Nonsolvable Signalizer Functors on Finite Groups

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1. INTRODUCTION

Recently Gorenstein and Lyons obtained the first nonsolvable signalizer functor theorems [9]. They pinpointed certain “unbalancing” problems. This paper grew from an attempt to manage such problems. Theorem A is the result. Theorem B and Corollary C give some measure of the practical scope of Theorem A.

Suppose $p$ is a prime, $A$ is an elementary Abelian $p$-subgroup of a finite group $G$, and $\theta$ is an $A$-signalizer functor on $G$. The unbalancing difficulties, referred to above, occur only if there are “certain” nonidentity subgroups $X$ of $G$, such that $C_X(A)$ is solvable. Using methods of Glauberman [5, Lemma 2.11 and Theorem 4.5] we are able to reduce the problem: either the “unbalancing” problems vanish or $\theta(C_G(A))$ is solvable. The latter case is treated in [13]. The rest of the work is treated here. This work pivots on Theorems 5.16 and 6.5, results which closely resemble [5, Lemma 2.11 and Theorem 4.5].

2. NOTATION, DEFINITIONS, AND CONVENTIONS

Conventions. All groups treated in this paper are finite. All simple groups are nonabelian. We shall reserve $p$ and $r$ for primes.

Suppose $A, B$ are groups and $B$ acts on $A$. Then $AB$ is the usual product if $A$ and $B$ are subgroups of a common group; otherwise $AB$ is the semidirect product of $A$ by $B$.

Suppose a group $G$ is the direct product of subgroups $A_1, A_2, \ldots, A_n$. Let $X$ be a subset of $G$. Then $\text{Proj}_{A_i}(X)$ is the usual projection map of $X$ on $A_i$. We often write $\text{Proj}_{A_i}(X)$ when $G = A \times C_G(A)$. Then projections are taken with respect to the pair $(A, C_G(A))$. If $X$ is contained in a subgroup $N$, we may* Partially supported by NSF Grant MCS 76-06626.
apply the above conventions to \( N \). We do so by stating that projections are being taken in \( N \).

**Notations and Definitions.** Our notation for groups of Lie type agrees with [2]. Most of the specialized notation is taken from [5, 7, 13]. For the convenience of the reader we shall repeat many of these. What is not explained can be found in [2, 8], or is hopefully self-explanatory.

1. Let \( S \) be a finite set. When the members of \( S \) are sets, \( \cap S \) is the intersection of the members of \( S \). When the members of \( S \) are groups, \( \times S \) is the direct product of the members of \( S \). When the members of \( S \) are real numbers, \( \sum S \) is the sum of the members of \( S \).

2. A section of a group \( G \) is a quotient group \( K/L \) of a subgroup \( K \) of \( G \) by a normal subgroup \( L \) of \( K \).

3. A simple group \( G \) is outer \( p \)-cyclic means that the outer automorphism group of \( G \), \( \text{Out}(G) \), has cyclic Sylow \( p \)-subgroups.

4. The group \( G \) is near \( p \)-solvable means that \( G \) is a \( p' \)-group, and any simple section of \( G \) is isomorphic to \( A_1(q) \), \( A_1(3^p) \), \( 2B_2(q) \), or \( 2A_2(q^2) \), where \( q = 2^p \).

5. A localized subgroup of a group \( G \) is any subgroup which normalizes a nonidentity solvable subgroup of \( G \).

6. Hypothesis A (applied to a pair \( (G, p) \)).
   
   (A.1) \( p \) is a prime and \( G \) is a simple \( p' \)-group.
   (A.2) \( G \) is outer \( p \)-cyclic.
   (A.3) \( G \) is near \( p \)-solvable, or the following three conditions apply to any automorphism \( f \) of \( G \) of order \( p \).
      
      (A.3.1) Let \( C = C_{G}(f) \). Then \( C \) is not a localized subgroup of \( G \).
      (A.3.2) \( f^*(C) \) is simple.
      (A.3.3) Any \( p' \)-automorphism of \( G \) which centralizes \( C \) is trivial.

7. Hypothesis B (applied to a pair \( (G, p) \)). \( p \) is a prime. \( G \) is a \( p' \)-group. Hypothesis A applies to \( (K, p) \) for all simple sections \( K \) of \( G \).

