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On Multivariable Sheffer Sequences

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1. INTRODUCTION

Let $\{B_n(x)\}_{n=0}^{\infty}$ be a polynomial sequence which is simple, one where each $B_n(x)$ is of degree precisely n. We say that $\{B_n(x)\}_{n=0}^{\infty}$ is a *binomial* sequence if

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(x)\, B_k(y) \qquad (n = 0, 1,\ldots), \tag{1}$$

and we call a simple polynomial sequence $\{P_n(x)\}_{n=0}^{\infty}$ a *Sheffer* sequence if there is a binomial sequence $\{B_n(x)\}_{n=0}^{\infty}$ such that

$$P_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(x)\, P_k(y) \qquad (n = 0, 1,\ldots). \tag{2}$$

These two classes of polynomial sequences, which were first studied to any great extent by Sheffer [12], have been characterized in a variety of other ways involving generating relations, differential operators, and recurrence relations. Other terminology has also been used. Sequences having property (1) were actually called basic sequences by Sheffer in [12] and interpolation sequences by R. Lagrange [6]. Those having property (2) were called zero type sequences by Sheffer and poweroids by Steffensen [13]. Our approach, as well as our terminology, is that used by Mullin and Rota [7] and Rota, Kahaner, and Odlyzko [11] in their recent and exhaustive studies of Sheffer sequences. The latter reference summarizes and extends the former and has been reprinted in [10]. For convenience, we shall cite only [10] when referring to the work of Rota *et al.*

In [10, pp. 34, 36] Rota *et al.* generalized relations (1) and (2) so as to apply to polynomial sequences which are simple in each of two variables x_1 and x_2. Using somewhat different notation, they termed such a sequence $\{B_n(x_1, x_2)\}_{n=0}^{\infty}$ a *cross* sequence if

$$B_n(x_1 + y_1, x_2 + y_2) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(x_1, x_2)\, B_k(y_1, y_2) \qquad (n = 0, 1,\ldots) \tag{3}$$

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and called $\{P_n(x_1, x_2)\}_{n=0}^{\infty}$ a *Steffensen* sequence if, in addition,

$$P_n(x_1 + y_1, x_2 + y_2) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(x_1, x_2) P_k(y_1, y_2) \qquad (n = 0, 1, \ldots). \tag{4}$$

While it is such sequences which are of main concern to us here, we shall with very little additional effort present our discussion of them in the context of polynomial sequences which are simple in each of N ($N \geqslant 1$) variables. For brevity, we write $\mathbf{x} = (x_1, \ldots, x_N)$ and

$$B_n(\mathbf{x}) = B_n(x_1, \ldots, x_N), \qquad P_n(\mathbf{x}) = P_n(x_1, \ldots, x_N).$$

In vector notation, then, we say that $\{B_n(\mathbf{x})\}_{n=0}^{\infty}$ is a sequence *of type* $\mathscr{B}_N$ if

$$B_n(\mathbf{x} + \mathbf{y}) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(\mathbf{x}) B_k(\mathbf{y}) \qquad (n = 0, 1, \ldots) \tag{5}$$

and that $\{P_n(\mathbf{x})\}_{n=0}^{\infty}$ is a sequence *of type* $\mathscr{S}_N$ if there is a sequence $\{B_n(\mathbf{x})\}_{n=0}^{\infty}$ of type $\mathscr{B}_N$ such that

$$P_n(\mathbf{x} + \mathbf{y}) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(\mathbf{x}) P_k(\mathbf{y}) \qquad (n = 0, 1, \ldots). \tag{6}$$

