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On the Univalence of Functions Defined by Certain Integral Transforms

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The integral transform

$$F(z) = \int_0^z (f'(t))^\alpha (g(t)/t)^\beta \, dt,$$

where α and β are real, of pairs of special analytic functions $f(z) = z + \cdots$, $g(z) = z + \cdots$, univalent in the open unit disc Δ is studied. The transform and our results extend some recent results due to Shirakova.

1. INTRODUCTION

Let $f(z) = z + \cdots$ be analytic and univalent in the open unit disc Δ in the complex plane. In a recent note, Shirakova [12] studied a transform of f given by

$$F(z) = \int_0^z (f'(t))^a (f(t)/t)^{1-a} \, dt,$$

where a is a real number, $0 \leqslant a \leqslant 1$. Some of his results are: (1) If f is convex, then F is convex. (2) If f is alpha-convex in the sense of Mocanu [8], then F is alpha-convex for all a, $0 \leqslant a \leqslant \alpha$. (3) If f is starlike, then F is close-to-convex. These results are certainly in the spirit of earlier ones due to Causey [1], Causey and White [2], Kim and Merkes [5], Merkes and Wright [6], Miller *et al.* [7], Royster [11], and Silverman [13], among others.

In this article we shall study a slightly more general transform, extend some of Shirakova's results, and note a possible further direction for study as one similar to one introduced by Hornich [3].

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2. Definitions and Known Results

We shall only be interested in the set S of functions $f(z) = z + \cdots$ that are analytic and univalent in the open unit disc Δ.

If $f \in S$, then f is starlike if and only if $\text{Re}\{zf'(z)/f(s)\} > 0$ holds in Δ. The set of all starlike functions f in S is denoted by S^*.

If $f \in S$, then f is convex if and only if $\text{Re}\{1 + zf''(z)/f'(z)\} > 0$ holds in Δ. The set of all such f in S is denoted by C.

If $f \in S$, then f is said to be α-convex, α real, if and only if

$$\text{Re}[(1-\alpha)\, zf'(z)/f(z) + \alpha(1 + zf''(z)/f'(z))] > 0$$

holds in Δ. The set of all α-convex functions is denoted by M_α. It is known that $M_\alpha \subset S^*$ holds for all α, $<\infty < \alpha < \infty$, and that $M_\alpha \subset C$ for all α, $1 \leqslant \alpha < \infty$ [8].

If $f \in S$, then f is said to be close-to-convex if and only if there exists $e^{ib}\phi \in C$, b real and $-\pi/2 < b < \pi/2$ such that $\text{Re}[f'(z)/\phi'(z)] > 0$ holds in Δ. The set of all such functions f is denoted by K. It is known that a necessary and sufficient condition for $f \in S$ to satisfy $f \in K$ is that

$$-\pi < \int_{\theta_1}^{\theta_2} d\arg(zf'(z)) = \int_{\theta_1}^{\theta_2} \text{Re}\left[1 + z\frac{f''(z)}{f'(z)}\right] d\theta < 3\pi, \tag{1}$$

$z = re^{i\theta}$, hold for all $0 \leqslant \theta_1 < \theta_2 \leqslant \theta_1 + 2\pi$, and for all $0 \leqslant r < 1$ [4].

If $\phi \in C$, then

$$\tfrac{1}{2}(\theta_2 - \theta_1) \leqslant \int_{\theta_1}^{\theta_2} \text{Re}\left[\frac{z\phi'(z)}{\phi(z)}\right] d\theta = \int_{\theta_1}^{\theta_2} d\arg\phi'(z) \leqslant \pi + \tfrac{1}{2}(\theta_2 - \theta_1), \tag{2}$$

$z = re^{i\theta}$, holds for all $0 \leqslant \theta_1 < \theta_2 \leqslant \theta_1 + 2\pi$, and all $0 \leqslant r < 1$ [5].

If $f \in K$, then

$$\begin{aligned} -\pi + \tfrac{1}{2}(\theta_2 - \theta_1) \leqslant \int_{\theta_1}^{\theta_2} d\arg f(z) = \int_{\theta_1}^{\theta_2} \text{Re}\left[\frac{zf'(z)}{f(z)}\right] d\theta \\ \leqslant 2\pi + \tfrac{1}{2}(\theta_2 - \theta_1), \end{aligned} \tag{3}$$

$z = re^{i\theta}$, holds for all $0 \leqslant \theta_1 < \theta_2 \leqslant \theta_1 + 2\pi$, and for all $0 \leqslant r < 1$ [5].