8. The group \( G \) is near \( A \)-solvable means that \( A \) is an elementary \( p \)-group, \( (G, p) \) satisfies Hypothesis B, and \( C_{G}(A) \) is solvable.

9. The statement "\( \theta \) is an \( A \)-signalizer functor on \( G \)" means that \( A \) is an Abelian \( p \)-subgroup of the group \( G \) for some prime \( p \), and that for each \( a \in A^* \) there is defined an \( A \)-invariant \( p' \)-subgroup \( \theta(C_{G}(a)) \) of \( C_{G}(a) \) such that

\[
\theta(C_{G}(a)) \cap C_{G}(b) \leq \theta(C_{G}(b)) \quad \text{for all } a, b \in A^*. \tag{\*}
\]

The property (\*) is called balance.

In definitions (10) through (18), let \( \theta \), \( G \), \( A \), and \( p \) be as in Definition 9.
(10) **Hypothesis (C)** (applied to \( \theta \)). The pairs \((\theta(C_\theta(a)), p)\) satisfy Hypothesis B for all \( a \in A^* \).

(11) The associated set of \( A \)-signalizers is the set of all \( A \)-invariant \( p' \)-subgroups \( X \) of \( G \) such that \( C_X(a) \leq \theta(C_\theta(a)) \) for all \( a \in A^* \), and such that \((X, p)\) satisfies Hypothesis B. It is denoted \( \mathcal{I}_\theta(A) \). The set of all maximal elements of \( \mathcal{I}_\theta(A) \) under inclusion is denoted by \( \mathcal{I}_\theta^*(A) \).

(12) We say that \( \theta \) is complete if \( G \) contains a unique maximal element of \( \mathcal{I}_\theta(A) \) under inclusion. This element is then denoted by \( \theta(G) \).

(13) We say that \( \theta \) is locally complete if, for every nonidentity element \( X \) of \( \mathcal{I}_\theta(A) \), \( N_\theta(X) \) contains a group \( \theta(N_\theta(X)) \) which is the unique maximal element among all elements of \( \mathcal{I}_\theta(A) \) contained in \( N_\theta(X) \). In this case, we put \( \theta(C_\theta(X)) = \theta(N_\theta(X)) \cap C_\theta(X) \).

(14) For every nonidentity subgroup \( B \) of \( A \), let

\[
\theta(C_\theta(B)) = \bigcap \{ \theta(C_\theta(b)) | b \in B^* \}.
\]

(15) The set of all elements of \( \mathcal{I}_\theta(A) \) which are \( \theta(C_\theta(A)) \)-invariant is denoted \( \bar{\mathcal{I}}_\theta(A) \).

(16) The set of all elements of \( \mathcal{I}_\theta(A) \) which contain \( \theta(C_\theta(A)) \) is denoted \( \bar{\mathcal{I}}_\theta(A) \).

(17) \( \pi(\theta) = \bigcup \{ \pi(\theta(C_\theta(a))) | a \in A^* \} \) and \( |\theta| = \sum_{a \in A^*} |\theta(C_\theta(a))| \).

(18) For any \( r \in \pi(\theta) \), let \( \mathcal{I}_\theta(A; r) \) be the set of all \( r \)-groups in \( \mathcal{I}_\theta(A) \), and let \( \mathcal{I}_\theta^*(A; r) \) be the set of maximal elements of \( \mathcal{I}_\theta(A; r) \). The elements of \( \mathcal{I}_\theta^*(A; r) \) are called \( S_r(A) \)-subgroups of \( G \).

(19) The solvable radical of a group \( G \) is the maximal solvable normal subgroup of \( G \). It is denoted \( \text{Sol}(G) \).

(20) The set of subnormal simple subgroups of a group \( G \) is denoted \( \mathcal{L}'(G) \). Let \( \bar{G} = G/\text{Sol}(G) \). Then \( \mathcal{N}(G) \) is the set of all subgroups \( X \) of \( G \), which contain \( \text{Sol}(G) \), and which satisfy \( \bar{X} \in \mathcal{L}'(\bar{G}) \).