As indicated in [10, pp. 34, 8], there are many instances of multivariable Sheffer sequences "unconsciously present in the literature," the theory being "largely undeveloped." The sequence $\{n! L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ of Laguerre polynomials is, for example, a Steffensen sequence in the two variables α and x (see Section 4); here $P_n(\mathbf{x}) = P_n((\alpha, x)) = n! L_n^{(\alpha)}(x)$. Very recently in [4] it has in effect been shown that when β and y are constants, the modification $\{n! L_n^{(\alpha+\beta n)}(x + yn)\}_{n=0}^{\infty}$, which is $\{P_n(\mathbf{x} + \mathbf{s}_n)\}_{n=0}^{\infty}$ where $\mathbf{x} = (\alpha, x)$ and $\mathbf{s}_n = (\beta, y)\, n$, remains a Steffensen sequence in α and x. Consideration of Laguerre polynomials thus suggests the possibility that the modification $\{P_n(\mathbf{x} + \mathbf{s}_n)\}_{n=0}^{\infty}$ of *any* sequence $\{P_n(\mathbf{x})\}_{n=0}^{\infty}$ of type $\mathscr{S}_N$ is itself of type $\mathscr{S}_N$ in $\mathbf{x}$ when $\mathbf{s}_n = \mathbf{y}n$ ($n = 0, 1, \ldots$), where $\mathbf{y}$ is independent of $\mathbf{x}$ and n. The purpose of the present paper is to show that this is indeed the case and, moreover, that there are essentially no other sequences $\{\mathbf{s}_n\}_{n=0}^{\infty}$, independent of $\mathbf{x}$, such that $\{P_n(\mathbf{x} + \mathbf{s}_n)\}_{n=0}^{\infty}$ is of type $\mathscr{S}_N$ in $\mathbf{x}$.

Our result appears to be new even for Sheffer sequences proper ($N = 1$), and we treat them separately in Section 2. The general result is given in Section 3 and is illustrated in Section 4.

2. The Single-Variable Case

In view of the special importance of Sheffer sequences ($N = 1$) and because the proof of our main result for arbitrary N is by an induction argument based on the case $N = 1$, we treat that single-variable case separately in this section.

THEOREM 1. *Let $\{P_n(x)\}_{n=0}^{\infty}$ be a Sheffer sequence and let $\{s_n\}_{n=0}^{\infty}$ be a sequence of constants. The modified sequence $\{P_n(x + s_n)\}_{n=0}^{\infty}$ remains a Sheffer sequence in the variable x if and only if $s_n = yn + z$ $(n = 1, 2,...)$ where y and z are constants.*

Note that the value of s_0 is immaterial since $\{P_n(x)\}_{n=0}^{\infty}$ is simple and $P_0(x)$ is therefore a constant. In proving that $\{P_n(x + s_n)\}_{n=0}^{\infty}$ is actually a Sheffer sequence when $s_n = yn + z$ $(n = 1, 2,...)$ we shall, for convenience, agree that $s_0 = z$. The proof involves a generating relation characterization of binomial sequences which is due orginally to R. Lagrange [6]. To be precise, the class of binomial sequences is the same as the class of polynomial sequences $\{B_n(x)\}_{n=0}^{\infty}$ generated by relations of the form

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \exp[xH(t)] \tag{7}$$

where

$$H(t) = \sum_{k=1}^{\infty} h_k t^k \qquad (h_1 \neq 0). \tag{8}$$

Here and elsewhere in our discussion it is to be understood that we are working in the context of *formal* power series. Questions of convergence are therefore of no concern to us, and we follow Rota *et al.* in this regard.

Setting $y = 0$ in (2), we see that

$$\frac{P_n(x)}{n!} = \sum_{k=0}^{n} g_k \frac{B_{n-k}(x)}{(n-k)!} \qquad (n = 0, 1,...) \tag{9}$$

where $g_k = P_k(0)/k!$ $(k = 0, 1,..., n)$. Hence any Sheffer sequence is generated by a relation of the form

$$\sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!} = G(t) \exp[xH(t)] \tag{10}$$

where $H(t)$ is given by (8) and

$$G(t) = \sum_{k=0}^{\infty} g_k t^k \qquad (g_0 \neq 0). \tag{11}$$

On the other hand, since (10) implies (9) and $\{B_n(x)\}_{n=0}^{\infty}$ is simple, any sequence $\{P_n(x)\}_{n=0}^{\infty}$ generated by a relation of the form (10) is a simple polynomial sequence; and (2) follows from (10) by the additivity property of the exponential function. Relation (10) is therefore an alternative characterization of Sheffer sequences, one of several that Sheffer originally gave for his sequences in [12]. This was demonstrated by Rota *et al.* [10, p. 24] in a completely different manner by operator methods.

