

## Separable Jordan Algebras over Commutative Rings. II

ROBERT BIX

*Department of Mathematics, The University of Michigan—Flint,  
Flint, Michigan 48503*

*Communicated by N. Jacobson*

Received August 25, 1981

Let  $J$  be a unital Jordan algebra over a commutative ring  $R$  containing  $\frac{1}{2}$ .  $J$  is called separable if its unital universal multiplication envelope  $U_R(J)$  is a separable associative  $R$ -algebra. In this paper, we continue the study of such algebras from [2].

In Section 1 we prove that a commutative associative  $R$ -algebra is separable in the associative sense if and only if it is separable considered as a Jordan  $R$ -algebra. We prove in Section 2 that  $J$  is separable over  $R$  if and only if  $J$  is separable over its center  $Z(J)$  and  $Z(J)$  is separable over  $R$ . Section 3 contains corollaries of this theorem, including the following result:  $J$  is a special separable Jordan  $R$ -algebra if and only if  $J \simeq H(A, j) \oplus K$ , where  $A$  is a separable associative  $R$ -algebra with involution  $j$ ,  $K$  is the Jordan  $S$ -algebra of a nondegenerate symmetric bilinear form on a finitely spanned projective  $S$ -module, and  $S$  is a commutative separable associative  $R$ -algebra. Moreover, an associative  $R$ -algebra  $A$  is separable in the associative sense if and only if  $A^+$  is a separable  $R$ -algebra.

In Section 4 we extend results of Harris [4] on centralizers of separable Jordan algebras over fields to algebras over commutative rings. In particular, if  $J \subset B$  are separable Jordan  $R$ -algebras, the centralizer of  $J$  in  $B$  is a separable Jordan  $R$ -algebra. In Section 5 we generalize results of Harris [5] and McCrimmon [11] on derivations of separable Jordan algebras over fields to algebras over commutative rings. If  $J$  is separable over  $R$ , we obtain necessary and sufficient conditions for all derivations of  $J$  into its bimodules to be inner; if  $\frac{1}{3} \in R$ , we prove that every derivation of  $J$  into its bimodules is a generalized inner derivation.

The results of Sections 1–3 are proved in [2] under the additional hypotheses that the algebras involved are finitely spanned. The results of the first three sections are used in Sections 4 and 5 to avoid finite spanning hypotheses in the theorems on centralizers and derivations.

PRELIMINARIES

All commutative rings are assumed to contain  $\frac{1}{2}$ . All algebras, subalgebras, modules, bimodules, and homomorphisms are assumed to be unital.

Throughout this paper, let  $R$  be a commutative ring, and let  $J$  be a Jordan  $R$ -algebra. Let  $U_R(J)$  be the *unital* universal multiplication envelope of  $J$  over  $R$  and let  $\rho$  be the canonical map from  $J$  to  $U_R(J)$  [6, p. 95]. Let  $S_R(J)$  be the *unital* special universal envelope of  $J$  over  $R$  and let  $\sigma$  be the canonical map from  $J$  to  $S_R(J)$  [6, p. 65].

If  $a, b, c \in J$ , let  $[a, b, c] = (a \cdot b) \cdot c - a \cdot (b \cdot c)$ . If the  $S_i$  are  $R$ -submodules of  $J$ , let  $[S_1, S_2, S_3]$  be the  $R$ -submodule of  $J$  spanned by all  $[a_1, a_2, a_3]$ ,  $a_i \in S_i$ . If  $M$  is a  $J$ -bimodule, the split null extension  $J \oplus M$  is a Jordan algebra such that  $J$  is a subalgebra,  $M^2 = 0$ , and  $J \cdot M \subset M$  via the bimodule action of  $J$  on  $M$  [6, p. 80]. Let

$$M^J = \{x \in M \mid [a, b, x] = 0 \text{ for all } a, b \in J\}, \tag{1}$$

where  $[a, b, x]$  is defined using the split null extension.

If  $B$  is a Jordan or associative  $R$ -algebra, let  $Z(B)$  be the center of  $B$ . If  $S$  is a subset of  $B$ , let  $\langle S \rangle$  be the subalgebra of  $B$  generated by  $S$ . We call  $B$  finitely spanned if it is finitely spanned as a  $R$ -module.

If  $A$  is an associative  $R$ -algebra and  $a, b \in A$ , let  $a \cdot b = \frac{1}{2}(ab + ba)$  and  $[a, b] = ab - ba$ . Let  $A^+$  be the Jordan  $R$ -algebra formed from the  $R$ -module  $A$  with multiplication  $a \cdot b$ . If  $A$  has an involution  $j$ , let  $H(A, j)$  be the Jordan subalgebra of  $A^+$  composed of elements fixed by  $j$ . If  $a, b, c \in A$ ,

$$[a, b] = -[b, a] \quad \text{and} \quad [ab, c] = [a, c]b + a[b, c]. \tag{2}$$

If  $A$  is a subalgebra of an associative algebra  $B$ , a derivation  $D: A \rightarrow B$  is an  $R$ -module homomorphism such that  $D(ab) = (Da)b + a(Db)$  for  $a, b \in A$ . If  $c \in B$ , (2) shows that the map from  $A$  to  $B$  taking  $a \in A$  to  $[a, c]$  is a derivation. A derivation of this form is called inner. If  $S$  and  $T$  are subsets of  $A$ , let  $[S, T]$  be the subset of  $A$  composed of all  $[s, t]$ ,  $s \in S$ ,  $t \in T$ . We say that  $S$  centralizes  $T$  if  $[S, T] = 0$ . The centralizer of  $S$  in  $A$  is the subalgebra of  $A$  composed of all  $a \in A$  such that  $[S, a] = 0$ .

Let  $A$  be an associative algebra. If  $\tau$  is an automorphism of  $A$ , let  $A^\tau$  be the subalgebra of  $A$  composed of elements fixed by  $\tau$ . If  $G$  is a group of automorphisms of  $A$ , let  $A^G = \bigcap A^\tau$  for  $\tau \in G$ . Let  $\pi$  be the automorphism of  $A \otimes_R A$  taking  $a \otimes b$  to  $b \otimes a$ ,  $a, b \in A$ . If  $x = \sum a_i \otimes b_i \in (A \otimes_R A)^\pi$ , then

$$\begin{aligned} x &= \frac{1}{2}(x + x^\pi) = \frac{1}{2} \sum (a_i \otimes b_i + b_i \otimes a_i) \\ &= \frac{1}{2} \sum [(a_i + b_i) \otimes (a_i + b_i) - a_i \otimes a_i - b_i \otimes b_i], \end{aligned}$$

so

$$(A \otimes_R A)^\pi = \left\{ \sum \alpha_i a_i \otimes a_i \mid \alpha_i \in R, a_i \in A \right\}. \quad (3)$$

If  $F$  is an algebraically closed field, define Jordan  $F$ -algebras  $F[p, q]$  as follows:  $F[1, 1] = F$ ;  $F[2, q] = F \oplus V$ , the Jordan algebra of a nondegenerate symmetric bilinear form on a vector space  $V$  of dimension  $q \geq 2$ ; and  $F[p, q] = H(M_p(C))$ , the Jordan algebra of symmetric  $p$ -by- $p$  matrices over a  $q$ -dimensional composition algebra  $C$  (unique up to isomorphism), where either  $(p, q) = (3, 8)$  or  $p \geq 3$  and  $q \in \{1, 2, 4\}$ . The  $F[p, q]$  represent the distinct isomorphism classes of finite-dimensional simple Jordan  $F$ -algebras [6, p. 204].

Certain properties of universal envelopes of Jordan algebras, modules over commutative rings, and separable associative algebras over commutative rings are summarized in [2, pp. 113–115]. References of the form  $[Ji]$ ,  $[Mi]$ , and  $[Ai]$ ,  $i$  an integer, refer to these. For ease of reference, we collect below some of the basic results on separable Jordan algebras over commutative rings proved in [2]. Let  $N \times N$  be the set of all ordered pairs of positive integers.

S1.  $J$  is called *separable* over  $R$  if  $U_R(J)$  is a separable associative  $R$ -algebra. If  $R$  is a field, a separable  $R$ -algebra  $J$  is finite-dimensional, and this definition of separability agrees with the classical one of remaining semisimple under arbitrary field extensions [2, p. 117].  $J$  is called *central* over  $R$  if the map  $a \rightarrow a1$  from  $R$  to  $Z(J)$  is bijective.

S2. If  $J$  is finitely spanned over  $R$ , then  $J$  is separable over  $R$  if and only if  $J/mJ$  is either zero or separable over  $R/m$  for every maximal ideal  $m$  of  $R$  [2, p. 118].

S3. If  $J$  is separable over  $R$ , then  $J$  is separable, finitely spanned, and projective over  $Z(J)$  [2, pp. 118, 122].

S4. If  $J$  is separable over  $R$ , there is an idempotent  $e \in U_R(J)$  called a separability idempotent such that  $eM = M'$  for every  $J$ -bimodule  $M$  [2, p. 116]. In particular,  $eJ = Z(J)$ .

S5. Let  $S$  be a commutative associative  $R$ -algebra and let  $J$  be an  $S$ -algebra. If  $J$  is a separable  $R$ -algebra via  $R1 \subset S$ , then  $J$  is a separable  $S$ -algebra [2, p. 118].

S6. Let  $J$  be separable over  $R$ . If  $S$  is a commutative associative  $R$ -algebra, then  $J \otimes_R S$  is either zero or separable over  $S$  and  $Z(J \otimes_R S) \simeq Z(J) \otimes_R S$ . If  $I$  is an ideal of  $R$ ,  $J/IJ$  is either zero or separable over  $R/I$  and  $Z(J/IJ) \simeq Z(J)/IZ(J)$ . If  $\phi$  is a homomorphism of  $J$  onto an  $R$ -algebra  $J'$ , then  $J'$  is separable over  $R$  and  $Z(J') = \phi Z(J)$  [2, pp. 117, 118].

S7. If  $J$  is separable over  $R$ , then  $J$  is a direct sum of ideals  $J(p, q)$  for  $(p, q) \in N \times N$  such that, if  $S$  is any commutative ring over which  $J$  is a separable algebra, if  $m$  is any maximal ideal of  $S$ , and if  $F$  is the algebraic closure of  $S/m$ , then

$$J(p, q)/mJ(p, q) \otimes_{S/m} F$$

is either zero or a finite direct sum of algebras isomorphic to  $F[p, q]$ . The ideals  $J(p, q)$  are uniquely determined, and only finitely many are nonzero [2, p. 137].

S8. Let  $J$  be separable over  $R$ . If  $S$  is a commutative associative  $R$ -algebra, then  $(J \otimes_R S)(p, q) \simeq J(p, q) \otimes_R S$ . If  $\phi$  is a homomorphism from  $J$  to another  $R$ -algebra, then  $(\phi J)(p, q) = \phi[J(p, q)]$  [2, p. 137].

## 1. CENTERS OF SEPARABLE JORDAN ALGEBRAS

We prove that a commutative associative  $R$ -algebra is separable as an associative  $R$ -algebra if and only if it is separable as a Jordan  $R$ -algebra. If  $J$  is separable over  $R$ , we show that  $Z(J)$  is separable over  $R$  and that  $\langle Z(J)^e \rangle$  is in the center of  $U_R(J)$ . The rest of the paper is based on these results.

**PROPOSITION 1.1.** *If  $J$  is separable over  $R$ , then  $Z(J)$  is separable as an associative  $R$ -algebra.*

*Proof.* We write  $U_R(J)$  as  $U$ .  $U$  is a separable associative  $R$ -algebra, so  $U$  is finitely spanned over  $Z(U)$  [A7]. Then  $J$  is finitely spanned over  $Z(U)$ , since  $J = U_R(J)1$  for  $1 \in J$ .  $Z(J) = eJ$ , where  $e \in U$  is a separability idempotent for  $J$  [S4]. Since the action of  $e$  on  $J$  commutes with the action of  $Z(U)$ ,  $Z(J)$  is a direct summand of  $J$  over  $Z(U)$ . Hence  $Z(J)$  is finitely spanned over  $Z(U)$ .

$J$  is naturally a  $Z(U)$ -algebra, since the action of  $Z(U)$  on  $J$  commutes with the action of  $J^e \subset U$ . Thus  $J$  is separable over  $Z(U)$  [S5]. Let  $m$  be a maximal ideal of  $Z(U)$ .  $J/mJ$  is either zero or finite-dimensional and separable in the classical sense over  $Z(U)/m$  [S1, S2]. Hence  $Z(J/mJ)$  is either zero or separable associative over  $Z(U)/m$  [6, p. 239]. Since  $J$  is separable over  $Z(U)$ ,  $Z(J/mJ)$  is isomorphic to  $Z(J)/mZ(J)$  [S6]. Hence  $Z(J)/mZ(J)$  is either zero or separable associative over  $Z(U)/m$ . Since this holds for every maximal ideal  $m$  of  $Z(U)$  and the preceding paragraph shows that  $Z(J)$  is finitely spanned over  $Z(U)$ , it follows that  $Z(J)$  is separable associative over  $Z(U)$  [A4]. Since  $U$  is separable associative over  $R$ , so is  $Z(U)$  [A1]. Hence  $Z(J)$  is separable associative over  $R$ , since  $Z(J)$  is separable over  $Z(U)$  and  $Z(U)$  is separable over  $R$  [3, p. 46]. ■

The main theorem of this section states that a commutative separable associative  $R$ -algebra  $S$  is separable as a Jordan  $R$ -algebra. The key fact required, that  $U_R(S)$  is commutative, is proved in Lemmas 1.2–1.7.

If  $J$  is any Jordan  $R$ -algebra and  $a, b, c \in J$ , then [6, pp. 95, 96] shows that

$$U_R(J) = \langle J^\rho \rangle, \tag{4}$$

$$[a, b, c]^\rho = [[c^\rho, a^\rho], b^\rho] = [b^\rho, [a^\rho, c^\rho]], \tag{5}$$

$$\begin{aligned} a^\rho b^\rho c^\rho + c^\rho b^\rho a^\rho + ((a \cdot c) \cdot b)^\rho \\ = a^\rho (b \cdot c)^\rho + b^\rho (a \cdot c)^\rho + c^\rho (a \cdot b)^\rho \\ = (b \cdot c)^\rho a^\rho + (a \cdot c)^\rho b^\rho + (a \cdot b)^\rho c^\rho. \end{aligned} \tag{6}$$

Set  $u(a) = 2(a^\rho)^2 - (a^2)^\rho$ ,  $u(a, b) = u(a + b) - u(a) - u(b)$ , and  $v(a) = 2a^\rho$ . Any identity satisfied by the  $U$ - and  $V$ -operators in every Jordan algebra also holds for the elements  $u(a)$ ,  $u(a, b)$ , and  $v(a)$  [6, p. 96].