(21) A group is semi-simple means that it is the direct product of its normal simple subgroups. This use is not in accord with [8, p. 501]. A group is perfect if it is its own derived group. A group is an \( E \)-group if it is perfect, and modulo its center is semi-simple. A group is a \( K \)-group if modulo its solvable radical it is semi-simple. Let \( G \) be a group. The Fitting subgroup of \( G \) is denoted \( F(G) \). The unique maximal normal \( E \)-subgroup of \( G \) is denoted \( E(G) \). The generalized Fitting subgroup of \( G \) equals \( E(G) F(G) \). It is denoted \( F^*(G) \). The unique maximal normal \( K \)-subgroup of \( G \) is denoted \( K(G) \). We define \( \bar{K}(G) = (\bigcap \{ N_\theta(M) | M \in \mathcal{N}(G) \}) \text{Sol}(G) \).

(22) Suppose \( A \) is an Abelian \( p \)-group acting on the \( p' \)-group \( G \). For
each subgroup $X$ of $G$, the smear of $X$ by $A$ is the subgroup $\langle X^A \rangle \cap C_G(A)$. It is denoted $X \ast A$. $\mathcal{L}^A(G) = \{L \in \mathcal{L}(G) | L \ast A \text{ is nonsolvable}\}$. $\mathcal{M}^A(G) = \{M \in \mathcal{M}(G) | M \ast A \text{ is nonsolvable}\}$. $K^A(G) = \langle \mathcal{M}^A(G) \rangle$. Finally, $K_A(G) = C_G(K^A(G)/\text{Sol}(G))$.

(23) We are interested in structures which are like wreathed structures. Suppose $G$ is a group. The expression $G = Hw(A, N, C)$ means: $A$ is an Abelian subgroup of $G$, $H$ is a subgroup of $G$, $G = (H, A)$, $\langle H^G \rangle = H^G \times N \times C(H)$, and $C = C_s(H)$.

(24) Suppose the group $G$ is the direct product of its subgroups $G_1, G_2, ..., G_n$. A diagonal subgroup of $G$, with respect to $\{G_1, G_2, ..., G_n\}$, is any subgroup $X$ such that $\text{Proj}_{G_i}: X \to G_i$ is an isomorphism.

(25) A direct factor of the group $G$ is any subgroup $K$ of $G$ such that $K \times L = G$ for some subgroup $L$ of $G$. We say $G$ is indecomposable if its only direct factors are $G$ and $1$. We denote the set of all indecomposable direct factors of $G$ by $\text{Ind}(G)$.

3. STATEMENT OF MAIN RESULTS

THEOREM A. Suppose $p$ is a prime, $A$ is an Abelian subgroup of a group $G$, $m(A) \geq 3$, and $\theta$ is an $A$-signalizer functor on $G$ which satisfies Hypothesis C. Then $\theta$ is complete.

THEOREM B. Suppose $p$ is a prime, $G$ is a simple $p'$-group, and at least one of the following conditions apply to $G$:

(a) $\text{Out}(G)$ is prime to $p$,
(b) $G$ is a Chevalley or a twisted Chevalley group, or
(c) $G$ has an Abelian Sylow 2-subgroup.

Then it follows that $(G,p)$ satisfies Hypothesis A.

COROLLARY C. Suppose $p = 2$ or $3$, $A$ is an Abelian $p$-subgroup of the finite group $G$, $m(A) \geq 3$, and $\theta$ is an $A$-signalizer functor on $G$. Then $\theta$ is complete.

4. PROOF OF THEOREM B AND COROLLARY C

We list the Lie notation used in this section. For greater detail see [2].

DEFINITION. Let $K$ be a finite field. We write $A(K)$ for any of the groups $A_n(K), B_n(K), C_n(K), D_n(K), G_2(K), F_4(K), E_6(K)$. In this section we
shall reserve $G(K)$ to mean $\mathcal{A}(K)$ or some twisted version $^t\mathcal{A}(K)$ of $\mathcal{A}(K)$. The root system and fundamental root system corresponding to $\mathcal{A}$ are given respectively by $\Phi$ and $\Pi$.