We now show that $\{P_n(x + yn + z)\}_{n=0}^{\infty}$ is also generated by a relation of the form (10). We use a technique that was used in [4] to generate the modifications $\{P_n(x + yn)\}_{n=0}^{\infty}$ of a number of particular Sheffer sequences. It is based on J. L. Lagrange's expansion formula [8, p. 146]

$$\sum_{n=0}^{\infty} \left\{\frac{d^n}{dt^n} [\Phi(t)(\phi(t))^n]\right\}_{t=0} \frac{1}{n!}\left(\frac{t}{\phi(t)}\right)^n = \frac{\Phi(t)}{1 - t\phi'(t)/\phi(t)}, \tag{12}$$

where $\Phi(t)$ and $\phi(t)$ have formal Maclaurin series expansions and $\phi(0) \neq 0$. While the desired generating relation for $\{P_n(x + yn + z)\}_{n=0}^{\infty}$ follows at once from a result in [4], it is almost as efficient to refer directly to (12). We do this in order to make our presention here more self-contained.

Putting

$$\Phi(t) = G(t)\, e^{xH(t)} \qquad \text{and} \qquad \phi(t) = e^{yH(t)}$$

in (12), we write

$$\sum_{n=0}^{\infty} \left\{\frac{d^n}{dt^n} [G(t)\, e^{(x+yn)H(t)}]\right\}_{t=0} \frac{(u(t))^n}{n!} = \frac{G(t)}{1 - tyH'(t)} \exp[xH(t)] \tag{13}$$

where

$$u(t) = t \exp[-yH(t)] \tag{14}$$

and $H'(t)$ denotes the formal derivative of $H(t)$. But according to (10), viewed as a formal Maclaurin series expansion,

$$\left\{\frac{d^n}{dt^n} [G(t)\, e^{xH(t)}]\right\}_{t=0} = P_n(x).$$

From (13), then, we find that

$$\sum_{n=0}^{\infty} P_n(x + yn)\frac{t^n}{n!} = \frac{G(\tilde{u}(t))}{1 - \tilde{u}(t)\, yH'(\tilde{u}(t))} \exp[xH(\tilde{u}(t))] \tag{15}$$

where $\tilde{u}(t)$ is the formal inverse of the function $u(t)$ defined in (14). Since this is of the form (10), we see that $\{P_n(x + yn)\}_{n=0}^{\infty}$ is a Sheffer sequence. It follows that $\{P_n(x + yn + z)\}_{n=0}^{\infty}$ is too; for replacing x by $x + z$ in (15) merely introduces the factor $\exp[zH(\tilde{u}(t))]$ on the right-hand side.

Turning to the proof of the converse, we assume that $\{P_n(x + s_n)\}_{n=0}^{\infty}$ is a Sheffer sequence. To show that $s_n = yn + z$ ($n = 1, 2,\ldots$), where y and z are constants, we appeal to the following observation of Sheffer's in [12]. Since $\{P_n(x)\}_{n=0}^{\infty}$ is generated by a relation of the form (10), differentiation of each side of that relation with respect to x yields the expression

$$P_n'(x) = nh_1P_{n-1}(x) + n(n-1)\, h_2P_{n-2}(x) + \cdots \qquad (n = 1, 2,\ldots) \tag{16}$$

where $P_n'(x)$ denotes the derivative of $P_n(x)$ and $\{h_k\}_{k=1}^{\infty}$ is the sequence of coefficients in (8). Under the assumption that $\{P_n(x + s_n)\}_{n=0}^{\infty}$ is a Sheffer sequence, there is also, then, a sequence $\{i_k\}_{k=1}^{\infty}$ of constants such that

$$P_n'(x + s_n) = ni_1 P_{n-1}(x + s_{n-1}) + n(n-1)\, i_2 P_{n-2}(x + s_{n-2}) + \cdots \qquad (n = 1, 2, \ldots). \tag{17}$$

Now when (7) is regarded as a formal Maclaurin series expansion, it is evident that $B_0(x) = 1$ and $B_1(x) = h_1 x$; and so, if we write (2) as

$$P_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} B_k(y)\, P_{n-k}(x) \qquad (n = 0, 1, \ldots)$$

and then put $y = s_n$, we find that

$$P_n(x + s_n) = P_n(x) + nh_1 s_n P_{n-1}(x) + \cdots \qquad (n = 0, 1, \ldots). \tag{18}$$