LEMMA 1.2. *If  $S$  is a commutative associative  $R$ -algebra, then*

$$[S^\rho, S^\rho] \subset Z[U_R(S)]. \tag{7}$$

*If  $a, b \in S$  and  $N$  is the ideal of  $U_R(S)$  generated by  $[(a^i)^\rho, (b^j)^\rho]$  for  $i, j \in \{1, 2\}$ , then*

$$u(a) u(b) \equiv u(ab) \pmod{N}. \tag{8}$$

*Proof.* Equation (5) implies that  $[[S^\rho, S^\rho], S^\rho] = 0$ , so (7) follows from (4). QJ3 [7, p. 1.10] shows that  $u(a) u(b) u(a) = u(aba) = u(a^2b)$ . Hence

$$u(a)^2 u(b) = u(a^2b) + u(a)[u(a), u(b)]. \tag{9}$$

Let  $S[\lambda]$  be the polynomial ring in an indeterminate  $\lambda$  over  $S$ . Since  $S[\lambda]$  is commutative and associative, (9) holds with  $S$  replaced by  $S[\lambda]$ ,  $R$  replaced by  $R[\lambda]$ , and  $a$  replaced by  $\lambda 1 + a$ . We can collect the terms of degree two in  $\lambda$  and obtain an identity of  $U_R(S)$ , since

$$U_{R[\lambda]}(S[\lambda]) \simeq U_{R[\lambda]}(S \otimes_R R[\lambda]) \simeq U_R(S) \otimes_R R[\lambda]$$

[J1]. This yields

$$2u(a) u(b) + v(a)^2 u(b) \equiv 4u(ab) + u(b, a^2b) \pmod{N}, \tag{10}$$

by (2). QJ19 [7, p. 1.19] shows that

$$v(a) \cdot u(b) = u(b, a \cdot b) = u(b, ab). \tag{11}$$

Hence

$$\begin{aligned} v(a) \cdot [v(a) \cdot u(b)] &= v(a) \cdot u(b, ab) && \text{(by (11))} \\ &= 2u(ab) + u(b, a^2b) && \text{(by (11) linearized).} \end{aligned}$$

It follows that  $v(a)^2 u(b) \equiv 2u(ab) + u(b, a^2b) \pmod{N}$ . Together with (10), this yields (8). ■

LEMMA 1.3. (i). *If  $G$  is a finite group of automorphisms of a commutative separable associative  $R$ -algebra  $S$ , then  $S^G$  is a separable associative  $R$ -algebra.*

(ii). *If  $S$  is a commutative separable associative  $R$ -algebra, then so is  $(S \otimes_R S)^\pi$ .*

*Proof.* (i).  $S$  is finitely spanned and projective over  $S^G$  [2, p. 128]. Hence  $S \otimes_R S$  is projective over  $S^G \otimes_R S^G$ .  $S$  is projective over  $S \otimes_R S$ , since  $S$  is a separable associative  $R$ -algebra [A1]. Then  $S$  is projective over  $S^G \otimes_R S^G$ , by the transitivity of projectivity [3, p. 5]. Since  $S$  is finitely spanned and projective over  $S^G$  and  $S^G \subset S$ , it follows that  $S^G$  is a direct summand of  $S$  over  $S^G$  [12, p. 2]. Then  $S^G$  is also a direct summand of  $S$  over  $S^G \otimes_R S^G$ . Since  $S$  is projective over  $S^G \otimes_R S^G$ , so is  $S^G$ . Thus  $S^G$  is separable associative over  $R$  [A1].

(ii). Since  $S$  is separable associative over  $R$ , so is  $S \otimes_R S$  [3, p. 43]. Then (ii) follows (i). ■

If  $A$  is an associative algebra, let  $I = I(A)$  be the ideal of  $A$  generated by  $[A, A]$ . Let  $a'$  denote the image in  $A/I$  of  $a \in A$ . Let  $b''$  denote the image in  $I/I^2$  of  $b \in I$ .  $I/I^2$  is a two-sided associative  $A/I$ -module such that

$$a'b'' = (ab)'' \quad \text{and} \quad b''a' = (ba)'' \tag{12}$$

for  $a \in A, b \in I$ .

LEMMA 1.4. *If  $S$  is a commutative separable associative  $R$ -algebra and  $I = I(U_R(S))$ , then  $U_R(S)/I$  is a separable associative  $R$ -algebra. There is an  $R$ -algebra isomorphism of  $(S \otimes_R S)^\pi$  onto  $U_R(S)/I$  taking  $a \otimes a$  to  $u(a)'$ ,  $a \in S$ .*

*Proof.* We write  $U_R(S)$  as  $U$ . There is an  $R$ -module homomorphism from  $S \otimes_R S$  to  $U/I$  taking  $a \otimes b$  to  $\frac{1}{2}u(a, b)'$ ,  $a, b \in S$ . This restricts to an  $R$ -module homomorphism  $\phi$  from  $(S \otimes_R S)^\pi$  to  $U/I$  such that  $\phi(a \otimes a) = u(a)'$ . Equations (3) and (8) imply that  $\phi$  is an  $R$ -algebra homomorphism. Since the identity map is an associative specialization of  $S$  in itself, there is an  $R$ -algebra homomorphism  $\tau$  from  $U$  to  $(S \otimes_R S)^\pi$  taking  $a^p$  to

$\frac{1}{2}(a \otimes 1 + 1 \otimes a)$  [6, p. 99]. Since  $(S \otimes_R S)^\tau$  is commutative,  $\tau$  induces an  $R$ -algebra homomorphism  $\psi$  from  $U/I$  to  $(S \otimes_R S)^\tau$  taking  $a^{\rho'}$  to  $\frac{1}{2}(a \otimes 1 + 1 \otimes a)$ . Then

$$\begin{aligned} \psi[u(a)'] &= \psi[2(a^{\rho'})^2 - (a^2)^{\rho'}] \\ &= \frac{1}{2}(a \otimes 1 + 1 \otimes a)^2 - \frac{1}{2}(a^2 \otimes 1 + 1 \otimes a^2) \\ &= a \otimes a, \end{aligned}$$

so

$$\psi\phi(a \otimes a) = a \otimes a \quad \text{and} \quad \phi\psi[u(a)'] = u(a)'. \tag{13}$$

Equation (4) and the equation  $a^\rho = \frac{1}{2}u(a, 1) = \frac{1}{2}[u(a + 1) - u(a) - u(1)]$  imply that

$$U = \langle \{u(a) \mid a \in S\} \rangle. \tag{14}$$

Since  $\phi$  and  $\psi$  are algebra homomorphisms, Eqs. (3), (13), and (14) imply that  $\phi$  and  $\psi$  are inverse isomorphisms. Then  $U/I$  is a separable associative  $R$ -algebra, since  $(S \otimes_R S)^\tau$  is [Lemma 1.3(ii)]. ■

LEMMA 1.5. *If  $S$  is a commutative separable associative  $R$ -algebra and  $I = I(U_R(S))$ , then  $I = I'$  for every positive integer  $t$ .*

*Proof.* We write  $U_R(S)$  as  $U$ . If  $a, b \in S$  and  $x, y \in U$ , (7) implies that

$$[x, [a^\rho, b^\rho]y] = [a^\rho, b^\rho][x, y] \in I^2.$$

Equations (2), (4), and (7) imply that  $I$  is spanned by elements of the form  $[a^\rho, b^\rho]y$ , so

$$[U, I] \subset I^2. \tag{15}$$

Fix  $z \in U$ . Equation (15) implies that there is a well-defined map  $\phi$  from  $U/I$  to  $I/I^2$  such that  $\phi(x') = [z, x]''$  for  $x \in U$ . If  $x, y \in U$ , then

$$\begin{aligned} \phi(x'y') &= \phi((xy)') = [z, xy]'' \\ &= [z, x]''y' + x'[z, y]'' \quad (\text{by (2) and (12)}) \\ &= \phi(x')y' + x'\phi(y'), \end{aligned}$$

so  $\phi$  is a derivation of  $U/I$  into  $I/I^2$ .  $U/I$  is a separable associative  $R$ -algebra [Lemma 1.4], so every derivation from  $U/I$  to a two-sided associative module is inner [12, p. 43]. The image of any inner derivation of  $U/I$  into  $I/I^2$  is contained in  $[U', I''] = [U, I]'' = 0$ , by (12) and (15). Hence  $\phi = 0$ , so  $[U, U] \subset I^2$ . Then  $I \subset I^2$ , whence the lemma follows. ■

LEMMA 1.6. *If  $S$  is a commutative associative  $R$ -algebra,  $a, b, c, d \in S$ , and  $\rho: S \rightarrow U_R(S)$ , then*

$$[a^\rho, b^\rho][a^\rho, c^\rho][a^\rho, d^\rho] = 0.$$

*Proof.* Identities QJ3 and QJ10 [7, pp. 1.10, 1.18] imply that

$$u(b)[u(a)]^2 u(b) = u(b) u(a^2) u(b) = u(a^2 b^2).$$

Then

$$u(b) u(a)^2 u(b) = u(a) u(b)^2 u(a),$$

since  $u(a^2 b^2)$  is symmetric in  $a$  and  $b$ . Hence

$$\begin{aligned} [u(a) u(b), [u(a), u(b)]] &= u(a) u(b) u(a) u(b) - u(a) u(b)^2 u(a) \\ &\quad - u(a) u(b) u(a) u(b) + u(b) u(a)^2 u(b) \\ &= 0. \end{aligned} \tag{16}$$

Replace  $a$  by  $\lambda 1 + a$  and  $b$  by  $\eta 1 + b$  for indeterminates  $\lambda$  and  $\eta$ , and collect the terms in (16) of degree one in  $\lambda$  and two in  $\eta$ . This yields

$$\begin{aligned} 0 &= [v(a), [u(a), u(b)]] + [u(a), [v(a), u(b)]] \\ &\quad + [v(a) v(b), [u(a), v(b)]] + [u(a) v(b), [v(a), v(b)]]. \end{aligned} \tag{17}$$

Since  $u(a)$  and  $v(a)$  commute [6, p. 96], one checks that

$$[v(a), [u(a), u(b)]] = [u(a), [v(a), u(b)]]. \tag{18}$$

Since (7) shows that  $[v(a), v(b)] \in Z[U_R(S)]$ ,

$$[u(a) v(b), [v(a), v(b)]] = 0. \tag{19}$$

Substituting (18) and (19) in (17) shows that

$$\begin{aligned} 0 &= 2[u(a), [v(a), u(b)]] + [v(a) v(b), [u(a), v(b)]] \\ &= 2[u(a), [2a^\rho, 2(b^\rho)^2 - (b^2)^\rho]] + [4a^\rho b^\rho, [2(a^\rho)^2 - (a^2)^\rho, 2b^\rho]] \\ &= 8[u(a), [a^\rho, (b^\rho)^2]] - 4[u(a), [a^\rho, (b^2)^\rho]] \\ &\quad + 16[a^\rho b^\rho, [(a^\rho)^2, b^\rho]] - 8[a^\rho b^\rho, [(a^2)^\rho, b^\rho]] \end{aligned}$$



$$\begin{aligned}
 &= 8[u(a), [a^\rho, (b^\rho)^2]] + 16[a^\rho b^\rho, [(a^\rho)^2, b^\rho]] \quad (\text{by (7)}) \\
 &= 8[u(a), b^\rho [a^\rho, b^\rho] + [a^\rho, b^\rho] b^\rho] \\
 &\quad + 16[a^\rho b^\rho, a^\rho [a^\rho, b^\rho] + [a^\rho, b^\rho] a^\rho] \quad (\text{by (2)}) \\
 &= 16[a^\rho, b^\rho][u(a), b^\rho] + 32[a^\rho, b^\rho][a^\rho b^\rho, a^\rho] \quad (\text{by (7)}) \\
 &= 16[a^\rho, b^\rho][2(a^\rho)^2 - (a^2)^\rho, b^\rho] + 32[a^\rho, b^\rho][a^\rho b^\rho, a^\rho] \\
 &= 64[a^\rho, b^\rho]^2 a^\rho - 16[a^\rho, b^\rho][(a^2)^\rho, b^\rho] \\
 &\quad + 32[a^\rho, b^\rho][b^\rho, a^\rho] a^\rho \quad (\text{by (2) and (7)}) \\
 &= 32[a^\rho, b^\rho]^2 a^\rho - 16[a^\rho, b^\rho][(a^2)^\rho, b^\rho].
 \end{aligned}$$

If  $d \in S$ , applying the map  $x \rightarrow [x, d^\rho]$  from  $U_R(S)$  to itself shows that

$$0 = 32[a^\rho, b^\rho]^2 [a^\rho, d^\rho],$$

by (7). Linearizing  $b \rightarrow b, c$  establishes the lemma, by (7) and the fact that  $\frac{1}{2} \in R$ . ■

If  $A$  is an  $R$ -algebra, the  $R$ -algebra  $A$ -opposite  $A^\circ = \{a^\circ \mid a \in A\}$  has operations  $r(a^\circ) = (ra)^\circ$ ,  $a^\circ + b^\circ = (a + b)^\circ$ , and  $a^\circ b^\circ = (ba)^\circ$ , for  $r \in R$ ,  $a, b \in A$ . If  $B$  is an  $R$ -algebra and  $f: A \rightarrow B$ , define  $f^\circ: A^\circ \rightarrow B^\circ$  by  $f^\circ(a^\circ) = (fa)^\circ$ ,  $a \in A$ .

LEMMA 1.7. *If  $S$  is a commutative separable associative  $R$ -algebra, then  $U_R(S)$  is commutative.*

*Proof.* First assume that  $R$  is a field. If  $T$  is the algebraic closure of  $R$ ,  $S \otimes_R T$  is a direct sum of ideals isomorphic to  $T$  as  $T$ -algebras [A3]. Then  $U_T(S \otimes_R T)$  is a direct sum of ideals isomorphic to  $T$  [6, p. 105]. Since  $U_T(S \otimes_R T)$  is isomorphic to  $U_R(S) \otimes_R T$  [J1],  $U_R(S)$  is commutative when  $R$  is a field.

Now let  $R$  be arbitrary.  $(S \otimes_R S)^\pi$  is a separable associative  $R$ -algebra [Lemma 1.3(ii)]. Let

$$e \in (S \otimes_R S)^\pi \otimes_R (S \otimes_R S)^{\pi\circ}$$

be a separability idempotent for  $(S \otimes_R S)^\pi$  [A2]. By (3), we can write

$$e = \sum \alpha_i (s_i \otimes s_i) \otimes (t_i \otimes t_i)^\circ, \quad s_i, t_i \in S, \alpha_i \in R.$$

Since  $e$  is a separability idempotent for  $(S \otimes_R S)^\pi$ ,

$$1 \otimes 1 = \sum \alpha_i (s_i \otimes s_i)(t_i \otimes t_i) = \sum \alpha_i s_i t_i \otimes s_i t_i \quad (20)$$

in  $(S \otimes_R S)^\pi$ , and, for all  $a \in S$ ,

$$\begin{aligned} 0 &= |(a \otimes a) \otimes (1 \otimes 1)^\circ - (1 \otimes 1) \otimes (a \otimes a)^\circ|e \\ &= \sum \alpha_i (as_i \otimes as_i) \otimes (t_i \otimes t_i)^\circ - \alpha_i (s_i \otimes s_i) \otimes (t_i a \otimes t_i a)^\circ \end{aligned} \quad (21)$$

in  $(S \otimes S)^\pi \otimes (S \otimes S)^{\pi^\circ}$  [A2].