We include a short proof of (3), much like that of Kim and Merkes [5]. Since $f \in K$, there exists $e^{ib}\phi \in C$, b real and $-\pi/2 < b < \pi/2$, such that $f'(z) = \phi'(z)\,\rho(z)$, where $\rho(z) = 1 + \cdots$ is analytic and has positive real part in Δ. Hence, $d\arg f'(z) = d\arg\phi'(z) + d\arg p(z)$ and, this, with (2) and the relation $-\pi \leqslant d\arg p(z) \leqslant \pi$, yields (3).

3. Generalizations of Shirakova's Results

We shall study the transform of pairs (f, g),

$$F(z) = \int_0^z (f'(t))^{\alpha} (g(t)/t)^{\beta}\, dt, \tag{4}$$

where f and g are elements of certain subsets of S, and where α and β are real constants. The special cases $\alpha = 0$ or $\beta = 0$ have been well studied by a number of authors [1, 2, 5–7, 11, 13], so that our results include many due to them.

Lemma 1. *Let f and g be fixed elements in S. Then the set of all (α, β) for which the transform* (4) *is a convex (close-to-convex) function is a closed convex set in the (α, β)-plane.*

Proof. Let (α_1, β_1) and (α_2, β_2) be pairs such that

$$F_i(z) = \int_0^z (f'(t))^{\alpha_i} (g(t)/t)^{\beta_i}\, dt \tag{5}$$

is convex in Δ for $i = 1, 2$. Then, for the function

$$F_\lambda(z) = \int_0^z (f'(t))^{\lambda_1 \alpha_1 + \lambda_2 \alpha_2} (g(t)/t)^{\lambda_1 \beta_1 + \lambda_2 \beta_2}\, dt,$$

where $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0$, and $\lambda_1 + \lambda_2 = 1$, we have

$$1 + zF_\lambda''(z)/F_\lambda'(z) = \lambda_1(1 + (zF_1''(z)/F_1'(z))) + \lambda_2(1 + (zF_2''(z)/F_2'(z))), \tag{6}$$

from which we conclude that F_λ is convex if F_1 and F_2 are convex. If $F_1 \in K$ and $F_2 \in K$, then we use (1) and (6) to conclude that F_λ is close-to-convex. This completes the proof.

Corollary. *Suppose F_1 and F_2 are close-to-convex and satisfy*

$$\operatorname{Re}\{F_i'(z)/\phi_i'(z)\} > 0, \qquad z \in \Delta, \qquad i = 1, 2,$$

for $e^{ib_1}\phi_1$ and $e^{ib_2}\phi_2$ in C, b_i real and $-\pi/2 < b_i < \pi/2$. Then F_λ satisfies the inequality

$$\operatorname{Re}\{F_\lambda'(z)/\phi_\lambda'(z)\} > 0, \qquad z \in \Delta,$$

where

$$\phi_\lambda = \int_0^z (\phi_1'(t))^{\lambda_1} (\phi_2'(z))^{\lambda_2}\, dt.$$

Proof. Since ϕ_λ is convex [5], it follows that F_λ is indeed close-to-convex (*with respect to* $\phi_\lambda(z)$).

We now state and prove our main results.

THEOREM 1. (i) *The transform* F *in* (4) *is convex for all pairs* $(f(z), g(z))$ *of convex functions only for those* (α, β) *in the closed convex hull of the points* $(0,0)$, $(1,0)$, *and* $(0,2)$.

(ii) *The transform* F *in* (4) *is close-to-convex for all pairs* $(f(z), g(z))$ *of convex functions only for those* (α, β) *in the closed convex hull of the points* $(\frac{3}{2}, 0)$, $(0,3)$, $(-\frac{1}{2}, 3)$, $(-\frac{1}{2}, 0)$, $(0,-1)$, *and* $(\frac{3}{2}, -1)$.

These results are sharp.