Replacing all the polynomials in (17) by means of (18), we have the identity

$$P_n'(x) + nh_1 s_n P_{n-1}'(x) + \cdots = ni_1 P_{n-1}(x) + n(n-1)\,(h_1 i_1 s_{n-1} + i_2)\, P_{n-2}(x) + \cdots \qquad (n = 2, 3, \ldots),$$

which, after the derivatives on its left-hand side are replaced by means of (16), becomes

$$n(h_1 - i_1)\, P_{n-1}(x) + n(n-1)\,(h_1{}^2 s_n - h_1 i_1 s_{n-1} + h_2 - i_2)\, P_{n-2}(x) + \cdots = 0 \qquad (n = 2, 3, \ldots). \tag{19}$$

Finally, since a Sheffer sequence is simple, the polynomials in (19) are linearly independent for any fixed n, and the coefficients of $P_{n-1}(x)$ and $P_{n-2}(x)$ are therefore zero. Hence

$$s_n - s_{n-1} = (i_2 - h_2)/h_1{}^2 \qquad (n = 2, 3, \ldots),$$

and $s_1, s_2, \ldots$ is evidently an arithmetic progression. That is, $s_n = yn + z$ $(n = 1, 2, \ldots)$; and the proof of the theorem is complete.

Remark 1. We observe in passing that the function $\tilde{u}(t)$ in generating relation (15) can be written explicitly as follows:

$$\tilde{u}(t) = t + \sum_{n=2}^{\infty} \left[\sum_{j=1}^{n-1} \frac{(yn)^j}{j!\,n} \sum h_{k_1} h_{k_2} \cdots h_{k_j} \right] t^n \tag{20}$$

where the h's are the coefficients in (8) and the innermost summation extends over all sets $\{k\}$ of j positive integers such that

$$k_1 + k_2 + \cdots + k_j = n - 1.$$

To show this, we need to find the coefficients c_n in the expansion

$$\tilde{u}(t) = t + \sum_{n=2}^{\infty} c_n t^n. \tag{21}$$

Note that the coefficient of t here is unity since the same is true of the formal Maclaurin series expansion of the function $u(t)$, defined in (14). Our approach is to first show that

$$c_n = \frac{B_{n-1}(yn)}{n!} \qquad (n = 2, 3,\ldots), \tag{22}$$

where $\{B_n(x)\}_{n=0}^{\infty}$ is the binomial sequence generated by (7), and then to take advantage of an explicit expression for $B_n(x)$ that has already been given in [2].

Expression (22) can be obtained as follows. Since $u(\tilde{u}(t)) = t$, we known that $\tilde{u}'(t) = 1/u'(\tilde{u}(t))$; thus, if we differentiate each side of (14) and then replace t by $\tilde{u}(t)$, we readily find that

$$\tilde{u}'(t) = \frac{\exp[yH(\tilde{u}(t))]}{1 - \tilde{u}(t)\, yH'(\tilde{u}(t))}. \tag{23}$$

On the other hand, if we set $G(t) = 1$ in (15), so that $P_n(x + yn) = B_n(x + yn)$ there, and then put $x = y$, it follows that

$$\sum_{n=1}^{\infty} B_{n-1}(yn) \frac{t^{n-1}}{(n-1)!} = \frac{\exp[yH(\tilde{u}(t))]}{1 - \tilde{u}(t)\, yH'(\tilde{u}(t))}. \tag{24}$$

So, equating the left-hand sides of (23) and (24), we have the expansion

$$\tilde{u}'(t) = \sum_{n=1}^{\infty} B_{n-1}(yn) \frac{t^{n-1}}{(n-1)!}$$

which, upon comparison with the differentiated form of (21), gives us (22).

Now, according to Boas and Buck [2, p. 18],

$$\frac{B_n(x)}{n!} = \sum_{j=1}^{n} \frac{x^j}{j!} \sum h_{k_1} h_{k_2} \cdots h_{k_j} \qquad (n = 1, 2,\ldots)$$

where the inner summation is as in (20) except that

$$k_1 + k_2 + \cdots + k_j = n.$$

Using this to substitute for $B_{n-1}(yn)$ in (22), we arrive at the desired expression for the coefficients in (21).