Let  $V = \{s_i, t_i, s_i^2, t_i^2\}$ . Let  $K$  be the ideal of  $U_R(S)$  generated by all  $[a^\circ, b^\circ]$ ,  $a \in V$ ,  $b \in S$ . We write  $U_R(S)/K$  as  $A$ . If  $a, b \in S$ , let  $\bar{u}(a)$  and  $\bar{u}(a, b)$  denote the images of  $u(a)$  and  $u(a, b)$  in  $A$ . The  $R$ -module homomorphism from  $S \otimes_R S$  to  $A$  taking  $a \otimes b$  to  $\frac{1}{2}\bar{u}(a, b)$  restricts to a homomorphism  $\phi$  from  $(S \otimes_R S)^\pi$  to  $A$  taking  $a \otimes a$  to  $\bar{u}(a)$ .

Set  $f = \sum \alpha_i \bar{u}(s_i) \otimes \bar{u}(t_i)^\circ \in A \otimes_R A^\circ$ . We claim that  $f$  is a separability idempotent for  $A$  as an associative  $R$ -algebra. Let  $v: A \otimes A^\circ \rightarrow A$  be the  $R$ -module homomorphism taking  $x \otimes y^\circ$  to  $xy$ ,  $x, y \in A$ . Then

$$\begin{aligned} vf &= \sum \alpha_i \bar{u}(s_i) \bar{u}(t_i) \\ &= \sum \alpha_i \bar{u}(s_i t_i) \quad (\text{by (8)}) \\ &= \phi \left( \sum \alpha_i s_i t_i \otimes s_i t_i \right) \\ &= \phi(1 \otimes 1) \quad (\text{by (20)}) \\ &= 1. \end{aligned} \quad (22)$$

If  $a \in S$ , the following equations hold in  $A \otimes_R A^\circ$ :

$$\begin{aligned} &|\bar{u}(a) \otimes 1^\circ - 1 \otimes \bar{u}(a)^\circ|f \\ &= \sum \alpha_i [\bar{u}(a) \bar{u}(s_i)] \otimes \bar{u}(t_i)^\circ - \alpha_i \bar{u}(s_i) \otimes [\bar{u}(t_i) \bar{u}(a)]^\circ \\ &= \sum \alpha_i \bar{u}(as_i) \otimes \bar{u}(t_i)^\circ - \alpha_i \bar{u}(s_i) \otimes \bar{u}(t_i a)^\circ \quad (\text{by (8)}) \\ &= (\phi \otimes \phi^\circ) \left[ \sum \alpha_i (as_i \otimes as_i) \otimes (t_i \otimes t_i)^\circ \right. \\ &\quad \left. - \alpha_i (s_i \otimes s_i) \otimes (t_i a \otimes t_i a)^\circ \right] \\ &= (\phi \otimes \phi^\circ)(0) \quad (\text{by (21)}) \\ &= 0. \end{aligned} \quad (23)$$

If we write  $\bar{u}(a_j)$  as  $c_j$  for  $a_j \in S$ , it follows by induction on  $n$  that

$$|(c_1 \cdots c_n) \otimes 1^\circ|f = |1 \otimes (c_1 \cdots c_n)^\circ|f,$$

since

$$\begin{aligned}
 [(c_1 \cdots c_n) \otimes 1^\circ]f &= [(c_1 \cdots c_{n-1}) \otimes 1^\circ][c_n \otimes 1^\circ]f \\
 &= [(c_1 \cdots c_{n-1}) \otimes 1^\circ][1 \otimes c_n^\circ]f \quad (\text{by (23)}) \\
 &= [(c_1 \cdots c_{n-1}) \otimes c_n^\circ]f \\
 &= [1 \otimes c_n^\circ][(c_1 \cdots c_{n-1}) \otimes 1^\circ]f \\
 &= [1 \otimes c_n^\circ][1 \otimes (c_1 \cdots c_{n-1})^\circ]f \quad (\text{by induction}) \\
 &= [1 \otimes (c_1 \cdots c_n)^\circ]f.
 \end{aligned}$$

Hence  $(x \otimes 1^\circ)f = (1 \otimes x^\circ)f$  for all  $x \in A$ , by (14). Together with (22), this proves that  $f$  is a separability idempotent for  $A$ , so  $A$  is a separable associative  $R$ -algebra [A2].

We write  $Z(A)$  as  $Z$ . Let  $m$  be a maximal ideal of  $Z$ .  $S \otimes_R Z/m$  is a commutative separable associative  $Z/m$ -algebra [A5]. Then  $U_{Z/m}(S \otimes_R Z/m)$  is commutative, by the first paragraph of the proof. [J1] shows that

$$U_{Z/m}(S \otimes_R Z/m) \simeq U_R(S) \otimes_R Z/m.$$

$U_R(S) \otimes_R Z/m$  has  $A \otimes_Z Z/m$  as a homomorphic image, since  $A$  is naturally a  $Z$ -algebra. Because  $A \otimes_Z Z/m \simeq A/mA$ , it follows that  $A/mA$  is commutative. Since the preceding paragraph shows that  $A$  is separable over  $R$ , the center of  $A/mA$  is the image of  $Z$  [A5]. Hence  $A = Z + mA$  for every maximal ideal  $m$  of  $Z$ .  $A$  is finitely spanned over  $Z$ , since  $A$  is separable over  $R$  [A1, A7]. Then  $A = Z$ , by Nakayama's Lemma [M6]. Hence  $A$  is commutative.

Define  $I = I(U_R(S))$  as before Lemma 1.4. Since  $A$  is commutative, it follows that  $I = K$ . If  $V$  has  $n$  elements, (7) implies that  $K^{2n+1}$  is contained in the ideal generated by

$$\{[a^\circ, b_1^\circ][a^\circ, b_2^\circ][a^\circ, b_3^\circ] \mid a \in V, b_i \in S\}.$$

Then  $K^{2n+1} = 0$ , by Lemma 1.6. By Lemma 1.5,  $I = I^{2n+1} = K^{2n+1} = 0$ , so  $U_R(S)$  is commutative. ■

**THEOREM 1.8.** *Let  $S$  be a commutative associative  $R$ -algebra. Then  $S$  is separable as a Jordan  $R$ -algebra if and only if  $S$  is separable as an associative  $R$ -algebra. If so,  $U_R(S)$  is commutative and there is an  $R$ -algebra isomorphism of  $(S \otimes_R S)^\pi$  onto  $U_R(S)$  taking  $a \otimes a$  to  $u(a)$ ,  $a \in S$ .*

*Proof.* If  $S$  is separable as a Jordan  $R$ -algebra, Proposition 1.1 shows that  $S$  is separable as an associative  $R$ -algebra. If  $S$  is separable as an associative  $R$ -algebra, then  $U_R(S)$  is commutative [Lemma 1.7]. It follows

that  $U_R(S)$  is a separable associative  $R$ -algebra isomorphic to  $(S \otimes_R S)^n$  [Lemma 1.4], so  $S$  is separable as a Jordan  $R$ -algebra. ■

The next theorem is stated in several forms to facilitate later use. If  $S$  is a Jordan subalgebra of  $J$ , we define  $J^S$  as in (1), considering  $J$  as an  $S$ -bimodule.

**THEOREM 1.9.** (i). *Let  $S$  be a commutative separable associative  $R$ -algebra. Considering  $S$  as a Jordan algebra, assume that  $S$  is an  $R$ -subalgebra of  $J$ . If  $\rho: J \rightarrow U_R(J)$ , then  $\langle S^\rho \rangle$  is a separable associative  $R$ -algebra centralizing  $(J^S)^\rho$ .*

(ii). *If  $Z(J)$  is a separable associative  $R$ -algebra, then  $\langle Z(J)^\rho \rangle$  is a separable associative  $R$ -subalgebra of the center of  $U_R(J)$ .*

(iii). *If  $J$  is a separable  $R$ -algebra, then  $\langle Z(J)^\rho \rangle$  is a separable associative  $R$ -subalgebra of the center of  $U_R(J)$ .*

*Proof.* (i). By Theorem 1.8,  $U_R(S)$  is a commutative separable associative  $R$ -algebra. The inclusion  $S \subset J$  induces an  $R$ -algebra homomorphism from  $U_R(S)$  to  $U_R(J)$  having image  $\langle S^\rho \rangle$  [J3]. Thus  $\langle S^\rho \rangle$  is a commutative separable associative  $R$ -algebra [A5]. We must show that  $[\langle S^\rho \rangle, (J^S)^\rho] = 0$ .

Let  $E$  be the centralizer of  $S^\rho$  in  $U_R(J)$ . Equations (1) and (5) imply that

$$0 = [S, S, J^S]^\rho = [S^\rho, [S^\rho, (J^S)^\rho]]$$

so  $[S^\rho, (J^S)^\rho] \subset E$ .  $\langle S^\rho \rangle$  is a subalgebra of  $E$ , since  $\langle S^\rho \rangle$  is commutative. It follows that  $[\langle S^\rho \rangle, (J^S)^\rho] \subset E$ , by (2). Thus, if  $a \in J^S$ , the map  $z \rightarrow [z, a^\rho]$  is a derivation of  $\langle S^\rho \rangle$  into  $E$ . Since  $\langle S^\rho \rangle$  is a separable associative  $R$ -subalgebra of  $E$ , every derivation of  $\langle S^\rho \rangle$  into  $E$  is inner [12, p. 43]. Then every derivation of  $\langle S^\rho \rangle$  into  $E$  is zero, since  $\langle S^\rho \rangle$  is in the center of  $E$ . Hence  $[\langle S^\rho \rangle, (J^S)^\rho] = 0$ . Part (ii) follows from (i) and (4). Part (iii) follows from (ii) and Proposition 1.1. ■

## 2. SEPARABILITY AND CENTRAL SEPARABILITY

We prove that  $J$  is separable over  $R$  if and only if  $J$  is separable over  $Z(J)$  and  $Z(J)$  is separable over  $R$ . In proving the “if” implication, we use Theorem 1.9(ii) to consider  $U_R(J)$  as an algebra over  $\langle Z(J)^\rho \rangle$ . This lets us reduce to the case where  $R$  is a field, once we prove in Proposition 2.2 that  $U_R(J)$  is finitely spanned over  $\langle Z(J)^\rho \rangle$ .

**LEMMA 2.1.** *If  $Z(J)$  is a separable associative  $R$ -algebra,  $x \in J$ , and  $\rho: J \rightarrow U_R(J)$ , then  $\langle Z(J)^\rho \rangle \langle Z(J)x^\rho \rangle$  is finitely spanned as a module over  $\langle Z(J)^\rho \rangle$ .*

*Proof.* We write  $Z(J)$  as  $Z$  and  $\langle Z, x \rangle$  as  $T$ .  $T$  is a commutative associative  $R$ -algebra, since Jordan algebras are power-associative [6, p. 36]. If  $a, b, c \in Z$ , linearizing  $b \rightarrow b, cx$  in (8) yields

$$u(a) u(b, cx) = u(ab, acx) \tag{24}$$

in  $U_R(T)$ , since Theorem 1.9(i) shows that  $[Z^\rho, T^\rho] = 0$  in  $U_R(T)$ . Then (24) holds in  $U_R(J)$ , since the inclusion  $T \subset J$  induces a homomorphism from  $U_R(T)$  to  $U_R(J)$  [J3].

By Theorem 1.8, there is an  $R$ -algebra isomorphism  $\phi$  of  $(Z \otimes_R Z)^\pi$  onto  $U_R(Z)$  such that  $\phi(a \otimes a) = u(a)$ ,  $a \in Z$ . Let  $p$  be the canonical  $R$ -algebra homomorphism from  $U_R(Z)$  to  $U_R(J)$  [J3]. There is an  $R$ -module homomorphism  $\psi$  from  $Z \otimes_R Z$  to  $U_R(J)$  such that  $\psi(b \otimes c) = u(b, cx)$ ,  $b, c \in Z$ . Then (24) shows that

$$p\phi(a \otimes a) \psi(b \otimes c) = \psi(ab \otimes ac), \quad a, b, c \in Z.$$

Together with (3), this shows that

$$(p\phi w)(\psi y) = \psi(wy), \quad w \in (Z \otimes_R Z)^\pi, \quad y \in Z \otimes_R Z. \tag{25}$$

$\psi(Z \otimes_R Z)$  is contained in  $\langle Z^\rho \rangle(Zx)^\rho$ , since  $[Z^\rho, J^\rho] = 0$  [Theorem 1.9(ii)]. Since  $p\phi$  maps  $(Z \otimes_R Z)^\pi$  onto  $\langle Z^\rho \rangle$  and  $\psi$  maps  $1 \otimes Z$  onto  $(Zx)^\rho$ , (25) implies that

$$\psi(Z \otimes_R Z) = \langle Z^\rho \rangle(Zx)^\rho. \tag{26}$$

Since  $Z$  is a commutative separable associative  $R$ -algebra, so is  $Z \otimes_R Z$  [3, p. 43]. It follows that  $Z \otimes_R Z$  is finitely spanned over  $(Z \otimes_R Z)^\pi$  [2, p. 128]. Then (25) and (26) imply that  $\langle Z^\rho \rangle(Zx)^\rho$  is finitely spanned over  $\langle Z^\rho \rangle$ . ■

**PROPOSITION 2.2.** *If  $J$  is separable over  $Z(J)$  and  $Z(J)$  is a separable associative  $R$ -algebra, then  $U_R(J)$  is finitely spanned over  $\langle Z(J)^\rho \rangle$ .*

*Proof.* We write  $Z(J)$  as  $Z$ . Since  $Z$  is a separable associative  $R$ -algebra,

$$[\langle Z^\rho \rangle, U_R(J)] = 0 \tag{27}$$

[Theorem 1.9(ii)]. Since  $J$  is separable over  $Z$ , there are finitely many  $x_i \in J$  such that  $J = \sum Zx_i$  [S3]. Each  $\langle Z^\rho \rangle(Zx_i)^\rho$  is finitely spanned over  $\langle Z^\rho \rangle$  [Lemma 2.1]. Hence  $\langle Z^\rho \rangle J^\rho$  is finitely spanned over  $\langle Z^\rho \rangle$ . Consequently, there are finitely many  $y_i \in J$  such that

$$\langle Z^\rho \rangle J^\rho = \sum \langle Z^\rho \rangle y_i^\rho. \tag{28}$$

We refer to an element of  $U_R(J)$  of the form

$$v = wy_{i_1}^p y_{i_2}^p \cdots y_{i_d}^p, \quad w \in \langle Z^p \rangle \tag{29}$$

as a monomial of degree  $d$ . If  $1 \leq j \leq d - 2$ , let  $v_j$  be the monomial obtained from  $v$  by interchanging  $y_{i_j}$  and  $y_{i_{j+2}}$ . Taking  $a = y_{i_j}$ ,  $b = y_{i_{j+1}}$ , and  $c = y_{i_{j+2}}$  in (6) and using (27) and (28) shows that

$$v = -v_j + (\text{monomials of degree } < d). \tag{30}$$

If  $i_j > i_{j+2}$ , (30) shows that  $v$  equals a sum of monomials of lower degree plus a monomial of the same degree with  $i_j < i_{j+2}$  and the other  $i_s$  unchanged. If  $i_j = i_{j+2}$ , then  $v = v_j$ , so (30) implies that  $v$  equals a sum of monomials of lower degree. Hence induction on the degree shows that every monomial equals a sum of monomials of form (29) satisfying

$$i_1 < i_3 < \cdots \quad \text{and} \quad i_2 < i_4 < \cdots. \tag{31}$$

Equations (4), (27), and (28) imply that  $U_R(J)$  is spanned over  $R$  by monomials. Hence  $U_R(J)$  is spanned over  $R$  by monomials of form (29) satisfying (31). Thus  $U_R(J)$  is finitely spanned over  $\langle Z^p \rangle$ . ■

Lemmas 2.3 and 2.4 show that, if  $J$  is separable over  $Z(J)$ ,  $Z(J)$  is separable over  $R$ , and  $R$  is a field, then  $J$  is separable over  $R$ .