Let $\mathbb{Z}$ be the integers. Then $\hat{H}$ is the set of automorphisms of $\mathcal{A}(K)$ of the form $h(\chi)$, $\chi \in \text{Hom}(\mathbb{Z}\Phi, K)$, defined by $h(\chi): x_r(s) \to x_r(s\chi(r))$ for $r \in \Phi$. The group of field automorphisms of $\mathcal{A}(K)$ is denoted $\mathcal{F}$. Let $A_1$ be the inner automorphism group of $G(K)$, $A_2$ the automorphism group induced by $N_\mathbb{Z}(G(K))$ on $G(K)$, $A_3 = \mathcal{F}$, and $A_4 = $ the automorphism group generated by the graph automorphism of $G(K)$. By [14, 15, 17], $\text{Aut}(G(K)) = A_1 A_2 A_3 A_4$, $A_2 \cong N_\mathbb{Z}(G(K))$, and $A_3 \cong N_\mathcal{F}(G(K))$. Hence we shall identify $A_2$ and $A_3$ with $N_\mathbb{Z}(G(K))$ and $N_\mathcal{F}(G(K))$, respectively. Also when convenient we may identify $\text{Aut}(^t\mathcal{A}(K))$ with a subgroup of $\text{Aut}(\mathcal{A}(K))$ and $G(K)$ with $A_1$.

Let $U$ (resp. $V$) be the positive (resp. negative) unipotent subgroups of $\mathcal{A}(K)$.

**Lemma 4.1.** Suppose $G(K)$ is a $p'$-group. Then $A_2$ contains a Sylow $p$-subgroup $P$ of $\text{Aut}(G)$. Moreover $P$ is cyclic.

**Proof.** $\pi(A_1) = \pi(G)$, $\pi(A_2) = \pi(K^e) \subseteq \pi(G)$, and $\pi(A_4) \subseteq \pi(G)$. Hence by Sylow's theorems, $A_3$ contains a Sylow $p$-subgroup of $\text{Aut}(G)$. Since $A_3$ is cyclic, the result follows.

**Lemma 4.2.** Let $K$ have characteristic $r$. Suppose $T$ is a subgroup of $U$, such that for all $s \in \Pi$, $T$ contains an element $\prod_{r \in \Phi} x_r(b_r)$, for which $b_r \neq 0$. Then $U$ is the unique Sylow $r$-subgroup of $\mathcal{A}(K)$ which contains $T$.

**Proof.** The proof of [1, Lemma 1.1] is based on these conditions and shows $N(T) \leq N(U)$. Since the conditions are inherited by $N_U(T)$, the result follows by induction on $|U: T|$.

**Lemma 4.3.** Suppose $G(K)$ is a $p'$-group and $f$ is an automorphism of $G(K)$ of order $p$. Let $C = C_{G(K)}(f)$ and $D = C_{\text{Aut}(G(K))}(C)$. Then $D = \langle f \rangle$.

**Proof.** Let $r$ be the characteristic of $K$. By Lemma 4.1 we may suppose $f$ is a field automorphism. Then by Lemma 4.2, $U$ is the unique Sylow $r$-subgroup of $\mathcal{A}(K)$ containing $U \cap C$. Since $U \cap C$ and $V \cap C$ are conjugate, it follows that $V$ is the unique Sylow $r$-subgroup of $\mathcal{A}(K)$ containing $V \cap C$. Hence $D \subseteq N(U) \cap N(V) \cap \text{Aut}(G(K)) = A_2 A_3 A_4$. Since $A_2 A_3$ normalizes each root group it follows that $D \subseteq A_2 A_3$. Now straightforward calculations assisted by [2, Theorem 5.3.3(ii), Proposition 13.6.1] yield the result.

**Lemma 4.4.** Suppose $G \cong A_1(q), A_1(3^p), A_2(q), \text{ or } B_2(q)$, where $q = 2^p$. Suppose in addition that $G$ is a $p'$-group. Then $G$ is near $p$-solvable.
Proof. When $G \cong \mathcal{B}_2(q)$, the result is given by [17, Theorem 9]. Otherwise the result follows from [4, Sects. 8.4 and 8.5].

Proof of Theorem B. (a) In this case all the conditions are vacuously satisfied.