Note that in the case of Appell sequences [1], occurring when $H(t) = t$, (20) reduces to

$$\tilde{u}(t) = t + \sum_{n=2}^{\infty} \frac{(yn)^{n-1}}{n!} t^n. \tag{25}$$

Remark 2. In [10] Rota *et al.* made extensive use of the two differential operators

$$J(D) = \sum_{k=1}^{\infty} j_k D^k \quad (j_1 \neq 0) \qquad \text{and} \qquad F(D) = \sum_{k=0}^{\infty} f_k D^k \quad (f_0 \neq 0) \tag{26}$$

where $J(t)$ is the formal power series inverse of the function $H(t)$ in generating relation (10) and $F(t)$ is then defined in terms of the function $G(t)$ there as $F(t) = 1/G(J(t))$. It follows, of course, that $H(t)$ is the inverse of $J(t)$ and that $G(t) = 1/F(H(t))$. More recently, in [3] the one to one correspondence between pairs $[J(D), F(D)]$ of such operators and Sheffer sequences was used to study certain group-theoretic aspects of the latter. In view of generating relation (15), we find that if $[J(D), F(D)]$ is the operator pair associated with a given Sheffer sequence $\{P_n(x)\}_{n=0}^{\infty}$, the corresponding pair for the modified sequence $\{P_n(x + yn + z)\}_{n=0}^{\infty}$ has the relatively simple form

$$\left[J(D)\, e^{-yD}, F(D) \left(1 - y \frac{J(D)}{J'(D)}\right) e^{-zD}\right].$$

For Appell sequences, whose operator pairs are $[D, F(D)]$, this becomes

$$[De^{-yD}, F(D)\,(1 - yD)\, e^{-zD}].$$

3. The Multivariable Case

We now extend Theorem 1 so as to make it applicable to sequences of type $\mathscr{S}_N$ when $N > 1$. We shall use the fact that sequences of type $\mathscr{S}_N$ can also be characterized as those sequences $\{P_n(\mathbf{x})\}_{n=0}^{\infty}$ in N variables generated by relations of the form

$$\sum_{n=0}^{\infty} P_n(\mathbf{x}) \frac{t^n}{n!} = G(t) \exp[\mathbf{x} \cdot \mathbf{H}(t)] \tag{27}$$

where $G(t)$ is as in (11) and $\mathbf{x} \cdot \mathbf{H}(t)$ denotes the inner product $\sum_{j=1}^{N} x_j H_j(t)$ of the two vectors

$$\mathbf{x} = (x_1, \ldots, x_N) \qquad \text{and} \qquad \mathbf{H}(t) = (H_1(t), \ldots, H_N(t)).$$

Here the functions $H_j(t)$ $(j = 1,..., N)$ are of the form indicated in (8). The sequence $\{B_n(\mathbf{x})\}_{n=0}^{\infty}$ of type $\mathscr{B}_N$ to which the sequence in (27) corresponds is generated by

$$\sum_{n=0}^{\infty} B_n(\mathbf{x}) \frac{t^n}{n!} = \exp[\mathbf{x} \cdot \mathbf{H}(t)]. \tag{28}$$

To show the validity of characterizations (28) and (27), we refer to the special case

$$B_n(\mathbf{x}) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(x_1 ,..., x_{N-1} , 0)\, B_k(0,..., 0, x_N)$$

of (5) and also note how it follows from (5) that

$$\{B_n(x_1 ,..., x_{N-1} , 0)\}_{n=0}^{\infty} \qquad \text{and} \qquad \{B_n(0,..., 0, x_N)\}_{n=0}^{\infty}$$

are of types $\mathscr{B}_{N-1}$ and $\mathscr{B}_1$, respectively. Then, by direct summation and induction on N, we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_n(\mathbf{x}) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} B_n(x_1 ,..., x_{N-1} , 0) \frac{t^n}{n!} \sum_{k=0}^{\infty} B_k(0,..., 0, x_N) \frac{t^k}{k!} \\ &= \exp\left[\sum_{j=1}^{N-1} x_j H_j(t)\right] \exp[x_N H_N(t)],\end{aligned}$$

or (28). Finally, letting y be the zero vector in (6) and using the resulting expression to sum $\sum_{n=0}^{\infty} P_n(\mathbf{x})\, t^n/n!$, we arrive at (27), where $G(t) = \sum_{k=0}^{\infty} P_k(\mathbf{0})\, t^k/k!$.

The converse, that (28) and (27) imply properties (5) and (6), is evident from the additivity property of the exponential function.