LEMMA 2.3. *Let  $S$  and  $T$  be commutative associative  $R$ -algebras. Let  $J$  be a separable  $S$ -algebra such that  $J$  is an  $R$ -algebra via  $R1 \subset S$ . Then  $J \otimes_R T$  is either zero or separable over  $S \otimes_R T$ .*

*Proof.* Assume that  $J \otimes_R T \neq 0$ . One checks that

$$\begin{aligned} U_{S \otimes_R T}(J \otimes_R T) &\simeq U_{S \otimes_R T}(J \otimes_S (S \otimes_R T)) \\ &\simeq U_S(J) \otimes_S (S \otimes_R T) \quad (\text{by [J1]}) \\ &\simeq U_S(J) \otimes_R T. \end{aligned} \tag{32}$$

Since  $J$  is separable over  $S$ ,  $U_S(J)$  is separable associative over  $S$ . Hence  $U_S(J) \otimes_R T$  is separable associative over  $S \otimes_R T$  [3, p. 43]. Then (32) shows that  $U_{S \otimes_R T}(J \otimes_R T)$  is separable associative over  $S \otimes_R T$ , so  $J \otimes_R T$  is separable over  $S \otimes_R T$ . ■

LEMMA 2.4. *Let  $S$  be a commutative separable associative algebra over a field  $R$ . Let  $J$  be a separable  $S$ -algebra such that  $J$  is an  $R$ -algebra via  $R1 \subset S$ . Then  $J$  is separable over  $R$ .*

*Proof.* Let  $F$  be the algebraic closure of  $R$ . Since  $S$  is separable

associative over  $R$ ,  $S \otimes_R F$  is the direct sum of a finite number of ideals  $F_i$  isomorphic to  $F$  as  $F$ -algebras [A3]. Since  $J$  is separable over  $S$ ,  $J \otimes_R F$  is separable over  $S \otimes_R F$  [Lemma 2.3]. For any integer  $j$ , let  $I_j \subset S \otimes_R F$  be the sum of the  $F_i$  for  $i \neq j$ . Since  $J \otimes_R F$  is separable over  $S \otimes_R F$ ,  $(J \otimes_R F)/I_j(J \otimes_R F)$  is separable over  $(S \otimes_R F)/I_j$  [S6]. Equivalently,  $F_j(J \otimes_R F)$  is separable over  $F_j \simeq F$ . Then  $F_j(J \otimes_R F)$  is finite-dimensional and semisimple over  $F$  [S1], and hence so is  $J \otimes_R F = \bigoplus F_j(J \otimes_R F)$ . Thus  $J$  is separable over  $R$  [S1]. ■

**THEOREM 2.5.** *The following conditions are equivalent:*

- (i).  $J$  is separable over  $R$ .
- (ii).  $J$  is separable over  $Z(J)$  and  $Z(J)$  is separable (as a Jordan or associative algebra) over  $R$ .

*Proof.* The two senses in which  $Z(J)$  may be separable over  $R$  are equivalent, by Theorem 1.8. [S3] and Proposition 1.1 show that (i)  $\Rightarrow$  (ii). Conversely, let  $J$  satisfy (ii). Set  $T = \langle Z(J)^\rho \rangle$  for  $\rho: J \rightarrow U_R(J)$ . Theorem 1.9(ii) shows that  $T$  is commutative. Let  $m$  be a maximal ideal of  $T$ .  $J \otimes_R T/m$  is zero or separable over  $Z(J) \otimes_R T/m$  [Lemma 2.3], and  $Z(J) \otimes_R T/m$  is zero or separable over  $T/m$  [A5]. Then  $J \otimes_R T/m$  is zero or separable over  $T/m$  [Lemma 2.4], so  $U_{T/m}(J \otimes_R T/m)$  is zero or separable associative over  $T/m$ . Since

$$U_{T/m}(J \otimes_R T/m) \simeq U_R(J) \otimes_R T/m$$

[J1], the latter is zero or separable associative over  $T/m$ . Because  $T$  is in the center of  $U_R(J)$  [Theorem 1.9(ii)],  $U_R(J)$  is naturally a  $T$ -algebra and there is a homomorphism of  $U_R(J) \otimes_R T/m$  onto

$$U_R(J) \otimes_T T/m \simeq U_R(J)/mU_R(J).$$

Thus  $U_R(J)/mU_R(J)$  is zero or separable associative over  $T/m$  [A5]. Since this holds for every maximal ideal  $m$  of  $T$  and since  $U_R(J)$  is finitely spanned over  $T$  [Proposition 2.2],  $U_R(J)$  is separable associative over  $T$  [A4]. Together with the fact that  $T$  is separable associative over  $R$  [Theorem 1.9(ii)], this implies that  $U_R(J)$  is separable associative over  $R$  [3, p. 46], so (ii)  $\Rightarrow$  (i). ■

### 3. CHARACTERIZATIONS OF SEPARABILITY

This section contains a number of consequences of Theorem 2.5, including the following results. A finite direct sum of separable Jordan  $R$ -algebras is separable over  $R$ . If  $J$  is separable over  $R$ , then  $S_R(J) \simeq S_{Z(J)}(J)$  is a

separable associative  $R$ -algebra. If  $A$  is an associative  $R$ -algebra, then  $A$  is separable as an associative  $R$ -algebra if and only if  $A^+$  is a separable Jordan  $R$ -algebra.  $J$  is special and separable over  $R$  if and only if  $J \simeq H(A, j) \oplus K$ , where  $A$  is a separable associative  $R$ -algebra with involution  $j$ ,  $K$  is the Jordan  $S$ -algebra of a nondegenerate symmetric bilinear form on a finitely spanned projective  $S$ -module, and  $S$  is a commutative separable associative  $R$ -algebra.

**THEOREM 3.1.** *Let  $S$  be a commutative separable associative  $R$ -algebra, and let  $J$  be a separable  $S$ -algebra such that  $J$  is an  $R$ -algebra via  $R1 \subset S$ . Then  $J$  is separable over  $R$ .*

*Proof.* Since  $J$  is separable over  $S$ ,  $Z(J)$  is separable associative over  $S$  [Proposition 1.1]. Since  $S$  is separable over  $R$ , the transitivity of separability for associative algebras implies that  $Z(J)$  is separable over  $R$  [3, p. 46]. Since  $J$  is separable over  $S$ ,  $J$  is separable over  $Z(J)$  [S3]. Thus Theorem 2.5 shows that  $J$  is separable over  $R$ . ■

**THEOREM 3.2.** *Let  $J = \oplus J_i$  be a finite direct sum of Jordan  $R$ -algebras. Then  $J$  is separable over  $R$  if and only if each  $J_i$  is separable over  $R$ .*

*Proof.* Each  $J_i$  is a homomorphic image of  $J$ . Thus, if  $J$  is separable over  $R$ , so is each  $J_i$  [S6]. Conversely, assume that each  $J_i$  is separable over  $R$ . We write  $Z(J_i)$  as  $Z_i$  and  $Z(J)$  as  $Z$ . Each  $J_i$  is separable over  $Z_i$  [S3], so  $U_{Z_i}(J_i)$  is separable associative over  $Z_i$ . Then  $\oplus U_{Z_i}(J_i)$  is separable associative over  $\oplus Z_i$  [3, p. 47].  $Z = \oplus Z_i$ , so  $U_Z(J) \simeq \oplus U_{Z_i}(J_i)$  is separable associative over  $Z$ . Thus  $J$  is separable over  $Z$ . Each  $Z_i$  is separable over  $R$  [Proposition 1.1], and hence so is  $Z = \oplus Z_i$  [3, p. 77]. Thus  $J$  is separable over  $R$  [Theorem 2.5]. ■

**THEOREM 3.3.** *If  $J$  is separable over  $R$ , then  $S_R(J) \simeq S_{Z(J)}(J)$  is separable associative over  $R$ .*

*Proof.*  $J \oplus R$  is a separable Jordan  $R$ -algebra [Theorem 3.2]. Since the center of  $J \oplus R$  is  $Z(J) \oplus R$ , Theorem 1.9(iii) shows that

$$\langle (Z(J) \oplus R)^\rho \rangle \subset Z[U_R(J \oplus R)] \tag{33}$$

for  $\rho: J \oplus R \rightarrow U_R(J \oplus R)$ . Ref. [6, p. 105] shows that

$$U_R(J \oplus R) \simeq U_R(J) \oplus [S_R(J) \otimes_R S_R(R)] \oplus U_R(R),$$

where the projection map from  $U_R(J \oplus R)$  to  $S_R(J) \otimes_R S_R(R)$  takes  $(a \oplus 0)^\rho$  to  $\frac{1}{2}a^\sigma \otimes 1$  for  $\sigma: J \rightarrow S_R(J)$  and  $a \in J$ . Since

$$S_R(J) \otimes_R S_R(R) \simeq S_R(J) \otimes_R R \simeq S_R(J),$$



there is an  $R$ -algebra homomorphism of  $U_R(J \oplus R)$  onto  $S_R(J)$  taking  $(a \oplus 0)^p$  to  $\frac{1}{2}a^\sigma$ ,  $a \in J$ . Then (33) implies that

$$\langle Z(J)^\sigma \rangle \subset Z[S_R(J)]. \tag{34}$$

By definition of  $S_R(J)$ ,  $(a \cdot b)^\sigma = \frac{1}{2}(a^\sigma b^\sigma + b^\sigma a^\sigma)$  for  $a, b \in J$  [6, p. 65]. Then (34) implies that  $(a \cdot b)^\sigma = a^\sigma b^\sigma = b^\sigma a^\sigma$  for  $a \in Z(J)$  and  $b \in J$ . Hence letting  $a \in Z(J)$  act on  $S_R(J)$  as multiplication by  $a^\sigma$  makes  $S_R(J)$  a  $Z(J)$ -algebra and  $\sigma$  an associative specialization of  $J$  over  $Z(J)$ . Thus there is an  $R$ -algebra homomorphism from  $S_{Z(J)}(J)$  to  $S_R(J)$  which is the inverse of the canonical one from  $S_R(J)$  to  $S_{Z(J)}(J)$  [6, pp. 65, 66]. Hence  $S_R(J)$  is isomorphic to  $S_{Z(J)}(J)$ . Since we have constructed an algebra homomorphism of  $U_R(J \oplus R)$  onto  $S_R(J)$  and  $U_R(J \oplus R)$  is separable associative over  $R$ , so is  $S_R(J)$  [A5]. ■

We recall definition [S7] of the components  $J(p, q)$  of a separable algebra  $J$ .

**COROLLARY 3.4.** *Let  $J$  be separable over  $R$ . Then  $J(3, 8)$  is the kernel of  $\sigma: J \rightarrow S_R(J)$ . In particular,  $J$  is special if and only if  $J = \oplus J(p, q)$  for  $(p, q) \neq (3, 8)$ .*

*Proof.* Theorem 3.3 implies that the kernel of  $\sigma_1: J \rightarrow S_R(J)$  equals the kernel of  $\sigma_2: J \rightarrow S_{Z(J)}(J)$ . Since  $J$  is finitely spanned and separable over  $Z(J)$  [S3], then kernel of  $\sigma_2$  is  $J(3, 8)$  [2, p. 138]. The corollary follows, since the components  $J(p, q)$  are the same whether  $J$  is considered over  $R$  or  $Z(J)$ . ■

**COROLLARY 3.5.** *Let  $A$  be an associative  $R$ -algebra. Then  $A$  is separable associative over  $R$  if and only if  $A^+$  is separable over  $R$ . In this case,  $Z(A) = Z(A^+)$ .*

*Proof.* First assume that  $A$  is separable associative over  $R$ .  $A$  is separable associative over  $Z(A)$ , and  $Z(A)$  is separable associative over  $R$  [A1]. Since  $A$  is finitely spanned over  $Z(A)$  [A7],  $A^+$  is separable over  $Z(A) = Z(A^+)$  [2, p. 119]. Thus  $A^+$  is separable over  $R$  [Theorem 2.5].

Next assume that  $A^+$  is finitely spanned and separable over  $R$ . Let  $m$  be a maximal ideal of  $R$ , and let  $F$  be the algebraic closure of  $R/m$ . Suppose that  $A/mA$  is neither zero nor separable associative over  $R/m$ . Since  $A/mA$  is finite-dimensional over  $R/m$ ,  $A/mA \otimes_{R/m} F$  contains a nonzero nilpotent ideal [A3]. The image of this ideal in

$$(A/mA \otimes_{R/m} F)^+ \simeq (A^+/mA^+) \otimes_{R/m} F$$

is nilpotent, contradicting the assumption that  $A^+$  is separable over  $R$  [S1, S2]. Thus  $A/mA$  is either zero or separable associative over  $R/m$  for

every maximal ideal  $m$  of  $R$ . Then  $A$  is separable associative over  $R$ , since  $A$  is finitely spanned over  $R$  [A4].

Finally, let  $A^+$  be separable over  $R$ . Since the identity map is an associative specialization of  $A^+$  in  $A$ , there is an  $R$ -algebra homomorphism of  $S_R(A^+)$  onto  $A$  taking  $a^\sigma$  to  $a$ ,  $a \in A$  [6, p. 65].  $[Z(A^+)]^\sigma$  is in the center of  $S_R(A^+)$  [Theorem 3.3], so  $Z(A^+)$  is in the center of  $A$ . Then  $Z(A) = Z(A^+)$ , and  $A$  is naturally a  $Z(A^+)$ -algebra. Since  $A^+$  is finitely spanned and separable over  $Z(A^+)$  [S3], the preceding paragraph shows that  $A$  is separable associative over  $Z(A^+)$ . Since  $Z(A^+)$  is separable over  $R$  [Proposition 1.1], it follows that  $A$  is separable associative over  $R$  [3, p. 46]. ■

Example 1.10(2) of [2] should have contained the additional hypothesis " $A$  is finitely spanned over  $R$ ."