Proof. (i) It is a simple matter to show that F is indeed convex for the pairs $(0,0)$, $(1,0)$, and $(0,2)$. Then Lemma 1 implies the first part of our result. The choice of $f(z) = g(z) = z/(1-z)$ shows that our result is sharp.

(ii) It is easy to verify that F is indeed close-to-convex for all pairs $(f(z), g(z))$ of close-to-convex for the pairs $(\frac{3}{2}, 0)$, $(0,3)$, $(-\frac{1}{2}, 3)$, $(-\frac{1}{2}, 0)$, $(0,-1)$, and $(\frac{3}{2}, -1)$ [6]. It is instructive, however, to use a technique used by Kim and Merkes [5] and Silverman [13] to show how those vertices were obtained.

From (4) we obtain

$$1 + \frac{zF''(z)}{F'(z)} = (1 - \alpha + \beta) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + \beta\frac{zg'(z)}{g(z)}, \tag{7}$$

which will be used to obtain criteria on α and β in order that the Kaplan inequality (1) holds for F. It is clear from (7) that since we plan to use (2) as related to both f and g, we must distinguish four cases.

Case A. $\alpha \geqslant 0$, $\beta \geqslant 0$. We use (2) and (7) to obtain

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left(1 + \frac{zF''(z)}{F'(z)}\right) d\theta \geqslant (1 - \alpha - \tfrac{1}{2}\beta)(\theta_2 - \theta_1),$$

$z = re^{i\theta}$. Hence, F satisfies (1) for all $0 \leqslant \theta_1 < \theta_2 \leqslant \theta_1 + 2\pi$ if and only if either $1 - \alpha - \frac{1}{2}\beta \geqslant 0$ or $1 - \alpha - \frac{1}{2}\beta \leqslant 0$ and $(2 - 2\alpha - \beta) \geqslant 1$ holds. Hence, it is clear that for pairs (α, β) in the first quadrant, for which F is certainly close-to-convex for all $f \in C$, $g \in C$, are those (α, β) in the closed triangle with vertices $(0,0)$, $(\frac{3}{2}, 0)$, and $(0,3)$.

Case B. $\alpha \leqslant 0$, $\beta \geqslant 0$. A similar calculation shows that in this case we have

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left(1 + \frac{zF''(z)}{F'(z)}\right) d\theta \geqslant (1 - \alpha - \tfrac{1}{2}\beta)(\theta_2 - \theta_1) + 2\pi\alpha,$$

$z = re^{i\theta}$. Hence, F satisfies (1) if (α, β) satisfies $1 - \alpha - \frac{1}{2}\beta \geqslant 0$ and $2\alpha \geqslant -1$ or if (α, β) satisfies $1 - \alpha - \frac{1}{2}\beta \leqslant 0$ and $\beta \leqslant 3$. Hence, in this case, F is close-to-convex for those (α, β) that lie in the closed rectangle whose vertices are $(0, 0)$, $(0, 3)$, $(-\frac{1}{2}, 3)$, and $(-\frac{1}{2}, 0)$.

Case C. $\alpha \leqslant 0, \beta \leqslant 0$, and *Case* D. $\alpha \geqslant 0, \beta \leqslant 0$ can be treated in the same way to yield the remaining vertices noted in conclusion (ii).

To show our result is sharp, we again appeal to the function $f(z) = g(z) = z/(1 - z)$ to obtain $F_0(z) = \int_0^z (1 - t)^{-2\alpha - \beta}\, dt$ and this is known to be close-to-convex only for $-3 \leqslant -2\alpha - \beta \leqslant 1$ [5]. We also appeal to now-classic results due to Merkes and Wright [6] that when $\beta = 0$, F in (4) is close-to-convex for all convex f only for $-\frac{1}{2} \leqslant \alpha \leqslant 2$ and for $\alpha = 0$, F is close-to-convex for convex g, only for $-1 \leqslant \beta \leqslant 3$. All these inequalities now support our statement that our result (ii) is sharp.

As we have already noted, our results include earlier ones due to Merkes and Wright [6]. Moreover, our results overlap earlier ones due to Silverman [13], who considered the transform (4) with $f \in C$ and $g \in S^*$. It is interesting to note that Silverman's range of (α, β) for the close-to-convexity of the transform (4) is the same as our range even though he permits a larger class of competitive g to enter into his considerations.