THEOREM 2. *Let $\{P_n(\mathbf{x})\}_{n=0}^{\infty}$ be a sequence of type $\mathscr{S}_N$ and let $\{\mathbf{s}_n\}_{n=0}^{\infty}$ be a sequence of N-dimensional vectors which are independent of* $\mathbf{x}$. *The modified sequence $\{P_n(\mathbf{x} + \mathbf{s}_n)\}_{n=0}^{\infty}$ remains a sequence of type $\mathscr{S}_N$ in the vector* $\mathbf{x}$ *if and only if* $\mathbf{s}_n = \mathbf{y}n + \mathbf{z}$ *$(n = 1, 2,...)$ where* $\mathbf{y}$ *and* $\mathbf{z}$ *are independent of* $\mathbf{x}$ *and n.*

As noted earlier when $N = 1$, $P_0(\mathbf{x})$ is a constant; and, in proving that $\{P_n(\mathbf{x} + \mathbf{s}_n)\}_{n=0}^{\infty}$ is of type $\mathscr{S}_N$ when $\mathbf{s}_n = \mathbf{y}n + \mathbf{z}$ $(n = 1, 2,...)$, we agree to put $\mathbf{s}_0 = \mathbf{z}$. The proof rests on the following generalization of (15):

$$\sum_{n=0}^{\infty} P_n(\mathbf{x} + \mathbf{y}n) \frac{t^n}{n!} = \frac{G(\bar{U}(t))}{1 - \bar{U}(t)\, [\mathbf{y} \cdot \mathbf{H}'(\bar{U}(t))]} \exp[\mathbf{x} \cdot \mathbf{H}(\bar{U}(t))] \tag{29}$$

where $\mathbf{H}'$ denotes the derivative of the vector $\mathbf{H}$ and $\bar{U}(t)$ is the formal inverse of

$$U(t) = t \exp[-\mathbf{y} \cdot \mathbf{H}(t)]. \tag{30}$$

We obtain (29) via induction on N. It follows from the nature of (27) that $\{P_n(x_1, \ldots, x_{N-1}, x_N)\}_{n=0}^{\infty}$ is of type $\mathscr{S}_{N-1}$ in the variables $x_1, \ldots, x_{N-1}$; and, by the inductive hypothesis,

$$\sum_{n=0}^{\infty} P_n(x_1 + y_1 n, \ldots, x_{N-1} + y_{N-1} n, x_N) \frac{t^n}{n!} = \frac{G(\tilde{v}(t)) \exp[\sum_{j=1}^{N-1} x_j H_j(\tilde{v}(t))]}{1 - \tilde{v}(t) \sum_{j=1}^{N-1} y_j H_j'(\tilde{v}(t))} \exp[x_N H_N(\tilde{v}(t))] \tag{31}$$

where $\tilde{v}(t)$ is the formal inverse of

$$v(t) = t \exp\left[-\sum_{j=1}^{N-1} y_j H_j(t)\right]. \tag{32}$$

Now the sequence generated by (31) is evidently a Sheffer sequence in the single variable x_N. Hence applying (15), which is (29) when $N = 1$, to (31), we have

$$\sum_{n=0}^{\infty} P_n(x_1 + y_1 n, \ldots, x_{N-1} + y_{N-1} n, x_N + y_N n) \frac{t^n}{n!}$$
$$= \frac{G(\tilde{U}(t)) \exp[\sum_{j=1}^{N-1} x_j H_j(\tilde{U}(t))]}{[1 - \tilde{U}(t) \sum_{j=1}^{N-1} y_j H_j'(\tilde{U}(t))]\,[1 - \tilde{w}(t)\, y_N H_N'(\tilde{U}(t))\, \tilde{v}'(\tilde{w}(t))]} \exp[x_N H_N(\tilde{U}(t))] \tag{33}$$

where $\tilde{w}(t)$ is the formal inverse of

$$w(t) = t \exp[-y_N H_N(\tilde{v}(t))] \tag{34}$$

and $\tilde{U}(t) = \tilde{v}(\tilde{w}(t))$. Note that $\tilde{U}(t)$ is the inverse of the function $U(t)$ in (30); for, in view of (32) and (34), expression (30) can be written

$$U(t) = v(t) \exp[-y_N H_N(t)] = w(v(t)).$$

To put (33) into the desired form (29), we evidently need to show that the denominator on the right-hand side of (33) is equal to