**PROPOSITION 3.6.** *If  $A$  is a separable associative  $R$ -algebra with involution  $j$ , then  $H(A, j)$  is separable over  $R$ . Moreover,  $A$  is finitely spanned over  $Z[H(A, j)] = Z(A) \cap H(A, j)$ .*

*Proof.* We write  $Z(A) \cap H(A, j)$  as  $Z^j$ .  $j$  is an involution of  $A$  as a  $Z^j$ -algebra.  $A$  is finitely spanned and separable associative over  $Z^j$ , by [2, p. 128; 3, p. 46]. It follows that  $H(A, j)$  is separable over  $Z^j$  and that  $Z^j = Z[H(A, j)]$  [2, p. 119]. Since  $Z(A)$  is commutative separable associative over  $R$  [A1], so is  $Z^j$  [Lemma 1.3(i)]. Hence Theorem 2.5 shows that  $H(A, j)$  is separable over  $R$ . ■

If  $(A, j)$  is a separable associative  $R$ -algebra with involution, then  $(A, j) = \bigoplus (A(p, q), j)$ , where the ideals  $A(p, q)$  correspond to the isomorphism classes of finite-dimensional simple associative algebras with involution over an algebraically closed field [2, p. 129]. The following result is proved in [2, p. 138] under the additional hypothesis that the algebras involved are finitely spanned over  $R$ . Let  $\pi$  be the main involution of  $S_R(J)$ , so  $\pi$  fixes  $a^\sigma$  for all  $a \in J$  [6, p. 65].

**THEOREM 3.7.** *There is a category isomorphism between the category of separable Jordan  $R$ -algebras  $J$  such that  $J = \bigoplus J(p, q)$  ( $p \geq 3$  and  $q \leq 4$ ) and the category of separable associative  $R$ -algebras with involution  $(A, j)$  such that  $(A, j) = \bigoplus (A(p, q), j)$  ( $p \geq 3$ ). This isomorphism takes  $J$  to  $(S_R(J), \pi)$  and  $(A, j)$  to  $H(A, j)$ . If  $J$  and  $(A, j)$  correspond, then  $J(p, q)$  and  $(A(p, q), j)$  also correspond for all  $(p, q)$ .*

*Proof.* Let  $J$  be a separable  $R$ -algebra such that  $J = \bigoplus J(p, q)$ ,  $p \geq 3$ ,  $q \leq 4$ . We write  $Z(J)$  as  $Z$ . Since  $J$  is finitely spanned and separable over  $Z$  [S3], [2, p. 138] shows that  $S_Z(J)$  is separable associative over  $Z$ ,

$$(S_Z(J), \pi) = (S_Z(J)(p, q), \pi),$$

and  $\sigma$  induces an isomorphism of  $J$  onto  $H(S_Z(J), \pi)$ .  $S_R(J) \simeq S_Z(J)$  is separable associative over  $R$  [Theorem 3.3], so

$$(S_R(J), \pi) = (S_R(J)(p, q), \pi)$$

and  $\sigma$  induces an isomorphism of  $J$  onto  $H(S_R(J), \pi)$ .

Conversely, let  $(A, j)$  be a separable associative  $R$ -algebra with involution such that  $(A, j) = (A(p, q), j)$ ,  $p \geq 3$ . If we write  $H(A, j)$  as  $J$ , then  $J$  is separable over  $R$ ,  $Z(J) \subset Z(A)$ , and  $A$  is finitely spanned over  $Z(J)$  [Proposition 3.6].  $A$  is separable associative over  $Z(J)$  with involution  $j$  [3, p. 46], so [2, p. 138] shows that  $J = J(p, q)$  and that the canonical map from  $(S_{Z(J)}(J), \pi)$  to  $(A, j)$  is an isomorphism. Hence Theorem 3.3 implies that the canonical map from  $(S_R(J), \pi)$  to  $(A, j)$  is an isomorphism.

The theorem follows by taking direct sums of the components  $J(p, q)$  and  $(A(p, q), j)$ , applying Theorem 3.2, [J5], and [3, p. 77]. ■

**COROLLARY 3.8.** *If  $J$  is a separable  $R$ -algebra such that  $J = \bigoplus J(p, q)$  for  $p \geq 3$  and  $q \leq 4$ , then  $J$  is special and reflexive.* ■

Let  $M$  be a module over a commutative ring  $S$ , and let  $Q: M \times M \rightarrow S$  be a symmetric bilinear form on  $M$ . Let  $J(Q, M, S)$  be the Jordan  $S$ -algebra determined by  $Q$ , i.e.,  $J(Q, M, S)$  is the  $S$ -module  $S \oplus M$  with multiplication

$$(\alpha, a) \cdot (\beta, b) = (\alpha\beta + Q(a, b), ab + \beta a),$$

$\alpha, \beta \in S$ ,  $a, b \in M$  [6, p. 13]. If  $S$  is an  $R$ -algebra, we consider  $J(Q, M, S)$  as a  $R$ -algebra via  $R1 \subset S$ . We recall definition [M5] of a nondegenerate symmetric bilinear form on a finitely spanned projective module over a commutative ring.

**THEOREM 3.9.** *The following conditions are equivalent:*

- (i).  $J$  is special and separable over  $R$ .
- (ii).  $J \simeq H(A, j) \oplus J(Q, M, S)$ , where  $A$  is a separable associative  $R$ -algebra with involution  $j$ ,  $S$  is a commutative separable associative  $R$ -algebra,  $M$  is a finitely spanned projective  $S$ -module, and  $Q$  is a nondegenerate symmetric bilinear form on  $M$  over  $S$ .

*Proof.* (i)  $\Rightarrow$  (ii). If  $J$  satisfies (i), then  $J = J_1 \oplus J_2 \oplus J_3$ , where  $J_1 = J(1, 1)$ ,  $J_2 = \bigoplus J(2, q)$ , and  $J_3 = \bigoplus J(p, q')$ ,  $q \geq 2$ ,  $p \geq 3$ ,  $q' \leq 4$  [S7, Corollary 3.4]. Each  $J_i$  is separable over its center  $Z_i$  and  $Z_i$  is separable associative over  $R$  [Theorem 2.5, Theorem 3.2].  $J_2 = J(Q, M, Z_2)$ , where  $Q$  is a nondegenerate symmetric bilinear form on a finitely spanned projective  $Z_2$ -module  $M$  [2, p. 140]. If  $m$  is any maximal ideal of  $Z_1$ ,  $J_1/mJ_1$  is commutative [S7], so  $J_1 = Z_1 + mJ_1$  [S6]. Since  $J_1$  is finitely spanned over

$Z_1$  [S3],  $J_1 = Z_1$  [M6]. Set  $A_1 = Z_1 \oplus Z_1$  and let  $j_1$  be the  $R$ -algebra involution of  $A_1$  taking  $a \oplus b$  to  $b \oplus a$ ,  $a, b \in Z_1$ . Clearly,  $J_1 \simeq H(A_1, j_1)$ . Since  $Z_1$  is separable associative over  $R$ , so is  $A_1$  [3, p. 77]. Theorem 3.7 implies that  $J_3 \simeq H(A_2, j_2)$ , where  $A_2$  is a separable associative  $R$ -algebra with involution  $j_2$ .  $A_1 \oplus A_2$  is a separable associative  $R$ -algebra [3, p. 77] with involution  $j_1 \oplus j_2$ , and  $J_1 \oplus J_3$  is isomorphic to  $H(A_1 \oplus A_2, j_1 \oplus j_2)$ . Hence  $J$  satisfies (ii).

(ii)  $\Rightarrow$  (i). Let  $J$  satisfy (ii). Write  $J(Q, M, S)$  as  $K$ . Proposition 3.6 shows that  $H(A, j)$  is a separable Jordan  $R$ -algebra.  $K$  is a separable Jordan  $S$ -algebra [2, p. 119], so Theorem 3.1 implies that  $K$  is separable over  $R$ . Thus  $J$  is separable over  $R$  [Theorem 3.2]. If  $m$  is any maximal ideal of  $S$ ,  $K/mK$  is the Jordan algebra of a quadratic form over  $S/m$ . Then  $K/mK$  is special [6, p. 261], so  $(K/mK)(3, 8) = 0$ . Thus  $K(3, 8)/mK(3, 8) = 0$  [S8]. Since  $K$  is finitely spanned over  $S$ , Nakayama's Lemma yields  $K(3, 8) = 0$  [M6]. Hence  $K$  is special over  $R$  [Corollary 3.4]. Since  $H(A, j)$  is also special over  $R$ , so is  $J$ . ■

In a subsequent article, we use generic minimum polynomials and the construction of Freudenthal–Springer–McCrimmon [9] to extend Theorem 3.9 to a determination of all separable Jordan algebras over commutative rings.

#### 4. CENTRALIZERS OF SEPARABLE ALGEBRAS

If  $J \subset B$  are Jordan  $R$ -algebras, the centralizer of  $J$  in  $B$  is

$$\begin{aligned} C_B(J) &= \{c \in B \mid [a, b, c] = 0 \text{ for all } a \in J, b \in B\} \\ &= \{c \in B \mid [a^\rho, c^\rho]B = 0 \text{ for all } a \in J, \rho: B \rightarrow U_R(B)\}. \end{aligned} \tag{35}$$

Centralizers of separable Jordan algebras over fields were studied by Harris in [4]. In this section we extend his results to separable Jordan algebras over commutative rings.

Let  $J \subset B$  be Jordan  $R$ -algebras and let  $J$  be separable over  $R$ . Define  $B^J$  as in (1). We prove that  $C_B(J) = B^J$ , whence  $C_B(J)$  is a functorial subalgebra of  $B$ . If  $B$  is also separable over  $R$ , then so is  $C_B(J)$ . If  $M$  is a  $J$ -bimodule, we prove that

$$M = M^{Z(J)} \oplus [Z(J), Z(J), M]$$

is a direct sum of  $J$ -bimodules, where  $M^{Z(J)}$  is a bimodule for  $J$  as a  $Z(J)$ -algebra.

Let  $J \subset B$  be Jordan  $R$ -algebras and let  $J$  be separable over  $R$ . Lemma 4.1

states that  $C_B(J) = B^J$  is a subalgebra of  $B$  if  $J^\rho$  centralizes  $(B^J)^\rho$  in  $U_R(B)$ . Theorem 1.9(i) implies that  $Z(J)^\rho$  centralizes  $(B^J)^\rho$ , and Lemma 4.2 shows that  $[J, J, J]^\rho$  centralizes  $(B^J)^\rho$  in  $U_R(B)$ . In Lemmas 4.3–4.5 we use the fact that  $U_R(J)$  is finitely spanned over  $\langle Z(J)^\rho \rangle$  [Proposition 2.2] to prove that  $U_R(J)$  is generated as an  $R$ -algebra by  $Z(J)^\rho$  and  $[J, J, J]^\rho$ . Together these results imply that  $C_B(J) = B^J$  is a subalgebra of  $B$ .

**LEMMA 4.1.** *If  $J \subset B$  are Jordan  $R$ -algebras such that  $J^\rho$  centralizes  $(B^J)^\rho$  for  $\rho: B \rightarrow U_R(B)$ , then  $C_B(J) = B^J$  is a subalgebra of  $B$ .*

*Proof.* Equations (1) and (35) show that  $C_B(J) \subset B^J$ . Since  $[J^\rho, (B^J)^\rho] = 0$ , (35) implies that  $B^J \subset C_B(J)$ , so  $C_B(J) = B^J$ . If  $x, y, a, b$  are elements of any Jordan algebra, one verifies that

$$[x, y, a \cdot b] = -[x \cdot y, a, b] + [x, y, a] \cdot b + [x, y \cdot a, b] + x \cdot [y, a, b]. \tag{36}$$

Since the equation  $C_B(J) = B^J$  implies that  $[J, B, B^J] = 0$ , taking  $x, y \in J$  and  $a, b \in B^J$  in (36) yields  $[x, y, a \cdot b] = 0$ . Then  $a \cdot b \in B^J$  for  $a, b \in B^J$ , so  $B^J$  is a subalgebra of  $B$ . ■

If  $a, b, c$  are elements of an associative algebra, one verifies the Jacobi identity

$$[c, [a, b]] = [[c, a], b] + [a, [c, b]]. \tag{37}$$

**LEMMA 4.2.** *If  $J \subset B$  are Jordan  $R$ -algebras, then  $[J, J, J]^\rho$  centralizes  $(B^J)^\rho$  for  $\rho: B \rightarrow U_R(B)$ .*

*Proof.* If  $a, b, c \in J$  and  $g \in B^J$ , Eqs. (2), (5), and (37) imply that

$$\begin{aligned} [g^\rho, [a, b, c]^\rho] &= [g^\rho, [[c^\rho, a^\rho], b^\rho]] \\ &= [[g^\rho, [c^\rho, a^\rho]], b^\rho] + [[c^\rho, a^\rho], [g^\rho, b^\rho]] \\ &= [[[g^\rho, c^\rho], a^\rho], b^\rho] + [[c^\rho, [g^\rho, a^\rho]], b^\rho] \\ &\quad + [[c^\rho, [g^\rho, b^\rho]], a^\rho] + [c^\rho, [a^\rho, [g^\rho, b^\rho]]] \\ &= [[c, a, g]^\rho, b^\rho] + [[g, c, a]^\rho, b^\rho] \\ &\quad + [[g, c, b]^\rho, a^\rho] + [c^\rho, [g, a, b]^\rho] \\ &= 0, \end{aligned}$$

since  $[J, J, B^J] = 0$ . ■

Let  $e_{ij}$  be the standard matrix units of  $M_p(C)$ , the nonassociative algebra of  $p$ -by- $p$  matrices over a composition algebra  $C$ . If  $c \in C$ ,  $\bar{c}$  is the conjugate of  $c$ , and  $i \neq j$ , let  $c[ij] = ce_{ij} + \bar{c}e_{ji} \in H(M_p(C))$ .