THEOREM 2. *The set of* (α, β) *for which the transform in* (4) *is close-to-convex for all close-to-convex f and g is the closed convex hull of the points* $(1, 0)$, $(0, 1)$, $(-\frac{1}{3}, 0)$, *and* $(0, -\frac{1}{2})$.

Proof. First, if $\beta = 0$, then Merkes and Wright [6] have shown that the transform F is close-to-convex for all close-to-convex f only for the range $-\frac{1}{3} \leqslant \alpha \leqslant 1$, and for $\alpha = 0$, the transform F is close-to-convex only for the range $-\frac{1}{2} \leqslant \beta \leqslant 1$. These considerations, plus the techniques used in the proof of Theorem 1 yield the (α, β) pairs noted.

To show our results are sharp, we must distinguish four cases.

Case A $\alpha \geqslant 0, \beta \geqslant 0$. We shall make use of the function

$$f_a(z) = (z - e^{2ai} \cos a z^2)/(1 - e^{ai} z)^2, \tag{8}$$

where a is a real constant, $0 < a < \pi$. The function f_a maps Δ one-to-one onto the plane slit along a vertical half-line extending upward from the tip

$$f_a(e^{-3ai}) = -(\cos a/2) - i(e^{-2ai}/4 \sin a).$$

Hence it is close-to-convex. It is geometrically clear that if the points $e^{i\theta_1}$, e^{-3ai}, $e^{i\theta_2}$, e^{-ia} appear on the unit circle in that order, then

$$\arg[e^{i\theta_2} f_a'(e^{i\theta_2}) - e^{i\theta_1} f_a'(e^{i\theta_1})] = -\pi$$

and

$$\arg f_a(e^{i\theta_2}) - \arg f_a(e^{i\theta_1}) = -\pi + \delta(a),$$

where $\delta(a) > 0$ and $\lim_{a\to 0} \delta(a) = 0$. Geometrically, the tip of the slit tends to $(-(\cos a/2) - i\infty)$ as $a \downarrow 0$.

Now consider $\alpha \geqslant 0$, $\beta \geqslant 0$, $\alpha + \beta = 1 + \varepsilon$, $\varepsilon > 0$, and the transform (4) with $f(z) = g(z) = f_a(z)$. We obtain

$$\int_{\theta}^{\theta_2} d\arg(e^{i\theta}F'(e^{i\theta})) = (1-\alpha-\beta)(\theta_2-\theta_1) - \pi(\alpha+\beta) + \delta(a)\pi$$

$$= \varepsilon(\theta_2-\theta_1) - \pi\varepsilon - \pi + \delta(a)\pi.$$

Now we may choose $\theta_2 - \theta_1$ and $\delta(a)$ as small as we wish to conclude, since $\varepsilon > 0$ is fixed, that the transform F here will satisfy

$$\int_{\theta_1}^{\theta_2} d\arg(e^{i\theta}F'(e^{i\theta})) < -\pi,$$

so that F cannot satisfy Kaplan condition (1). Hence, the transform (4) is not close-to-convex for all close-to-convex functions f and g for the (α, β) pairs satisfying $\alpha + \beta > 1$. Thus $\alpha + \beta \leqslant 1$ is a necessary condition that (4) be close-to-convex for close-to-convex functions f and g.

Case B $\alpha \leqslant 0, \beta \geqslant 0$. We have already noted that the only pairs (α, β) we need consider are those for which $\alpha + \beta \leqslant 1$. We now show that if $-3\alpha + \beta > 1$, then there is a non close-to-convex transform (4). First we note that for the function (8), if the points $e^{i\theta_2}$, e^{-3ai}, $e^{i\theta_1}$, e^{-ai} appear in that order on the unit circle, then

$$\arg e^{i\theta_2}f_a'(e^{i\theta_2}) - \arg e^{i\theta_1}f_a'(e^{i\theta_1}) = 3\pi.$$