(b) Suppose $G(K)$ is a $p'$-group. We must show $(G(K),p)$ satisfies Hypothesis A. By (a), we may suppose that $p \in \pi(\text{Aut}(G(K)))$. By Lemma 4.1, $G(K)$ is outer $p$-cyclic. Let $f$ be an automorphism of $G(K)$ of order $p$ and let $C = C_{G(K)}(f)$. By Lemma 4.3, any $p'$-automorphism of $G(K)$ which centralizes $C$ is trivial.

By Lemma 4.4, we may suppose $G = A_1(q), A_1(3^p), \mathcal{B}_2(q)$, or $\mathcal{B}_2(q)$ for $q = 2^p$. By [1], $C$ is a maximal subgroup of $G(K)$. Hence it suffices to show $F^*(C)$ is simple. By [2, Theorems 21.1.2, 14.4.1, comments on p. 175, and the note on p. 268] it suffices to show $\mathcal{B}_2(3)$ and $\mathcal{B}_2(2)$ have trivial center. The argument on [2, p. 173] carries over to the above two situations. This completes (b).

(c) Let $G$ be a $p'$-simple group with Abelian Sylow 2-subgroup. We must show $(G,p)$ satisfies Hypothesis A. By parts (a), (b), and [10, 20], we may suppose $G$ has an elementary Abelian Sylow 2-subgroup $P$ of order 8, that $C_G(j) \cong \mathbb{Z}_2 \times A_1(q)$ where $q = 3^n$ for some odd integer $n$ at least 3, and that $G$ has an automorphism of order $p$. Such groups have been studied extensively [11, 15, 19, 21]. Let $N = N_G(P), A_1$ be the group of inner automorphisms of $G$, and $B_2$ the group of automorphisms centralizing $N$. By [20, p. 335], there follows

$$\text{Aut}(N) = \text{Inn}(N) \cong N.$$  

(4.1)

So by the Frattini argument

$$\text{Aut}(G) = A_1B_2.  \quad (4.2)$$

$G$ does not have a strongly embedded subgroup, and $N$ is transitive on $P^\#$, whence $G = \langle N, C_G(j) \rangle$ for any $j \in P^\#$. Hence

$$B_2 \text{ acts faithfully on } (C_G(j))' \cong A_1(3^n) \text{ for any } j \in P^\#. \quad (4.3)$$

Now suppose $f$ is an automorphism of $G$ of order $p$. By (4.2), we may suppose $f \in B_2$. Hence by (4.3), $G$ is outer $p$-cyclic. Let $C = C_G(f)$. Any automorphism $k$ of $G$ which centralizes $C$ must centralize $N$. Hence $k \in B_2$.

By (4.3) and (b), it follows that $k \in \langle f \rangle$.

Let $j \in P^\#$. Then $C_N(j)$ normalizes no nontrivial subgroup of odd order of $C_G(j)$. Hence $N$ normalizes no nontrivial subgroup of odd order. Since $N$ is transitive on $P^\#$ we obtain

Suppose $N \leq H \leq G$. Then $H = N$ or $F^*(H)$ is simple.  \hspace{1cm} (4.4)
To complete the proof it suffices by (4.4) to show that $N \neq C$. The order of $N$ is 168. Let $e$ be an element of $N$ of order 3, and $t$ an involution of $N$ centralizing $e$. By [11, 21], $e$ is contained in a unique Sylow 3-subgroup $R$ of $G$. So $t$ and $f$ normalize $R$. Now $e \in C_R(t) \leq R'$. Hence it suffices to show $C_{R^f}(f) \neq 1$. However, $N_{G}(R)$ is transitive on $(R/\Phi(R))^*$ whence $(e) < C_R(f)$. Hence $N \neq C$. This completes the proof of Theorem B.


5. PRELIMINARY LEMMAS

**Lemma 5.1.** Suppose the Abelian $p$-group $A$ acts on the $p'$-group $X$. Then $X = \langle C_x(A) \mid A/A_0$ is cyclic$\rangle$.

*Proof.* See [7, Lemma 2.1].

**Lemma 5.2 (Glauberman).** Suppose the $\pi$-group $A$ acts on the $\pi'$-group $K$. Suppose $K$ is generated by $A$-invariant pairwise permuting subgroups $K_1, K_2, \ldots, K_n$. Then $C_K(A) = C_{K_1}(A) C_{K_2}(A) \cdots C_{K_n}(A)$.