LEMMA 4.3. *If  $J$  is separable over an algebraically closed field  $R$ , then  $J$  is spanned over  $R$  by  $Z(J)$ ,  $[J, J, J]$ , and*

$$\{(a_1 \cdot a_2) \cdot a_3 \mid a_i \in [J, J, J] \text{ and } a_i \cdot a_j \in [J, J, J] \text{ for } i \neq j\}. \quad (38)$$

*Proof.*  $J$  is a direct sum of finite-dimensional simple  $R$ -algebras [S1]. Considering each component separately, we can assume that  $J$  is finite-dimensional and simple. We apply the classification of such algebras, using the notation of the Preliminaries. If  $J \simeq R$  or  $J = R[2, q]$ ,  $q \geq 2$ , it is clear that  $J = R1 + [J, J, J]$  for  $1 \in J$ . Assume that  $J \simeq H(M_p(C))$ ,  $p \geq 3$ . If  $c \in C$  and  $i \neq j$ ,

$$c[ij] = [4c[ij], e_{ii}, e_{jj}] \quad \text{and} \quad e_{ii} - e_{jj} = [2[ij], 1[ij], e_{ii}]. \quad (39)$$

If  $i, j, k$  are distinct,

$$e_{ii} + e_{jj} = (2[ik] \cdot 1[kj]) \cdot 1[ij], \quad (40)$$

where (39) implies that  $a_1 = 2[ik]$ ,  $a_2 = 1[kj]$ , and  $a_3 = 1[ij]$  satisfy condition (38). The lemma follows, since  $H(M_p(C))$  is spanned over  $R$  by elements of the forms (39) and (40). ■

LEMMA 4.4. *If  $J$  is separable over a field  $R$ , then  $U_R(J)$  is generated as an  $R$ -algebra by  $Z(J)^p$  and  $[J, J, J]^p$ .*

*Proof.* By field extension, we can assume that  $R$  is algebraically closed. Equation (6) implies that  $[J, J, J]^p$  contains all elements  $x^p$  such that  $x = (a_1 \cdot a_2) \cdot a_3$  has form (38). Then Lemma 4.3 implies that the subalgebra of  $U_R(J)$  generated by  $Z(J)^p$  and  $[J, J, J]^p$  contains  $J^p$ . We are done by (4). ■

LEMMA 4.5. *If  $J$  is separable over  $R$ , then  $U_R(J)$  is generated as an  $R$ -algebra by  $Z(J)^p$  and  $[J, J, J]^p$ .*

*Proof.* We write  $\langle Z(J)^p \rangle$  as  $S$ . Since  $S$  is in the center of  $U_R(J)$  [Theorem 1.9(iii)],  $U_R(J)$  is naturally an  $S$ -algebra. Let  $m$  be a maximal ideal of  $S$ .  $J \otimes_R S/m$  is either zero or a separable  $S/m$ -algebra whose center is  $Z(J) \otimes_R S/m$  [S6]. By Lemma 4.4,  $U_{S/m}(J \otimes_R S/m)$  is generated as an  $S/m$ -algebra by  $(Z(J) \otimes_R S/m)^p$  and

$$[J \otimes_R S/m, J \otimes_R S/m, J \otimes_R S/m]^p.$$

$U_{S/m}(J \otimes_R S/m)$  is isomorphic to  $U_R(J) \otimes_R S/m$  [J1]. Since  $U_R(J)$  is an  $S$ -algebra, there is a homomorphism of  $U_R(J) \otimes_R S/m$  onto

$$U_R(J) \otimes_S S/m \simeq U_R(J)/mU_R(J).$$

Composing these maps gives a homomorphism of  $U_{S/m}(J \otimes_R S/m)$  onto  $U_R(J)/mU_R(J)$  as  $S/m$ -algebras. It follows that  $U_R(J)/mU_R(J)$  is generated as a unital  $S/m$ -algebra by the image of  $[J, J, J]^\rho$ . Thus  $U_R(J) = T + mU_R(J)$ , where  $T$  is the  $R$ -subalgebra of  $U_R(J)$  generated by  $Z(J)^\rho$  and  $[J, J, J]^\rho$ .  $U_R(J)$  is finitely spanned over  $S$  [Proposition 2.2, Theorem 2.5] and  $T$  is an  $S$ -submodule of  $U_R(J)$ , so  $U_R(J) = T$ , by Nakayama's Lemma [M6]. ■

**THEOREM 4.6.** *If  $J \subset B$  are Jordan  $R$ -algebras and  $J$  is separable over  $R$ , then  $C_B(J) = B^J$  is an  $R$ -subalgebra of  $B$ . Moreover,  $(C_B(J))^\rho$  centralizes  $J^\rho$  for  $\rho: B \rightarrow U_R(B)$ .*

*Proof.* If  $\rho: B \rightarrow U_R(B)$ , Proposition 1.1 and Theorem 1.9(i) imply that  $\langle Z(J)^\rho \rangle$  centralizes  $(B^{Z(J)})^\rho \supset (B^J)^\rho$ . Lemma 4.2 shows that  $[J, J, J]^\rho$  centralizes  $(B^J)^\rho$ . The canonical homomorphism from  $U_R(J)$  to  $U_R(B)$  has image  $\langle J^\rho \rangle$  [J3], so Lemma 4.5 implies that  $\langle J^\rho \rangle$  is generated as an  $R$ -algebra by  $Z(J)^\rho$  and  $[J, J, J]^\rho$ . Then  $\langle J^\rho \rangle$  centralizes  $(B^J)^\rho$ , and we are done by Lemma 4.1. ■

**COROLLARY 4.7.** *If  $J \subset B \subset D$  are Jordan  $R$ -algebras and  $J$  is separable over  $R$ , then  $C_B(J) = C_D(J) \cap B$ .*

*Proof.* Equation (1) implies that  $B^J = D^J \cap B$ . Then  $C_B(J) = C_D(J) \cap B$ , by Theorem 4.6. ■

**COROLLARY 4.8.** *Let  $J \subset B$  be Jordan  $R$ -algebras such that  $J$  is separable over  $R$ .*

(i). *If  $S$  is a commutative associative  $R$ -algebra and  $f: J \otimes_R S \rightarrow B \otimes_R S$  is the canonical map, then*

$$C_{B \otimes_R S}[f(J \otimes_R S)] \simeq C_B(J) \otimes_R S.$$

(ii). *If  $I$  is an ideal of  $R$ , then*

$$C_{B/IB}[J/(J \cap IB)] \simeq C_B(J)/IC_B(J).$$

(iii). *If  $\phi$  is an algebra homomorphism from  $B$  to another  $R$ -algebra, then  $C_{\phi B}(\phi J) = \phi[C_B(J)]$ .*

*Proof.* (i). Let all tensor products be taken over  $R$ .  $B^J = eB$ , where  $e \in U_R(J)$  is a separability idempotent for  $J$  [S4].  $e \otimes 1 \in U_R(J) \otimes S \simeq$

$U_S(J \otimes S)$  is a separability idempotent for  $J \otimes S$  as an  $S$ -algebra [2, p. 117]. Then  $(B \otimes S)^{f(J \otimes S)} = (e \otimes 1)(B \otimes S)$  is the image in  $B \otimes S$  of  $eB \otimes S$ . Since  $eB$  is a direct summand of  $B$  as an  $R$ -module, the map from  $eB \otimes S$  to  $B \otimes S$  is injective. Hence  $(B \otimes S)^{f(J \otimes S)}$  is isomorphic to  $eB \otimes S = B^J \otimes S$ . Theorem 4.6 implies that  $C_B(J) = B^J$  and

$$C_{B \otimes S}[f(J \otimes S)] = (B \otimes S)^{f(J \otimes S)},$$

since  $J$  is separable over  $R$  and  $f(J \otimes S)$  is either zero or separable over  $S$  [S6]. Hence  $C_{B \otimes S}[f(J \otimes S)]$  is isomorphic to  $C_B(J) \otimes S$ . Parts (ii) and (iii) are proved similarly. ■

If  $J \subset B$  are Jordan algebras over a field  $R$  such that  $J$  is separable and  $B$  is central simple over  $R$ , Harris proved that  $C_B(J)$  is a separable  $R$ -algebra [4, p. 785].

**THEOREM 4.9.** *If  $J \subset B$  are separable Jordan  $R$ -algebras, then so is  $C_B(J)$ .*

*Proof.* First assume that  $B$  is central over  $R$ . By Theorem 4.6,  $C_B(J)$  is an  $R$ -algebra. Let  $m$  be a maximal ideal of  $R$ .  $J/(J \cap mB)$  is a homomorphic image of  $J/mJ$ , so  $J/(J \cap mB)$  is either zero or separable over  $R/m$  [S6].  $B/mB$  is central separable over  $R/m$  [S6]. Since  $B/mB$  is a direct sum of simple ideals [S1] and its center is a field,  $B/mB$  is central simple over  $R/m$ . Hence Harris' theorem shows that  $C_{B/mB}[J/(J \cap mB)]$  is separable over  $R/m$ . This algebra is isomorphic to  $C_B(J)/mC_B(J)$  [Corollary 4.8(ii)], so  $C_B(J)/mC_B(J)$  is separable over  $R/m$ .  $C_B(J) = B^J = eB$ , where  $e \in U_R(J)$  is a separability idempotent for  $J$  [Theorem 4.6, S4].  $eB$  is a direct summand of  $B$  and  $B$  is finitely spanned over  $R$  [S3], so  $C_B(J)$  is finitely spanned over  $R$ . Hence, since  $C_B(J)/mC_B(J)$  is separable over  $R/m$  for every maximal ideal  $m$  of  $R$ ,  $C_B(J)$  is separable over  $R$  [S2].

In the general case, we write  $Z(B)$  as  $Z$ . Let  $ZJ$  be the  $Z$ -submodule of  $B$  spanned by  $J$ .  $ZJ$  is a  $Z$ -subalgebra of  $B$  and a homomorphic image of  $J \otimes_R Z$ . It follows that  $ZJ$  is separable over  $Z$  [S6]. Then the preceding paragraph shows that  $C_B(ZJ)$  is separable over  $Z$ . Since  $Z$  is separable associative over  $R$  [Proposition 1.1],  $C_B(ZJ)$  is separable over  $R$  [Theorem 3.1]. The theorem follows, since (35) implies that  $C_B(J) = C_B(ZJ)$ . ■

**PROPOSITION 4.10.** *Let  $J$  be separable over  $R$  and write  $Z(J)$  as  $Z$ . Then  $U_R(J)$  is the direct sum of ideals  $A$  and  $B$  such that, if  $M$  is a  $J$ -bimodule, then  $AM = M^2$  and  $BM = [Z, Z, M]$ . Thus  $M = M^2 \oplus [Z, Z, M]$  is a direct sum of  $J$ -bimodules. Moreover,  $M^2$  is a bimodule for  $J$  as a  $Z$ -algebra.*

*Proof.* Let  $\rho: J \rightarrow U_R(J)$ . Let  $v: U_R(J) \rightarrow J$  take  $x \in U_R(J)$  to  $x1$ ,  $1 \in J$ .



Since  $J$  is separable over  $R$ ,  $Z$  is a separable Jordan  $R$ -algebra [Theorem 2.5]. Let  $e \in U_R(Z)$  be a separability idempotent for  $Z$  as a Jordan  $R$ -algebra [S4]. Let  $f \in U_R(J)$  be the image of  $e$  under the canonical homomorphism from  $U_R(Z)$  to  $U_R(J)$  [J3]. Then  $f \in \langle Z^\rho \rangle$ ,  $f$  is an idempotent,  $\psi^f = 1$ , and

$$xf = (\psi x)^\rho f, \quad x \in \langle Z^\rho \rangle. \quad (41)$$

Let  $A = fU_R(J)$  and  $B = (1 - f)U_R(J)$ .  $A$  and  $B$  are ideals of  $U_R(J)$ , since Theorem 1.9(iii) implies that

$$f \in Z[U_R(J)]. \quad (42)$$

$U_R(J) = A \oplus B$ , since  $f$  is an idempotent. If  $M$  is a  $J$ -bimodule, it suffices to prove that  $fM = M^Z$  is a bimodule for  $J$  as a  $Z$ -algebra and that  $(1 - f)M = [Z, Z, M]$ .

Considering  $M$  as a  $Z$ -bimodule yields  $eM = M^Z$  [S4], so  $fM = M^Z$ . Equation (42) implies that  $fM = M^Z$  is a  $J$ -subbimodule of  $M$ . Since  $Z$  is a separable Jordan subalgebra of the split null extension  $J \oplus M$ , Theorem 4.6 implies that  $C_{J \oplus M}(Z) = (J \oplus M)^Z = J \oplus M^Z$ . Then  $[Z, J, M^Z] = 0 = [Z, M^Z, J]$ , so  $M^Z$  is a bimodule for  $J$  as a  $Z$ -algebra.

If  $a, b \in Z$  and  $m \in M$ ,

$$\begin{aligned} f[a, b, m] &= f((ab)^\rho - a^\rho b^\rho)m \\ &= ((ab)^\rho - a^\rho b^\rho)fm \quad (\text{by (42)}) \\ &= 0 \quad (\text{by (41)}). \end{aligned}$$

Thus  $f[Z, Z, M] = 0$ , so  $[Z, Z, M] \subset (1 - f)M$ . To prove that  $(1 - f)M \subset [Z, Z, M]$ , since  $\psi^f = 1$ , it suffices to prove that

$$((\psi x)^\rho - x)M \subset [Z, Z, M] \quad (43)$$

for all  $x \in \langle Z^\rho \rangle$ . We can assume that  $x = a_1^\rho \cdots a_d^\rho$ ,  $a_i \in Z$ . We induct on  $d$ . If  $d = 1$ , (43) is clear. If  $d > 1$  and  $m \in M$ , let  $z = a_2^\rho \cdots a_d^\rho$  and  $y = \psi z$ . Then

$$\begin{aligned} ((\psi x)^\rho - x)m &= (a_1 \cdot y)^\rho m - a_1^\rho z m \\ &= [a_1, y, m] + a_1^\rho y^\rho m - a_1^\rho z m \\ &= [a_1, y, m] + (y^\rho - z) a_1^\rho m \quad (\text{by Theorem 1.9(iii)}) \\ &\subset [Z, Z, M], \end{aligned}$$

since  $(y^\rho - z)M \subset [Z, Z, M]$  by induction. Hence  $(1 - f)M = [Z, Z, M]$ . ■

If  $J$  is a separable  $R$ -algebra and  $M$  is a  $J$ -bimodule, Proposition 4.10 implies that

$$[Z, Z, M] = BM = B^2M = [Z, Z, BM] = [Z, Z, [Z, Z, M]].$$

## 5. DERIVATIONS OF SEPARABLE ALGEBRAS

A derivation of  $J$  into a bimodule  $M$  is an  $R$ -linear map  $D: J \rightarrow M$  such that  $D(a \cdot b) = a(Db) + (Da)b$  for  $a, b \in J$ . A map from  $J$  to  $M$  of the form  $x \rightarrow \sum [a_i, x, b_i]$ ,  $a_i \in J$ ,  $b_i \in M$ , is a derivation [6, p. 35]. Such maps are called inner derivations. If  $J$  is separable over a field  $R$ , Harris proved that all derivations of  $J$  into its bimodules are inner if and only if the characteristic of  $R$  does not divide the degree of any special simple component of  $J$  over its center [5, p. 502]. If  $J$  is separable over a field, McCrimmon proved that all derivations of  $J$  into its bimodules are "generalized inner derivations" [11, p. 955]. In this section we extend these results to separable algebras over commutative rings. If  $J$  is separable over a commutative ring  $R$ , we prove that all derivations of  $J$  into its bimodules are inner if and only if, for every maximal ideal  $m$  of  $Z(J)$  such that  $J/mJ$  is special, the characteristic of  $Z(J)/m$  does not divide the degree of  $J/mJ$  as a central simple  $Z(J)/m$ -algebra. If  $J$  is separable over a commutative ring containing  $1/3$ , we prove that all derivations of  $J$  into its bimodules are "generalized inner derivations." We first extend Harris' theorem to the case where  $J$  is finitely spanned.