If we select θ_1 and θ_2 so that $e^{i\theta_1} = e^{-3bi}$, $0 < b < a$, $e^{i\theta_2} = e^{-bi}$ and if we introduce the function

$$f_b(z) = (z - e^{2bi}\cos bz^2)/(1-e^{ib}z)^2,$$

then

$$\arg f_b(e^{i\theta_2}) - \arg f_b(e^{i\theta_1}) = -\pi + \delta(a, b)\pi,$$

$\delta(a, b) > 0$ and $\lim_{a\to 0} \delta(a, b) = 0$. Hence, for the transform F, with $f = f_a$ and $g = g_b$, we have

$$\int_{\theta_1}^{\theta_2} d\arg(e^{i\theta}F'(e^{i\theta})) = (1-\alpha-\beta)(\theta_2-\theta_1) + 3\pi\alpha - \beta\pi + \delta(a, b)\beta\pi$$

$$= (1-\alpha-\beta)(\theta_2-\theta_1) - (1+\varepsilon+\delta\beta)\pi,$$

where we set $-3\alpha+\beta=1+\varepsilon$, $\varepsilon>0$, with ε fixed. Now $\lim_{a\to 0}\delta(a,b)=0$ and this carries with it $\lim(\theta_2-\theta_1)=0$, too. Hence, for a sufficiently small, and $\theta_2-\theta_1\neq 0$, we conclude

$$\int_{\theta_1}^{\theta_2} d\arg(e^{i\theta}F'(e^{i\theta}))<-\pi$$

so that the transform of the particular pair (f_a, f_b), with $0<b<a$ and a sufficiently small, is not close-to-convex for (α,β) satisfying $-3\alpha+\beta>1$. Hence, a necessary condition for the transform (4) to be close-to-convex for all close-to-convex f and g is that $-3\alpha+\beta\leqslant 1$ holds.

Case C $\alpha\leqslant 0$, $\beta\leqslant 0$. We wish to show that if $-3\alpha-2\beta>1$, then there is a transform F, for close-to-convex functions f and g, that is not close-to-convex. We again use the function f_a given in (8). We select $e^{i\theta_1}$ and $e^{i\theta_2}$, close to and straddling e^{-ia}, so that e^{-3ai}, $e^{i\theta_1}$, e^{-ai}, $e^{i\theta_2}$ appear in that order on the unit circle. Then it is geometrically clear we can choose $e^{i\theta_1}$ and $e^{i\theta_2}$ so that $f_a(e^{i\theta_1})=f_a(e^{i\theta_2})$. Then the transform F with $f=g=f_a$ satisfies the relation

$$\begin{aligned}\int_{\theta_1}^{\theta_2} d\arg(e^{i\theta}F'(e^{i\theta}))&=(1-\alpha-\beta)(\theta_2-\theta_1)+3\pi\alpha+2\pi\beta\\&=(1-\alpha-\beta)(\theta_2-\theta_1)-(1+\varepsilon)\pi,\end{aligned}$$

where we have set $-3\alpha-2\beta=1+\varepsilon$, $\varepsilon>0$. Since ε is fixed and $\theta_2-\theta_1$ can be made as small as we wish, it follows that F satisfies

$$\int_{\theta_1}^{\theta_2} d\arg(e^{i\theta}F'(e^{i\theta}))<-\pi,$$

for θ_1 and θ_2 near to and separated by e^{-ia}. Hence, for each pair (α,β) that satisfies $-3\alpha-2\beta>1$, there is a transform (4) of close-to-convex functions f and g that is not itself close-to-convex.

Case D $\alpha\geqslant 0$, $\beta\leqslant 0$. Function (8) maps Δ onto the plane slit vertically, with the tip of the slit at $f_a(e^{-3ai})$ and the end of the slit at $f_a(e^{-ai})$. It is geometrically clear that if $e^{i\theta_1}$ and $e^{i\theta_2}$ straddle e^{-3ai}, but near e^{-3ai}, and if $f(e^{i\theta_1})=f(e^{i\theta_2})$, then

$$\arg e^{i\theta_2}f_a'(e^{i\theta_2})-\arg e^{i\theta_1}f_a'(e^{i\theta_1})=-\pi.$$