*Proof.* See [9, Lemma 2.1].

**Lemma 5.3.** Suppose $\theta$ is an $A$-signalizer functor on a group $G$, $P \in I_0(A; r)$ and $B$ is a noncyclic subgroup of $A$. Then the following statements are equivalent:

1. $P \in I^+_0(A; r)$
2. $C_p(b)$ is an $S_r(A)$-subgroup of $\theta(C_0(b))$ for all $b \in B^r$.

*Proof.* See [7, Lemma 3.2].

**Lemma 5.4.** Let $G$ be a group and $\bar{G} = G/\text{Sol}(G)$. Then the functors $F^*, K, E$, and $\text{Sol}$ satisfy:

1. $\text{Sol}(\bar{G}) = \overset{\sim}{1}$,
2. $C_G(F^*(G)) \subseteq F^*(G)$, and
3. $K(G) = K(\bar{G}) = E(\bar{G}) = F^*(\bar{G})$ is semi-simple.

*Proof.* See [13, Lemma 2.4].

**Lemma 5.5.** Suppose the Abelian group $A$ acts on the group
$G = G_1 \times G_2 \times \cdots \times G_n$. Suppose $A$ acts on $\{G_1, G_2, \ldots, G_n\}$ via the induced action of $A$ on subgroups. Then

$$\text{Proj}_{G_i}(C_G(A)) = C_{G_i}(N_A(G_i))$$

where projections are taken with respect to $\{G_1, G_2, \ldots, G_n\}$.

**Proof.** See [13, Lemma 2.9].

**Lemma 5.6.** Suppose the group $A$ acts on the group $G$, $G$ is a direct product of a set $S$ of subgroups of $G$ on which $A$ acts semi-regularly. Suppose $W$ is a subgroup of $C_G(A)$, and $T \leq K \in S$ satisfies $T^* A \leq W$. Then $T \leq \text{Proj}_K(W)$ when projections are taken with respect to $S$.

**Proof.** Let $t \in T$ and $y = \prod_{a \in A} t^a$. The elements of $t^a$ commute pairwise; so $y$ is well defined and centralized by $A$. So $t = \text{Proj}_K(y) \in \text{Proj}_K(W)$ as required.

**Lemma 5.7.** Suppose the group $G$ acts faithfully on the set $\Omega$, $G$ has a Sylow $p$-subgroup $S$ acting transitively on $\Omega$, and $O^p(G) = O_p(G)$. Then $G = S$.

**Proof.** [See 13, Lemma 2.6].

**Lemma 5.8.** Suppose $M$ is a group of operators of the semi-simple group $K$. Then $[K, M] = \text{the product of components of } K \text{ not centralized by } M$.

**Proof.** Suppose that $K$ has a component $L$ centralized by $M$. Then $[K, M] = [C_K(L) \times L, M] \leq C_K(L) < K$.

Now let $K_1 = [K, M]$ and $K_2 = C_K(K_1)$. Then both $K_1$ and $K_2$ are normal in $KM$, and $K = K_1 \times K_2$. Hence $[K_2, M] \leq K_1 \cap K_2 = 1$. So $K_1 = [K, M] = [K_1 \times K_2, M] = [K_1, M]$. The previous paragraph implies that $K_1$ has no component centralized by $M$.

**Lemma 5.9.** Suppose $G$ is a group and $K(G) \leq X \leq G$. Then $K(G) = K(X)$.

**Proof.** See [13, Lemma 2.15].

**Lemma 5.10.** Suppose the group $G$ is semisimple. Let $\mathcal{L} = \mathcal{L}(G)$. Suppose $H$ is a subgroup of $G$ such that $\text{Proj}_T(H) = L$ for all $L \in \mathcal{L}$. For each nonempty subset $T$ of $\mathcal{L}$ let $G_T = \langle T \rangle$ and $H_T = H \cap G_T$. Then