**THEOREM 5.1.** *If  $J$  is a finitely spanned, separable  $R$ -algebra, the following conditions are equivalent:*

- (i). *All derivations of  $J$  into its bimodules are inner.*
- (ii). *For every maximal ideal  $m$  of  $R$ , all derivations of  $J/mJ$  as an  $R/m$ -algebra into its bimodules are inner.*
- (iii). *For every maximal ideal  $m$  of  $R$ , the characteristic of  $R/m$  does not divide the degree of any special simple component of  $J/mJ$  over its center.*
- (iv). *For every ordered pair  $(p, q) \neq (3, 8)$  such that  $J(p, q)$  is nonzero,  $p1_{pq}$  is a unit in  $R1_{pq}$ , where  $1_{pq}$  is the unit element of  $J(p, q)$ .*

*Proof.* (i)  $\Rightarrow$  (ii). Let  $D$  be a derivation of  $J/mJ$  as an  $R/m$ -algebra into a bimodule  $M$ . Let  $f$  be the canonical map of  $J$  onto  $J/mJ$ .  $M$  becomes a bimodule for  $J$  as an  $R$ -algebra if we define  $ab$  as  $(fa)b$  and  $rb$  as  $(f(r))b$  for  $a, 1 \in J$ ,  $b \in M$ , and  $r \in R$ . Then  $Df$  is a derivation of  $J$  as an  $R$ -algebra into its bimodule  $M$ , so (i) implies that there are  $a_i \in J$  and  $b_i \in M$  such that  $Df(x) = \sum [a_i, x, b_i]$  for  $x \in J$ . Then  $Dx = \sum [fa_i, x, b_i]$  for  $x \in J/mJ$ , so  $D$  is inner. (ii)  $\Leftrightarrow$  (iii) is Harris' theorem [5, p. 502].

(iii)  $\Rightarrow$  (iv). Let  $(p, q) \neq (3, 8)$  be an ordered pair such that  $J(p, q)$  is nonzero. Let  $I = \{a \in R \mid aJ(p, q) = 0\}$ . If  $pR \neq R$ , let  $m$  be a maximal ideal of  $R$  containing  $pR$ .  $J(p, q)/mJ(p, q)$  is a direct sum of special simple algebras of degree  $p$  over their centers [S7]. Since  $pR \subset m$ , the characteristic of  $R/m$  is nonzero and divides  $p$ . Then (iii) implies that  $J(p, q)/mJ(p, q)$  is zero. Thus there is  $a \in m$  such that  $(1 - a)J(p, q) = 0$ , since  $J(p, q)$  is finitely spanned over  $R$  [12, p. 1].  $1 - a \in I$ , so  $I$  is not contained in  $m$ . Hence  $I + pR$  is an ideal of  $R$  contained in no maximal ideal, so  $R = I + pR$ . Then  $R1_{pq} = pR1_{pq}$ , so  $p1_{pq}$  is a unit in  $R1_{pq}$ .

(iv)  $\Rightarrow$  (iii). Let  $m$  be a maximal ideal of  $R$  and let  $B$  be a special simple component of  $J/mJ$ . If  $B$  has degree  $p$  over its center,  $B$  is contained in  $J(p, q)/mJ(p, q)$  for some  $(p, q) \neq (3, 8)$  [S7]. Since  $J(p, q)/mJ(p, q)$  is nonzero, (iv) implies that the image of  $p1_{pq}$  in  $J(p, q)/mJ(p, q)$  is nonzero. Hence the characteristic of  $R/m$  does not divide  $p$ .

(iv)  $\Rightarrow$  (i). Let  $D$  be a derivation of  $J$  into a bimodule  $M$ . Since  $J$  is finitely spanned over  $R$ , so is  $U_R(J)$  [J6]. Replacing  $M$  with  $U_R(J)DJ$ , we can assume that  $M$  is finitely spanned over  $R$ .

First assume that  $(R, m)$  is complete local Noetherian.  $D$  induces a derivation  $D'$  of  $J/mJ$  as an  $R/m$ -algebra into  $M/mM$ .  $D'$  is inner, by the implications (iv)  $\Rightarrow$  (iii)  $\Rightarrow$  (ii) already established. Taking preimages shows that there is an inner derivation  $I_1$  of  $J$  into  $M$  such that  $(D - I_1)J \subset mM$ . Hence it follows by induction that for every positive integer  $t$  there is an inner derivation  $I_t$  of  $J$  into  $m^{t-1}M$  such that

$$(D - I_1 - \dots - I_t)J \subset m^tM.$$

Let  $a_1, \dots, a_n$  span  $J$  over  $R$ . For each  $t$ , there are  $b_{1t}, \dots, b_{nt} \in m^{t-1}M$  such that  $I_t x = \sum [a_i, x, b_{it}]$  for  $x \in J$ .  $M$  is complete in the  $m$ -topology, since it is finitely spanned over  $R$  [1, p. 108]. For  $1 \leq i \leq n$ , let  $b_i = \sum_t b_{it}$ . Then  $Dx = \sum [a_i, x, b_i]$  for  $x \in J$ , and  $D$  is inner.

Next assume that  $R$  is Noetherian. Let  $m$  be a maximal ideal of  $R$ , and let  $R_m^*$  be the completion of  $R$  in the  $m$ -topology.  $R_m^*$  is complete local Noetherian [1, pp. 109, 113], and  $J \otimes R_m^*$  satisfies condition (iv) as an  $R_m^*$ -algebra [S8]. Let  $D(J, M)$  be the  $R$ -module composed of all derivations of  $J$  into  $M$  and let  $I(J, M)$  be the  $R$ -submodule of  $D(J, M)$  composed of all inner derivations.  $R_m^*$  is a flat  $R$ -module [1, p. 109], so we identify  $D(J, M) \otimes R_m^*$  and  $I(J, M) \otimes R_m^*$  with their images in  $\text{Hom}_R(J, M) \otimes R_m^*$ .  $J$  is finitely presented as an  $R$ -module, since  $J$  is finitely spanned over  $R$  and  $R$  is Noetherian. Since  $R_m^*$  is flat and  $J$  is finitely presented over  $R$ , it follows that

$$\text{Hom}_R(J, M) \otimes R_m^* \simeq \text{Hom}_{R_m^*}(J \otimes R_m^*, M \otimes R_m^*)$$

[8, p. 15]. This isomorphism takes  $D(J, M) \otimes R_m^*$  to a submodule of

$D(J \otimes R_m^*, M \otimes R_m^*)$  and  $I(J, M) \otimes R_m^*$  onto  $I(J \otimes R_m^*, M \otimes R_m^*)$ . The preceding paragraph shows that

$$D(J \otimes R_m^*, M \otimes R_m^*) = I(J \otimes R_m^*, M \otimes R_m^*).$$

Since  $D(J, M)$  contains  $I(J, M)$ , it follows that  $D(J, M) \otimes R_m^*$  equals  $I(J, M) \otimes R_m^*$ . Thus

$$[D(J, M)/I(J, M)] \otimes R_m^* = 0.$$

Then  $[D(J, M)/I(J, M)] \otimes R_m = 0$ , where  $R_m$  is the localization of  $R$  at  $m$  [1, pp. 110, 114–115]. Since this holds for every maximal ideal  $m$  of  $R$ ,  $D(J, M)/I(J, M) = 0$  [1, p. 40], and  $D$  is inner.

Finally let  $R$  be arbitrary. There is a Noetherian subring  $R'$  of  $R$  and a finitely spanned  $R'$ -subalgebra  $J'$  of  $J$  such that  $J'$  is separable over  $R'$  and  $J = RJ'$  [2, p. 134]. For each nonzero  $J(p, q)$  with  $(p, q) \neq (3, 8)$ , take a preimage of  $(p1_{pq})^{-1}$  in  $R$  and adjoin it to  $R'$ . Since only a finite number of  $J(p, q)$  are nonzero,  $R'$  remains Noetherian [1, p. 81]. [S8] implies that  $J'$  satisfies condition (iv) as an  $R'$ -algebra.  $D$  induces a derivation  $D'$  of  $J'$  as an  $R'$ -algebra into its bimodule  $U_{R'}(J') D(J')$ , where this bimodule is finitely spanned over  $R'$ . The preceding paragraph shows that  $D'$  is inner. Since  $J = RJ'$ , it follows that  $D$  is inner. ■

We consider next which implications in Theorem 5.1 remain valid without the hypothesis that  $J$  is finitely spanned over  $R$ . The proof of the theorem shows that the implications

$$(i) \Rightarrow (ii) \Leftrightarrow (iii) \Leftarrow (iv) \tag{44}$$

hold without this hypothesis. We use the following lemma to study the remaining implications.

**LEMMA 5.2.** *Let  $R$  be an integral domain and let  $F$  be its quotient field. If  $J$  is a separable  $F$ -algebra, then  $J$  is separable over  $R$  and  $U_R(J)$  is isomorphic to  $U_F(J)$ . If  $M$  is a bimodule for  $J$  as an  $R$ -algebra, then  $M$  is naturally a bimodule for  $J$  as an  $F$ -algebra and any derivation of  $J$  as an  $R$ -algebra into  $M$  is naturally a derivation of  $J$  as an  $F$ -algebra.*

*Proof.* Let  $\rho: J \rightarrow U_R(J)$ ,  $a, b \in R$ ,  $b \neq 0$ ,  $1, x \in J$ . Since  $\rho$  is linear over  $R$ ,

$$(b1)^\rho (b^{-1}1)^\rho = b(b^{-1}1)^\rho = (bb^{-1}1)^\rho = 1^\rho,$$

so  $[(b1)^\rho]^{-1} = (b^{-1}1)^\rho$ . Similarly,

$$(b1)^\rho (ab^{-1}x)^\rho = b(ab^{-1}x)^\rho = (ax)^\rho,$$

so

$$(ab^{-1}x)^\rho = [(b1)^\rho]^{-1}(ax)^\rho = (b^{-1}1)^\rho(ax)^\rho = (ab^{-1}1)^\rho x^\rho.$$

Likewise  $(ab^{-1}x)^\rho = x^\rho(ab^{-1}1)^\rho$ . It follows that  $U_R(J)$  becomes an  $F$ -algebra and  $\rho$  becomes  $F$ -linear if we let  $\alpha \in F$  act on  $U_R(J)$  as multiplication by  $(\alpha 1)^\rho$ . Then  $\rho$  is a multiplicative specialization of  $J$  over  $F$ , so there is a homomorphism from  $U_F(J)$  to  $U_R(J)$  which is the inverse of the canonical homomorphism from  $U_R(J)$  to  $U_F(J)$  [6, p. 88]. Thus  $U_R(J)$  and  $U_F(J)$  are isomorphic.

Since  $J$  is separable over  $F$ ,  $U_F(J)$  is separable associative over  $F$ .  $F$  is separable associative over  $R$ , since  $F$  is the quotient field of  $R$  [8, p. 74]. Then  $U_F(J)$  is separable associative over  $R$ , by the transitivity of separability [3, p. 46]. Since  $U_R(J)$  is isomorphic to  $U_F(J)$ ,  $U_R(J)$  is separable associative over  $R$ , so  $J$  is separable over  $R$ . The last statement of the lemma follows from the isomorphism of  $U_R(J)$  and  $U_F(J)$ . ■

We now show that condition (i) of Theorem 5.1 does not imply (iv) if  $J$  is not finitely spanned over  $R$ . (By (44), this also shows that (ii) and (iii) do not imply (iv).) Let  $Q$  be the rational numbers, and let  $p$  be an odd prime. Let  $R$  be the subring of  $Q$  consisting of fractions whose denominators are not divisible by  $p$ . Let  $J = M_p(Q)^+$ , where  $M_p(Q)$  is the associative algebra of  $p$ -by- $p$  matrices over  $Q$ .  $J$  is separable over  $R$  [Lemma 5.2, S1].  $J = J(p, 2)$  and  $p1$  is not a unit in  $R1 \subset J$ , so (iv) is not satisfied. If  $D$  is a derivation of  $J$  as an  $R$ -algebra into a bimodule  $M$ ,  $D$  is naturally a derivation of  $J$  as a  $Q$ -algebra into  $M$  [Lemma 5.2]. Then Harris's theorem shows that  $D$  is inner, so (i) is satisfied.

Parts (ii) and (iii) do not imply (i) if  $J$  is not finitely spanned over  $R$ . To see this, let  $p$  be an odd prime, and let  $R$  be the polynomial ring in one indeterminate over the integers modulo  $p$ . Let  $F$  be the quotient field of  $R$ , and let  $J = M_p(F)^+ \oplus M_2(R)^+$ .  $J$  is separable over  $R$  [Theorem 3.2, Lemma 5.2, S1]. For every maximal ideal  $m$  of  $R$ ,  $F/mF = 0$ , so  $J/mJ$  is isomorphic to  $M_2(R/m)^+$ . Since  $R/m$  has characteristic  $p \neq 2$ , (ii) and (iii) are satisfied. By Harris' theorem, there is a derivation  $D$  of  $M_p(F)^+$  as an  $F$ -algebra into a bimodule  $M$  such that  $D$  is not inner. Extend  $M$  to a  $J$ -bimodule by defining  $[M_2(R)^+]M = 0$ , and extend  $D$  to a derivation  $D'$  of  $J$  into  $M$  by defining  $D'[M_2(R)^+] = 0$ . Since  $D$  is not inner, neither is  $D'$ , so (i) is not satisfied.

Finally, the implication (iv)  $\Rightarrow$  (i) remains valid if  $J$  is not finitely spanned over  $R$ . This follows from the implication (v)  $\Rightarrow$  (i) of Theorem 5.4. Theorem 5.4 extends Harris' theorem to separable algebras which are not necessarily finitely spanned.

**PROPOSITION 5.3.** *If  $D$  is a derivation of a separable  $R$ -algebra  $J$  into a*

bimodule  $M$ , there is an inner derivation  $I$  of  $J$  into  $M$  such that  $D - I$  is a derivation of  $J$  as a  $Z(J)$ -algebra into its bimodule  $M^{Z(J)}$ .

*Proof.* We write  $Z(J)$  as  $Z$ . The restriction of  $D$  to  $Z$  extends to a derivation  $D_1$  of the split null extension  $Z \oplus M$  into itself such that  $D_1 M = 0$ .  $D_1$  induces a derivation  $D_2$  of the associative algebra  $U_R(Z \oplus M)$  into itself such that  $D_2(a^\rho) = (D_1 a)^\rho$ ,  $a \in Z \oplus M$ ,  $\rho: Z \oplus M \rightarrow U_R(Z \oplus M)$  [6, p. 97].  $D_2$  restricts to a derivation  $D_3$  of  $\langle Z^\rho \rangle$  into  $U_R(Z \oplus M)$ .  $\langle Z^\rho \rangle$  is a separable associative  $R$ -algebra [Proposition 1.1, Theorem 1.9(i)]. Then  $D_3$  is inner [12, p. 43], and there is  $d \in U_R(Z \oplus M)$  such that  $D_3 x = [d, x]$  for  $x \in \langle Z^\rho \rangle$ . If  $a \in Z$  and  $1 \in Z \oplus M$ ,

$$Da = D_1 a = (D_1 a)^\rho 1 = [D_2(a^\rho)]1 = [D_3(a^\rho)]1 = [d, a^\rho]1.$$

By (4),  $U_R(Z \oplus M) = \langle Z^\rho \rangle + N + P$ , where  $N = \langle Z^\rho \rangle M^\rho \langle Z^\rho \rangle$  and  $P = \sum N^i$  for  $i \geq 2$ . Let  $d = f + g + h$ ,  $f \in \langle Z^\rho \rangle$ ,  $g \in N$ ,  $h \in P$ . Since  $J^2 \subset J$  and  $M^2 = 0$  in  $J \oplus M$ , the relation  $DJ \subset M$  implies that  $Da = [g, a^\rho]1$  for  $a \in Z$ .