It is also geometrically clear that if $e^{i\theta_3}$ and $e^{i\theta_4}$ straddle e^{-ai}, but near e^{-ai}, and if $f_a(e^{i\theta_3})=f_a(e^{i\theta_4})$, then

$$\arg f_a(e^{i\theta_4})-\arg f_a'(e^{i\theta_3})=2\pi.$$

We now construct a function g_a of the form (8) with the end of its slit at $f_a(e^{-3ai})$, the tip of the slit of the mapping discussed above. Such a function is

$$g_a(z) = (z - e^{6ai} \cos 3az^2)/(1 - e^{3ai}z)^2.$$

And for θ_1 and θ_2 close to and straddling e^{-3ai}, we have

$$\arg g_a(e^{i\theta_2}) - \arg g_a(e^{i\theta_1}) = 2\pi.$$

Hence, for the transform (4) with $f=f_a$ and $g=g_a$, we have

$$\int_{\theta_1}^{\theta_2} d \arg(e^{i\theta}F'(e^{i\theta})) = (1 - \alpha - \beta)(\theta_2 - \theta_1) - \alpha\pi + 2\beta\pi. \tag{9}$$

To show the sharpness of our result concerning the (α, β) pairs, we assume $\alpha - 2\beta = 1 + \varepsilon$, where $\varepsilon > 0$. Then (9) yields

$$\int_{\theta_1}^{\theta_2} d \arg(e^{i\theta}F'(e^{i\theta})) = (1 - \alpha - \beta)(\theta_2 - \theta_1) - \pi(1 + \varepsilon).$$

Now $\theta_2 - \theta_1$ may be taken as small as we wish, so that this last equation shows that

$$\int_{\theta_1}^{\theta_2} d \arg(e^{i\theta}F'(e^{i\theta})) < -\pi \tag{10}$$

will hold for $e^{i\theta_1}$ and $e^{i\theta_2}$ sufficiently close to e^{-3ai}. Hence (1) is violated, so that the transform F here is not close-to-convex for $\alpha \geqslant 0$, $\beta \leqslant 0$, and $\alpha - 2\beta > 1$.

4. Improvements of Some of Shirokova's Results

Let $B_{1/k}$ denote the class of all functions $f(z) = z + \cdots$, analytic and univalent in Δ, that satisfy the inequality

$$\int_{\theta_1}^{\theta_2} d \arg(zf'f^{1/k-1}) > -\pi, \qquad z = re^{i\theta},$$

for all $0 \leqslant \theta_1 < \theta_2 \leqslant \theta_1 + 2\pi$ and for all $0 \leqslant r < 1$, where k is a fixed real, $0 \leqslant k \leqslant 1$. Shirakova proposed the problem of finding the range of k, $0 \leqslant k \leqslant 1$, for which the Shirakova transform

$$F_\alpha(z) = \int_0^z (f'(t))^\alpha (f(t)/t)^{1-\alpha}\, dt, \tag{11}$$

where $0 \leqslant \alpha \leqslant 1$, is close-to-convex in Δ for $f \in B_{1/k}$. We offer some improvement of Shirakova's result.

LEMMA 2. *Let* $f \in B_m$, *where* $m > 0$. *Then*

$$2\pi + \frac{\pi}{m} > \int_{\theta_1}^{\theta_2} d \arg f(z) > \frac{-\pi}{m}. \tag{12}$$

Proof. Reade [10] has shown that if $f \in B_m$, $m > 0$, then there is a univalent starlike function $\sigma(z)$ such that

$$\operatorname{Re}\{zf'(z) f^{m-1}(z)/\sigma^m(z)\} > 0, \qquad z \in \Delta. \tag{13}$$

Now it follows from a result of Mocana [8] that

$$M(z) = \left[m \int_0^z \frac{\sigma^m(t)}{t} \, dt \right]^{1/m}$$

is a starlike (indeed m-convex) function in Δ. Hence, (13) may be written in the form

$$\operatorname{Re}\{f'(z) f^{m-1}(z)/M'(z) M^{m-1}(z)\} > 0, \qquad z \in \Delta.$$

This last, in turn, by a result due to Sakaguchi [9] yields

$$\operatorname{Re}\{f^m(z)/M^m(z)\} > 0, \qquad z \in \Delta,$$

which implies the relation (12). This completes our proof.

Our result (12) is an improvement of a result due to Shirakova. She proved that

$$\int_{\theta_1}^{\theta_2} d \arg f(z) > -2\pi.$$

holds for all $f \in B_m$, $m \geqslant 1$.