$$1 - \tilde{U}(t) \sum_{j=1}^{N} y_j H_j'(\tilde{U}(t)). \tag{35}$$

To accomplish this, we first replace t by $\tilde{v}(t)$ in (32) to write

$$t = \tilde{v}(t) \exp\left[-\sum_{j=1}^{N-1} y_j H_j(\tilde{v}(t))\right].$$

Taking natural logarithms of each side of this last equation, differentiating the result with respect to t, and then replacing t by $\tilde{w}(t)$, we arrive at the expression

$$\tilde{w}(t)\,\tilde{v}'(\tilde{w}(t)) = \frac{\tilde{U}(t)}{1 - \tilde{U}(t)\sum_{j=1}^{N-1} y_j H_j'(\tilde{U}(t))}.$$

If we use this to substitute for $\tilde{w}(t)\,\tilde{v}'(\tilde{w}(t))$ in the denominator on the right-hand side of (33), that denominator readily simplifies to become (35). The validity of (29) is now established. It is of the form (27), as is the generating relation obtained from it by replacing $\mathbf{x}$ by $\mathbf{x} + \mathbf{z}$. The modification $\{P_n(\mathbf{x} + \mathbf{y}n + \mathbf{z})\}_{n=0}^{\infty}$ is therefore of type $\mathscr{S}_N$.

To demonstrate the converse, that the vectors $\mathbf{s}_n$ must be of the form

$$\mathbf{s}_n = \mathbf{y}n + \mathbf{z} \qquad (n = 1, 2, \ldots) \tag{36}$$

if $\{P_n(\mathbf{x} + \mathbf{s}_n)\}_{n=0}^{\infty}$ is to remain of type $\mathscr{S}_N$, we need only observe from (27) that a sequence of type $\mathscr{S}_N$ is a Sheffer sequence in any one of its variables x_j ($j = 1, \ldots, N$). Thus, from Theorem 1, we know that when $\{P_n(\mathbf{x} + \mathbf{s}_n)\}_{n=0}^{\infty}$ is of type $\mathscr{S}_N$, each component $s_{n,j}$ of $\mathbf{s}_n$ ($n = 1, 2, \ldots$) must be of the form $s_{n,j} = y_j n + z_j$ ($j = 1, \ldots, N$). That is, $\mathbf{s}_n$ is of the form (36); and Theorem 2 is established.

4. Examples

We now illustrate how Theorem 2 can be applied. To do this, we first list several of the more prominent multivariable Sheffer sequences appearing in the literature, with special attention to Steffensen sequences ($N = 2$). The reader is referred to [2, pp. 29–42] for an extensive listing of Sheffer sequences proper ($N = 1$) to which Theorem 1 can be applied.

In each example we identify the sequence as a multivariable Sheffer sequence by exhibiting a known generating relation as a special case of (27).

EXAMPLE 1. The sequence $\{n!L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ of Laguerre polynomials is a Steffensen sequence in the variables α and x since [9, p. 202]

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n$$

$$= (1-t)^{-1-\alpha} \exp\left[\frac{-xt}{1-t}\right] = \frac{1}{1-t} \exp\left[\alpha \log\left(\frac{1}{1-t}\right) + x\left(\frac{-t}{1-t}\right)\right].$$

EXAMPLE 2. Recalling that the Jacobi polynomials are generated by [9, p. 271]

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, t^n$$

$$= \frac{1}{R(x,t)} \left[\frac{2}{1-t+R(x,t)}\right]^{\alpha} \left[\frac{2}{1+t+R(x,t)}\right]^{\beta}$$

$$= \frac{1}{R(x,t)} \exp\left\{\alpha \log\left[\frac{2}{1-t+R(x,t)}\right] + \beta \log\left[\frac{2}{1+t+R(x,t)}\right]\right\}$$

where

$$R(x,t) = (1 - 2xt + t^2)^{1/2},$$

we see that $\{n! P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ is a Steffensen sequence in the parameters α and β.