Since  $g \in N$ ,  $g = \sum s_i m_i^\rho t_i$  for  $s_i, t_i \in \langle Z^\rho \rangle$  and  $m_i \in M$ . If  $a \in Z$  and we write  $s_i, m_i$ , and  $t_i$  as  $s, m$ , and  $t$ , then

$$\begin{aligned} [sm^\rho t, a^\rho]1 &= sm^\rho ta^\rho 1 - a^\rho sm^\rho t1 \\ &= s(m \cdot (ta)) - a^\rho s(m \cdot (t1)) \\ &= s(ta)^\rho m - a^\rho s(t1)^\rho m \\ &= (ta)^\rho sm - (t1)^\rho a^\rho sm \end{aligned}$$

(by Proposition 1.1 and Theorem 1.9(i))

$$= (t1 \cdot a) \cdot sm - t1 \cdot (a \cdot sm)$$

(since  $a \in Z$ )

$$= [t1, a, sm].$$

Hence the map  $a \rightarrow [g, a^\rho]1$  is an inner derivation.

Together the two paragraphs above show that the restriction of  $D$  to  $Z$  is an inner derivation. This extends to an inner derivation  $I$  of  $J$  into  $M$ .  $(D - I)Z = 0$ , so  $(D - I)(z \cdot a) = z \cdot (D - I)a$  for  $z \in Z$  and  $a \in J$ . Then  $D - I$  is linear over  $Z$ , and

$$0 = (D - I)[Z, Z, J] = [Z, Z, (D - I)J].$$

Thus  $D - I$  maps  $J$  into  $M^Z$ , where  $M^Z$  is a bimodule for  $J$  as a  $Z$ -algebra [Proposition 4.10]. ■

**THEOREM 5.4.** *If  $J$  is separable over  $R$ , the following conditions are equivalent.*

- (i). *All derivations of  $J$  as an  $R$ -algebra into its bimodules are inner.*
- (ii). *All derivations of  $J$  as a  $Z(J)$ -algebra into its bimodules are inner.*
- (iii). *For every maximal ideal  $m$  of  $Z(J)$ , all derivations of  $J/mJ$  as a central simple  $Z(J)/m$ -algebra into its bimodules are inner.*
- (iv). *For every maximal ideal  $m$  of  $Z(J)$  such that  $J/mJ$  is special over  $Z(J)/m$ , the characteristic of  $Z(J)/m$  does not divide the degree of  $J/mJ$  as a central simple  $Z(J)/m$ -algebra.*
- (v). *For every ordered pair  $(p, q) \neq (3, 8)$  such that  $J(p, q)$  is nonzero, the image of  $p$  in  $Z[J(p, q)]$  is a unit.*

*Proof.* (i)  $\Rightarrow$  (ii) is obvious, and (ii)  $\Rightarrow$  (i) follows from Proposition 5.3. If  $m$  is a maximal ideal of  $Z(J)$ ,  $J/mJ$  is separable over  $Z(J)/m$ , and  $Z(J/mJ) \simeq Z(J)/m$  is a field [S6], so  $J/mJ$  is central simple over  $Z(J)/m$  [S1]. Since  $J$  is finitely spanned and separable over  $Z(J)$  [S3], Theorem 4.1 shows the equivalence of (ii) through (v). ■

**COROLLARY 5.5.** *Let  $J$  be separable over  $R$ , and assume that either  $J$  is special or  $1/3 \in R$ . Then all derivations of  $J$  into its bimodules are inner if and only if  $J = Z(J) + [J, J, J]$ .*

*Proof.* We first note that, if  $J$  is finite-dimensional central simple over a field  $R$ , then  $J = R1 + [J, J, J]$ ,  $1 \in J$ , if and only if the characteristic of  $R$  does not divide the degree of  $J$  over  $R$ . This follows by extending  $R$  to its algebraic closure and applying the classification of finite-dimensional simple Jordan algebras over an algebraically closed field [6, p. 204].

Theorem 5.4, Corollary 3.4, and the hypotheses on  $J$  imply that all derivations of  $J$  into its bimodules are inner if and only if the characteristic of  $Z(J)/m$  does not divide the degree of  $J/mJ$  over  $Z(J)/m$  for every maximal ideal  $m$  of  $Z(J)$ . The preceding paragraph, [S1], and [S6] imply that this holds if and only if

$$J/mJ = (Z(J)/m)1 + [J/mJ, J/mJ, J/mJ],$$

$1 \in J/mJ$ . This is equivalent to the condition that  $J = Z(J) + [J, J, J] + mJ$  for every maximal ideal  $m$  of  $J$ . Since  $J$  is finitely spanned over  $Z(J)$  [S3], this holds if and only if  $J = Z(J) + [J, J, J]$  [M6]. ■

Finally, we extend McCrimmon's results on generalized inner derivations to separable algebras over commutative rings containing  $1/3$ . An  $R$ -algebra  $J$

is called *semi-4-interconnected* if  $1 \in J$  can be written as a sum of orthogonal idempotents  $e_i$ , where for each  $e_i$  there are at least three other  $e_j$  such that

$$e_i \in U_{J_{ij}}(J_{jj}), \quad J_{jj} = U_{e_j}(J), \quad J_{ij} = U_{e_i, e_j}(J).$$

$J$  is called *latently 4-interconnected* if there is a faithfully flat, commutative associative  $R$ -algebra  $S$  such that  $J \otimes_R S$  is semi-4-interconnected.  $J$  is called *weakly 4-interconnected* if  $J \otimes_R R_m$  is latently 4-interconnected over  $R_m$  for every maximal ideal  $m$  of  $R$ , where  $R_m$  is the localization of  $R$  at  $m$ .

A weakly 4-interconnected algebra is special and reflexive [2, p. 139]. Thus, if  $J$  is weakly 4-interconnected and  $x_i \in J$ , let  $\langle x_1, \dots, x_n \rangle$  be the unique preimage in  $J$  of  $x_1^\sigma \cdots x_n^\sigma + x_n^\sigma \cdots x_1^\sigma$ ,  $\sigma: J \rightarrow S_R(J)$ . If  $J$  is latently 4-interconnected and one of the  $x_i$  belongs to an ideal  $N$  of  $J$ , then  $\langle x_1, \dots, x_n \rangle \in N$  [10, p. 927]. It follows by localization that this also holds if  $J$  is weakly 4-interconnected [1, p. 40].

Let  $J$  be the direct sum of ideals  $J_1$  and  $J_2$  such that  $J_1$  is weakly 4-interconnected over  $Z(J_1)$ . Let  $M$  be a bimodule for  $J$  over  $R$ , and let  $M_1$  be subbimodule of  $M$  such that  $M_1$  is a unital bimodule for  $J_1$  over  $Z(J_1)$ . Let  $a, b, x \in J_1$  and  $c \in M_1$ . The split null extension  $J_1 \oplus M_1$  is weakly 4-interconnected over  $Z(J_1)$ , so we can use it to define  $\langle a, b, c, x \rangle$  and  $\langle c, b, a, x \rangle$ . These are elements of  $M_1$ , since  $M_1$  is an ideal of  $J_1 \oplus M_1$ . Define  $D_{a,b,c} \in \text{Hom}_R(J, M)$  by  $D_{a,b,c}(J_2) = 0$  and

$$D_{a,b,c}(x) = \langle a, b, c, x \rangle - \langle c, b, a, x \rangle$$

for  $x \in J_1$ .  $D_{a,b,c}$  is a derivation, since  $J_2 M_1 = 0$  and

$$(D_{a,b,c}(x))^\sigma = [a^\sigma b^\sigma c^\sigma - c^\sigma b^\sigma a^\sigma, x^\sigma],$$

$x \in J_1$ ,  $\sigma: J_1 \oplus M_1 \rightarrow S_{Z(J_1)}(J_1 \oplus M_1)$ . Define a generalized inner derivation of  $J$  into  $M$  to be a sum of inner derivations plus derivations of the form  $D_{a,b,c}$ . (This type of derivation is called a strong generalized inner derivation in [11].)

**THEOREM 5.6.** *All derivations of a separable  $R$ -algebra  $J$  into its bimodules are generalized inner derivations if  $1/3 \in R$ .*

*Proof.* Let  $D$  be a derivation of  $J$  into a bimodule  $M$ . By Proposition 5.3, we can assume that  $R = Z(J)$ . If  $1_{pq}$  is the identity element of  $J(p, q)$ , then  $D[J(p, q)] = D(1_{pq}J) \subset 1_{pq}M$ , where  $1_{pq}M$  is a unital  $J(p, q)$ -bimodule. It suffices to prove that the restriction of  $D$  to each  $J(p, q)$  is a generalized inner derivation, so we can assume that  $J = J(p, q)$ . If  $p \leq 3$ , then  $D$  is inner, by Theorem 5.4 and the hypothesis that  $1/3 \in R$ . Let  $p \geq 4$ .  $J$  is finitely spanned over  $R = Z(J)$  [S3]. Then  $J$  is weakly 4-interconnected over  $R$ , since  $J = J(p, q)$  for  $p \geq 4$  [2, p. 139]. McCrimmon proved that all derivations of



a separable Jordan algebra over a field of characteristic  $\neq 2, 3$  into its bimodules are generalized inner derivations [11, p. 955]. This implies that all derivations of  $J$  into its bimodules are generalized inner derivations, by the proof of the implication (iv)  $\Rightarrow$  (i) of Theorem 5.1 (replacing inner derivations with generalized inner derivations in that proof). ■

**COROLLARY 5.7.** *Let  $J \subset B$  be Jordan  $R$ -algebras,  $1/3 \in R$ . If  $J$  is separable and finitely spanned over  $R$ , then any derivation of  $J$  into  $B$  extends to a derivation of  $B$  into itself.*

*Proof.* Clearly any inner derivation of  $J$  into  $B$  extends to all of  $B$ . By the proof of Theorem 5.6, it suffices to prove that any derivation of the form  $D_{a,b,c}$  extends to all of  $B$ , where  $a, b \in J(p, q)$ ,  $p \geq 4$ , and  $c$  belongs to a  $J$ -subbimodule  $B'$  of  $B$  such that  $B'$  is a unital bimodule for  $J(p, q)$  over  $Z[J(p, q)]$ . Let  $e$  be the identity element of  $J(p, q)$  and let  $B_1(e)$  be the  $i$ -eigenspace of  $B$  under multiplication by  $e$ . By hypothesis,  $J(p, q)$  is finitely spanned over  $R$ , so it is weakly 4-interconnected over  $R$  [2, p. 139].  $J(p, q)$  is a unital subalgebra of  $B_1(e)$ , so  $B_1(e)$  is weakly 4-interconnected over  $R$ . Considering  $B_1(e)$  as a bimodule for itself, let  $B_1(e) \oplus B_1(e)$  be the split null extension. If  $x \in B_1(e)$ , identify  $a, b, c$ , and  $x$ , with  $a \oplus 0, b \oplus 0, 0 \oplus c$ , and  $x \oplus 0$ , respectively, in  $B_1(e) \oplus B_1(e)$ , and define  $\langle a, b, c, x \rangle$  and  $\langle c, b, a, x \rangle$  using  $S_R(B_1(e) \oplus B_1(e))$ . Define an  $R$ -module endomorphism  $D$  of  $B$  by

$$D(B_0(e)) = 0,$$

$$Dx = \langle a, b, c, x \rangle - \langle c, b, a, x \rangle, \quad x \in B_1(e),$$

$$Dx = V_a V_b V_c x - V_c V_b V_a x, \quad x \in B_{1/2}(e).$$

For any maximal ideal  $m$  of  $R$ ,  $B_1(e) \otimes R_m$  is latently 4-interconnected, and McCrimmon has proved that  $D \otimes 1$  is a derivation of  $B \otimes R_m$  into itself [11, p. 948]. It follows that  $D$  is a derivation [1, p. 40]. If  $J(p, q) \oplus B'$  is the split null extension and  $x \in J(p, q)$ ,  $\langle a, b, c, x \rangle$  and  $\langle c, b, a, x \rangle$  are the same whether they are defined using

$$S_{Z[J(p,q)]}(J(p, q) \oplus B'), S_R(J(p, q) \oplus B'),$$

or

$$S_R(B_1(e) \oplus B_1(e)),$$

because of the canonical homomorphisms from the second of these algebras to the first and third [6, pp. 65, 66]. Hence  $D$  extends  $D_{a,b,c}$ . ■

## ACKNOWLEDGMENTS

Portions of this paper are contained in the author's dissertation written at Yale University under the direction of Professor Nathan Jacobson. The author would like to thank Professor Jacobson for his guidance and encouragement.

## REFERENCES

1. M. F. ATIYAH AND I. G. MACDONALD, "Introduction to Commutative Algebra," Addison-Wesley, Reading, Mass., 1969.
2. R. BIX, Separable Jordan algebras over commutative rings, I, *J. Algebra* **57** (1979), 111-143.
3. F. DEMEYER AND E. INGRAHAM, "Separable Algebras over Commutative Rings," Lecture Notes in Mathematics No. 181, Springer-Verlag, New York/Berlin/Heidelberg, 1971.
4. B. HARRIS, Centralizers in Jordan algebras, *Pacific J. Math.* **8** (1958), 757-790.
5. B. HARRIS, Derivations of Jordan algebras, *Pacific J. Math.* **9** (1959), 495-512.
6. N. JACOBSON, "Structure and Representations of Jordan Algebras," Colloquium Publications No. 39, Amer. Math. Soc., Providence, R.I., 1968.
7. N. JACOBSON, "Lectures on Quadratic Jordan Algebras," Tata Institute of Fundamental Research, Bombay, 1969.
8. M.-A. KNUS AND M. OJANGUREN, "Théorie de la Descente et Algèbres d'Azumaya," Lecture Notes in Mathematics No. 389, Springer-Verlag, New York/Berlin/Heidelberg, 1974.
9. K. MCCRIMMON, The Freudenthal-Springer-Tits constructions of exceptional Jordan algebras, *Trans. Amer. Math. Soc.* **139** (1969), 495-510.
10. K. MCCRIMMON, Specialty and reflexivity of quadratic Jordan algebras, *Comm. Algebra* **5** (1977), 903-935.
11. K. MCCRIMMON, Malcev's theorem for Jordan algebras, *Comm. Algebra* **5** (1977), 937-967.
12. M. ORZECH AND C. SMALL, "The Brauer Group of Commutative Rings," Lecture Notes in Pure and Applied Mathematics No. 11, Dekker, New York, 1975.