THEOREM 3. *Let* α *be fixed*, $0 \leqslant \alpha \leqslant 1$. *If* $f \in B_{1/k}$, *then* $F_\alpha(z)$ *is close-to-convex for all* k *satisfying*

$$\alpha - (3 - \sqrt{9 - 4\alpha + \alpha^2})/2 \leqslant k \leqslant 1. \tag{14}$$

Proof. From (11) we obtain

$$\int_{\theta_1}^{\theta_2} d \arg(zF'_\alpha(z)) = \alpha \int_{\theta_1}^{\theta_2} d \arg(zf'(z) + (1 - \alpha)) \int_{\theta_1}^{\theta_2} d \arg f(z),$$

$z = re^{i\theta}$, so that F_α is close-to-convex if and only if

$$\int_{\theta_1}^{\theta_2} d\arg(zf'(z)) + ((1-\alpha)/\alpha)\int_{\theta_1}^{\theta_2} d\arg f(z) > -\pi/\alpha \tag{15}$$

holds for all $0 \leqslant \theta_1 < \theta_2 \leqslant \theta_1 + 2\pi$ and for all $0 \leqslant r < 1$.

If $f \in B_{1/k}$, then

$$\begin{aligned} 3\pi > \int_{\theta_1}^{\theta_2} d\arg(zf'(z)) + \frac{1-k}{k}\int_{\theta_1}^{\theta_2} d\arg f(z) > -\pi \\ (2+k)\pi > \int_{\theta_1}^{\theta_2} d\arg f(z) > -\pi k \end{aligned} \tag{16}$$

both hold for all $0 \leqslant \theta_1 < \theta_2 \leqslant \theta_1 + 2\pi$ and for all $0 \leqslant r < 1$. Here we have used (10) and (12).

If $k \geqslant \alpha$, then the inequalities (16) yield

$$\int_{\theta_1}^{\theta_2} d\arg(zf'(z)) + \frac{1-k}{k}\int_{\theta_1}^{\theta_2} d\arg f(z) + \frac{k-\alpha}{\alpha k}\int_{\theta_1}^{\theta_2} d\arg f(z) > -\pi(1 + (k-\alpha)/\alpha)$$

or

$$\int_{\theta_1}^{\theta_2} d\arg(zf'(z)) + \frac{1-\alpha}{\alpha}\int_{\theta_1}^{\theta_2} d\arg f(z) > -\frac{\pi k}{\alpha}.$$

Hence, (15) is satisfied so that F_α is close-to-convex for all k satisfying $1 \geqslant k \geqslant \alpha$.

If $k < \alpha$, then we have

$$\int_{\theta_1}^{\theta_2} d\arg(zf'(z)) + \frac{1-k}{k}\int_{\theta_1}^{\theta_2} d\arg f(z) - \frac{\alpha-k}{\alpha k}\int_{\theta_1}^{\theta_2} d\arg f(z) > -\pi[1 + ((\alpha-k)(2+k)/\alpha k)]$$

or

$$\int_{\theta_1}^{\theta_2} d\arg(zf'(z)) + \frac{1-\alpha}{\alpha}\int_{\theta_1}^{\theta_2} d\arg f(z) > -\pi\left[\frac{2\alpha k + 2\alpha - 2k - k^2}{\alpha k}\right].$$

Now (15) will hold provided

$$(2\alpha + 2\alpha k - 2k - k^2)/\alpha k \leqslant 1/\alpha,$$

i.e., provided

$$k \geqslant \alpha - (3 - \sqrt{9 - 4\alpha + 4\alpha^2})/2.$$

We have thus established (14) which represents an improvement of another result due to Shirakova.

5. Concluding Remarks

The various integral transforms have led to considerations of *Hornich Spaces* [3]. We propose to study the vector space V of all functions $f(z) = z + \cdots$, analytic in Δ, with

$$\alpha[f_1] + \beta[f_2] \equiv \int_0^z f_1'(t)^\alpha (f_1(t)/t)^{1-\alpha} f_2'(t)^\beta (f_2(t)/t)^{1-\beta}\, dt,$$

where α and β are real. We propose to study the possible metrics on V and the attendant topologies [3].

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