac{1}{2})!$
5	$0.491496\text{E } 01 \times (-1/0.05)^n n!$
10	$0.977591\text{E } 01 \times (-1/0.025)^n n!$
100	$0.972859\text{E } 02 \times (-1/0.0025)^n n!$

case $D=0.0001$. The series can be written

$$E[x^2]=\sum_{n=0}^{5} a_n\varepsilon^n-(0.720118E\text{-}05)\sum_{n=6}^{\infty}(n+\tfrac{1}{2})!\left(\frac{\varepsilon}{0.3}\right)^n. \tag{50}$$

Using Dingle's terminant for a single-sign asymptotic series, equation (50) can be written in the much more useful form

$$E[x^2]=\sum_{n=0}^{5} a_n\varepsilon^n-(0.720118E-05)(6+\tfrac{1}{2})!\left(\frac{\varepsilon}{0.3}\right)^6\bar{\Lambda}_{6+1/2}\left(-\frac{0.3}{\varepsilon}\right) \tag{51}$$

where the terminant $\bar{\Lambda}$, a tabulated function, is given by

$$\bar{\Lambda}_m(-x)=\frac{1}{m!}P\int_0^{\infty}\frac{\zeta^m e^{-\zeta}}{1-\zeta/x}\mathrm{d}\zeta \tag{52}$$

P denoting principal value. The results for the other values of D are given in Table 3.

Table 3. Terminated asymptotic series representing the mean square response for various values of the intensity D

D	$E[x^2]$
0.001	$\sum_{n=0}^{5} a_n\varepsilon^n-(0.719998\text{E-}04)(6+\frac{1}{2})!\left(\frac{\varepsilon}{0.3}\right)^6\bar{\Lambda}_{6+1/2}(-0.3/\varepsilon)$
0.01	$\sum_{n=0}^{5} a_n\varepsilon^n-(0.718805\text{E-}03(6+\frac{1}{2})!\left(\frac{\varepsilon}{0.3}\right)^6\bar{\Lambda}_{6+1/2}(-0.3/\varepsilon)$
0.1	$\sum_{n=0}^{5} a_n\varepsilon^n-(0.707305\text{E-}02)(6+\frac{1}{2})!\left(\frac{\varepsilon}{0.3}\right)^6\bar{\Lambda}_{6+1/2}(-0.3/\varepsilon)$
5	$\sum_{n=0}^{5} a_n\varepsilon^n+(0.491496\text{E }01)6!\left(\frac{\varepsilon}{0.05}\right)^6\Lambda_6(0.05/\varepsilon)$
10	$\sum_{n=0}^{5} a_n\varepsilon^n+(0.977591\text{E }01)6!\left(\frac{\varepsilon}{0.025}\right)^6\Lambda_6(0.025/\varepsilon)$
100	$\sum_{n=0}^{5} a_n\varepsilon^n+(0.972859\text{E }02)6!\left(\frac{\varepsilon}{0.0025}\right)^6\Lambda_6(0.0025/\varepsilon)$

In this table the terminant Λ, also a tabulated function, is given by

$$\Lambda_m(x)=\frac{1}{m!}\int_0^{\infty}\frac{\zeta^m e^{-\zeta}}{1+\zeta/x}\mathrm{d}\zeta. \tag{53}$$

Dingle has also developed absolutely convergent expansions for the terminants Λ and $\bar{\Lambda}$. Using them, absolutely convergent representations of the mean square response can be found and are given in Table 4.

In Table 4

$$\Sigma(\varepsilon)=\frac{-(0.3/\varepsilon)}{6.5}\left[1+\frac{(0.3/\varepsilon)}{5.5}+\frac{(0.3/\varepsilon)^2}{(5.5)(4.5)}+\dots\right] \tag{54}$$

$$\Omega(\varepsilon)=\frac{\pi(0.3/\varepsilon)^{7.5}\exp(-0.3/\varepsilon)}{(6.5)!\tan 6.5\pi} \tag{55}$$

$$\theta(f,\varepsilon)=6!\left(\frac{\varepsilon}{f}\right)^6\left\{\frac{(f/\varepsilon)}{6}\left[1-\frac{(f/\varepsilon)}{5}+\dots-\frac{(f/\varepsilon)^5}{5!}\right]-\frac{(-f/\varepsilon)^7}{6!}\sum_{t=0}^{\infty}\frac{\psi(t)-\ln(f/\varepsilon)}{t!}(f/\varepsilon)^t\right\} \tag{56}$$

$$\psi(t)=\frac{1}{\Gamma(t+1)}\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(t+1) \tag{57}$$

Γ denoting the gamma function.