(a) $\mathcal{L}$ is the disjoint union of subsets $\mathcal{L}_i$, $1 \leq i \leq k$,
(b) $H$ is the direct product of $H_{\mathcal{L}_i}$, $1 \leq i \leq k$, and
(c) $H_{\mathcal{L}_i}$ is a diagonal subgroup of $G_{\mathcal{L}_i}$.
Proof: Let $T$ be a nonempty subset of $\mathcal{L}$ of least possible order subject to $G_T \cap H \neq 1$. If $T = \mathcal{L}$, then $H$ is already a diagonal subgroup of $G$ and we are done. Suppose then that $T$ is a proper subset of $\mathcal{L}$. Let $H^* = \text{proj}_{G_T}(H)$. Now $H_T \supseteq [H, H_T] = [H^*, H_T]$. So $H_T < H^*$. Let $L \in T$. Then $1 \neq \text{proj}_T(H_T) \leq \text{proj}_T(H) = L$. Hence $H_T$ is a diagonal of $G_T$. So $H_T \leq H^* \leq N_{G_T}(H_T) = H_T$. Hence $H = H_T \times C_H(G_T)$. The result now follows by induction on $|\mathcal{L}|$.

Lemma 5.11. Suppose the elementary Abelian $p$-group $A$ acts on the $p'$-group $G$, $m(A) \geq 2$, and each member of $\mathcal{L}(G/\text{Sol}(G))$ is outer $p$-cyclic. Let $L = \langle K(C_A(a))|a \in A^* \rangle$. Then $K(L) = K(G)$.

Proof: By Lemma 5.9, it suffices to show that $L \supseteq K(G)$. We may now make the following sequence of reductions; first $G = K(G)$, then $\text{Sol}(G) = 1$, then $A$ is of order $p^2$, then $C_A(G) = 1$, and finally $A$ acts transitively on $\mathcal{L}(G)$. By the outer $p$-cyclic property and Lemma 5.1, we may suppose $B = A \cap K(GA) \cong \mathbb{Z}_p$. Let $K \in \mathcal{L}(G)$. Then $1 \neq K(C_K(B)) = K(C_A(B)) \cap K \leq L$. So $L \cap K \neq 1$. Let $E \in \mathcal{E}_1(A)$. Then $C_G(E) \leq L$. By Lemma 5.10, $L = G$.

Lemma 5.12. Suppose $H$ is a group of operators on the group $G = \times \Omega$. Suppose the action of $H$ on $G$ induces a semiregular action of $H$ on $\Omega$. Then $Z(C_H(H)) = C_{Z(G)}(H)$.

Proof: Let $C = C_H(H)$. Take projections in $G$ with respect to $\Omega$. Since $H$ acts semiregularly on $\Omega$, it follows that $\text{proj}_K(C) = K$ for all $K \in \Omega$. Hence

$$C_G(C) = \times \{C_K(K)|K \in \Omega\} = \times \{C_K(\text{proj}_K(C))|K \in \Omega\}$$

$$\times \{C_K(K)|K \in \Omega\} = Z(G).$$

Hence $Z(C) = C \cap Z(G)$ as required.

Lemma 5.13. Suppose $A$ is a $p$-group and $G$ is near $A$-solvable. Then $G$ is near $p$-solvable.

Proof: A simple section of $G$ is isomorphic to a simple section of some chief factor of $GA$ in $G$. Hence we may assume $G$ is nonsolvable and is minimal normal in $GA$. Then $G$ is near $(A \cap C(GA))$-solvable. Hence we may suppose that $G$ is simple. Since $C_G(A)$ is a localized subgroup of $G$ and $A/C_A(G) \cong \mathbb{Z}_p$, it follows by Hypothesis A.3. that $G$ is near $p$-solvable.

Lemma 5.14. Suppose $A$ is an elementary Abelian $p$-group of operators on the group $G$. Suppose $(G, p)$ satisfies Hypothesis B. Let $D = C_G(A)$. Suppose $X$ is a $DA$-invariant subgroup of $G$, and $\text{Sol}(G) = 1$. Let $J \in \mathcal{L}(G)$. Then $F^*(J ** A) \leq X$ or $J$ is centralized by $X$. 