EXAMPLE 3. The sequence $\{G_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ of actuarial polynomials is a Steffensen sequence in α and x since [13, 15]

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x) \frac{t^n}{n!} = \exp[\alpha t + x(1-e^t)].$$

EXAMPLE 4. In [10, p. 43] Rota *et al.* discussed the sequence $\{H_n^{(v)}(x)\}_{n=0}^{\infty}$ of Hermite polynomials of variance v and pointed out how it is a Steffensen sequence in x and v. It is generated by the relation

$$\sum_{n=0}^{\infty} H_n^{(v)}(x) \frac{t^n}{n!} = \exp[xt + v(-t^2/2)].$$

EXAMPLE 5. Carlson's [5] hypergeometric R-polynomials are generated by

$$\sum_{n=1}^{\infty} (\operatorname{tr} \mathbf{b})_n R_n(\mathbf{b}; \mathbf{z}) \frac{t^n}{n!} = \prod_{j=1}^{N} (1 - tz_j)^{-b_j} = \exp\left[\sum_{j=1}^{N} b_j \log\left(\frac{1}{1-tz_j}\right)\right]$$

where

$$\mathbf{b} = (b_1, ..., b_N), \qquad \mathbf{z} = (z_1, ..., z_N),$$

and $\operatorname{tr} \mathbf{b}$ denotes the trace $b_1 + \cdots + b_N$ of $\mathbf{b}$. The sequence $\{(\operatorname{tr} \mathbf{b})_n R_n(\mathbf{b}; \mathbf{z})\}_{n=0}^{\infty}$ thus provides an example which is of type $\mathscr{S}_N$ (in the vector $\mathbf{b}$) where N is arbitrary.

Note that the sequences in Examples 3 and 4 are, moreover, cross sequences and that the one in Example 5 is also of type $\mathscr{B}_N$.

Applying Theorem 2 to the Laguerre sequence in Example 1, we let $\{\sigma_n\}_{n=0}^{\infty}$ and $\{s_n\}_{n=0}^{\infty}$ be two sequences which are independent of both α and x. According to that theorem, where $P_n(\mathbf{x}) = P_n((\alpha, x)) = n! L_n^{(\alpha)}(x)$ and $\mathbf{s}_n = (\sigma_n, s_n)$, the sequence $\{n! L_n^{(\alpha+\sigma_n)}(x + s_n)\}_{n=0}^{\infty}$ continues to be a Steffensen sequence in α and x if and only if $\mathbf{s}_n = (\beta, y)\, n + (\gamma, z)$ $(n = 1, 2,\ldots)$, or

$$\sigma_n = \beta n + \gamma \qquad \text{and} \qquad s_n = yn + z \qquad (n = 1, 2,\ldots),$$

where β, γ, y, and z are constants. Serious interest in such modified Laguerre sequences was initiated by Toscano in [14], where he found generating relations for $\{L_n^{(\alpha+n)}(x)\}_{n=0}^{\infty}$ and $\{L_n^{(\alpha-2n)}(x)\}_{n=0}^{\infty}$. More recent references to such modifications of Laguerre and other polynomial sequences can be found in [4], where $\{L_n^{(\alpha+\beta n)}(x + yn)\}_{n=0}^{\infty}$ is generated.

Examples 2 through 4 are treated accordingly. Thus, if we agree that the modifying sequences are always to be independent of the variables with respect to which the given polynomial sequence is a Steffensen sequence, we see that

$$\{n! P_n^{(\alpha+\sigma_n, \beta+\tau_n)}(x)\}_{n=0}^{\infty}, \qquad \{G_n^{(\alpha+\sigma_n)}(x + s_n)\}_{n=0}^{\infty}, \qquad \{H_n^{(v+t_n)}(x + s_n)\}_{n=0}^{\infty}$$

all remain Steffensen sequences if and only if the terms in those modifying sequences depend linearly on the index n $(n = 1, 2,\ldots)$.

Finally, given a sequence $\{\mathbf{s}_n\}_{n=0}^{\infty}$ of N-dimensional vectors which are independent of the vector $\mathbf{b}$ in Example 5, we know that

$$\{(\mathrm{tr}(\mathbf{b} + \mathbf{s}_n))_n\, R_n(\mathbf{b} + \mathbf{s}_n;\, \mathbf{z})\}_{n=0}^{\infty}$$

remains of type $\mathscr{S}_N$ if and only if $\mathbf{s}_n = \mathbf{c}n + \mathbf{d}$ $(n = 1, 2,\ldots)$ where $\mathbf{c}$ and $\mathbf{d}$ are independent of $\mathbf{b}$ and n.

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