For purposes of comparison, calculations based on (i) linear system, (ii) first-order

Table 4. Absolutely convergent expansion representation of the mean square response for various values of the intensity D

D	$E[x^2]$
0.0001	$\sum_{n=0}^{5} a_n\varepsilon^n - (0.720118E\text{-}05)(6.5)!\left(\frac{\varepsilon}{0.3}\right)^6[\Sigma(\varepsilon)+\Omega(\varepsilon)]$
0.001	$\sum_{n=0}^{5} a_n\varepsilon^n - (0.719998E\text{-}04)(6.5)!\left(\frac{\varepsilon}{0.3}\right)^6[\Sigma(\varepsilon)+\Omega(\varepsilon)]$
0.01	$\sum_{n=0}^{5} a_n\varepsilon^n - (0.718805E\text{-}03)(6.5)!\left(\frac{\varepsilon}{0.3}\right)^6[\Sigma(\varepsilon)+\Omega(\varepsilon)]$
0.1	$\sum_{n=0}^{5} a_n\varepsilon^n - (0.707305E\text{-}02)(6.5)!\left(\frac{\varepsilon}{0.3}\right)^6[\Sigma(\varepsilon)+\Omega(\varepsilon)]$
5	$\sum_{n=0}^{5} a_n\varepsilon^n + (0.491496E\ 01)\theta(0.05\varepsilon)$
10	$\sum_{n=0}^{5} a_n\varepsilon^n + (0.977591E\ 01)\theta(0.025\varepsilon)$
100	$\sum_{n=0}^{5} a_n\varepsilon^n + (0.972859E\ 02)\theta(0.0025\varepsilon)$

expansion, (iii) second-order expansion, and (iv) terminant expansion, were carried out for $\varepsilon = 0.15$ and various values of D, and the results are shown in Table 5.

Several observations can be made. Note that for values of $D \leqslant 0.1$ the linear portion of the system is dominant and that the first-order expansion essentially gives the response of the nonlinear system. The second-order and terminant expansions adjust the accuracy for the third and fourth significant figures, respectively. As the excitation increases in intensity,

Table 5. A comparison of mean square response $E[x^2]$ according to the linear system, first-order expansion, second-order expansion and terminant expansion for various values of intensity D

		$E[x^2]$			
D	ε	Linear system	1st order expansion	2nd order expansion	Terminant expansion
0.0001	0.015	0.1000000E-03	0.9774932E-04	0.9763965E-04	0.9762209E-04
0.001	0.015	0.1000000E-02	0.9774325E-03	0.9763372E-03	0.9761616E-03
0.01	0.015	0.1000000E-01	0.9768250E-02	0.9757441E-02	0.9755693E-02
0.1	0.015	0.1000000E 00	0.9707500E-01	0.9698851E-01	0.9697149E-01
5	0.015	0.5000000E 01	0.3200000E 01	0.4234438E-01	0.3756974E 01
10	0.015	0.1000000E 02	0.3025000E 01	0.1118153E 02	0.3775957E 01
100	0.015	0.1000000E 03	−0.5772500E 03	0.7453455E 04	−0.3772789E 07

significant deviations from the linear response occur, as can be seen from the results for $D \geqslant 5$. In these cases, decisions based on truncating the expansions at second order can be quite misleading, as the terminant expansion shows. With increasing values of D, the influence of the non-linear portion of the system can be expected to increase until eventually the basic underlying assumption that the solution can be represented as a perturbation about the linear solution is questionable. This is presumably what occurs when $D = 100$, since the calculated negative value of the response is of course impossible, even though a linear pattern had been established on the inverted Domb–Sykes plot. To further pursue this point, the behavior of the system when $D = 100$ was investigated for four other values of ε and the results are shown in Table 6. As expected, all responses are seen to be positive. For $\varepsilon = 0.001$ and 0.005 note that the second-order expansions are inadequate and the terminant expansion is needed. Finally, observe that as ε gets progressively smaller, the linear portion of the system again begins to dominate.

Table 6. A comparison of mean square response $E[x^2]$ according to the linear system, first-order expansion, second-order expansion and terminant expansion for various values of ε

		$E[x^2]$			
D	ε	Linear system	1st order expansion	2nd order expansion	Terminant expansion
100	0.001	0.1000000E 03	0.5485000E 02	0.9054202E 02	0.6995111E 02
100	0.0005	0.1000000E 03	0.7742500E 02	0.8634801E 02	0.8323705E 02
100	0.00025	0.1000000E 03	0.8871250E 02	0.9094325E 02	0.9046707E 02
100	0.000125	0.1000000E 03	0.9435625E 02	0.9491394E 02	0.9484574E 02

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Resume:

Dans ce travail, on s'interesse aux solutions en developpement de fonctions propres de l'equation directe de Fokker - Planck associee a un systeme particulier non lineaire du premier ordre soumis a une excitation avec un bruit blanc. En utilisant un ordinateur, on a genere un nombre substantiel de nouveaux termes dans les developpements. Avec cette nouvelle information, des graphiques inverses de Domb - Sykes revelent un modele dans les coefficients pour certains domaines de valeurs des parametres. On a alors utilise avec ce modele la theorie de Dingle pour refondre les series sous une forme plus favorable au calcul.

Zusammenfassung:

Diese Arbeit befasst sich mit Losyngen mit Eigenfunktionsentwicklungen fur die vorwarts wirkende Gleichung nach Fokker und Planck, die ein bestimmtes, nichtlineares System erster Ordnung unter Erregung durch weisses Rauschen, beschreibt. Mit Hilfe eines Digitalrechners wurde eine betrachtliche Anzahl neuer Gleider in der Entwicklung bestimmt. Mit dieser neuen Information zeigte sich in umgekehrten Domb-Sykes Diagrammen ein Muster in den Koeffizienten fur gewisse Zahlenbereiche der Parameter. Dingles Terminantentheorie wurde benutzt um mit Hilfe dieses Musters due Reihen in eine fur die Brechnung besser geeignete Form zu bringen.