481/78/1-15
Proof. Let $L = \langle J^A \rangle$, $X_1 = X \cap K(G)$, and $X_2 = \text{Proj}_L(X_1)$, where projections are being taken in $K(G)$. Then $X_2$ is $C_L(A)$-invariant. First suppose $X_2 \cap (J ** A) = 1$. Thus $C_{X_2}(A) = 1$. By [12], $X_2$ is solvable. By Lemma 5.5 and Hypothesis A, $\text{Proj}_J(J ** A) = C_J(N_J(A))$ is not localized. Hence, $\text{Proj}_J(X_2) = 1$. Hence $X_2 = 1$. Hence $[J ** A, X] \subseteq X_1 \subseteq C_G(L)$. By the 3-subgroup lemma, $[F^*(J ** A), X] = [F^*(J ** A), F^*(J ** A), X] = 1$.

Since $L = \text{the product of components of } G \text{ not centralized by } F^*(J ** A)$, it follows that $L$ admits $X$. In particular $[J ** A, X] \subseteq X_1 \cap L = 1$. By Lemma 5.7, $X$ normalizes $J$. Hence by Lemma 5.5, $X$ centralizes $\text{Proj}_J(J ** A) = C_J(N_J(A))$. By Hypothesis A.3.3, $X$ centralizes $J$.

Next suppose $X_2 \cap (J ** A) \neq 1$. By Lemma 5.5, $J ** A \cong C_J(N_J(A))$ is simple and is the unique minimal normal subgroup of $J ** A$. In particular, $F^*(J ** A) \subseteq X_2$. So

$$F^*(J ** A) \leq \{ [F^*(J ** A) \leq [J ** A, X_1] = [J ** A, X_1] \}
\leq [D, X] \leq X.$$ 

This completes the proof of the lemma.

**Theorem 5.15.** Suppose $A$ is an elementary Abelian $p$-group of operators of the group $G$. Suppose $(G, p)$ satisfies Hypothesis B. Let $D = C_G(A)$. Suppose $X$ is a DA-invariant subgroup of $G$. Then all of the following hold.

1. $X \leq K_A(G)$ if $X$ is near $A$-solvable.
2. Suppose $J \in \mathcal{A}_p(A)$ and $X = \hat{K}(X)$. Then $J$ admits $X$.
3. Suppose $X = K(X)$. Then $X$ normalizes $K^A(G)$ and induces inner automorphisms on $K^A(G)/\text{Sol}(G)$.
4. Suppose $X = K(X)$. Then $X \leq K^A(G) K_A(G)$.
5. Suppose $K_A(G) = 1$ and $X = \hat{K}(X)$. Then $X \leq \hat{K}(G)$.
6. Suppose $K_A(G) = 1$ and $B \leq A$. Then $K(C_B(G)) = K(C_{K(G)}(B))$.

Moreover $\mathcal{L}(C_B(G)) = \{ F^*(J \ast B) \mid J \in \mathcal{L}(G) \}$.

7. Suppose $K_A(G) = 1$ and $B \leq A$. Then $\hat{K}(C_B(G)) - C_{\hat{K}(G)}(B)$.

**Proof.** (a) Without loss of generality assume $\text{Sol}(G) = 1$. If $K^A(G) = 1$, there is nothing to prove. Suppose $K^A(G) \neq 1$. Let $J \in \mathcal{L}^A(G)$. Then $F^*(J ** A) \not\leq X$. Hence by Lemma 5.14, $[J, X] = 1$. So $X \leq K_A(G)$.

(b) We may suppose $\text{Sol}(G) = 1$. Let $J \in \mathcal{L}^A(G)$. If $F^*(J ** A) \not\leq X$, then $[X, J] = 1$ by Lemma 5.14. Suppose then $F^*(J ** A) \leq X$. Let $L = \langle J^A \rangle$ and $X_1 = K(X \cap L)$. By Lemma 5.14, $F^*(J ** A) \leq X_1$. By (a), $\text{Sol}(X_1) \leq K_A(L) = 1$. Also by (a), $\text{Sol}(X) \leq K_A(G) \leq C_G(L)$. Hence $X_1 = (X_1 \times \text{Sol}(X))^{\infty}$ admits $X$. Since $L$ is the product of components of $G$
not centralized by $X_1$, it follows that $L$ admits $X$. By Lemma 5.7, $J$ admits $X$.

(c) By (b), $X$ normalizes $K^d(G)$. Hence we may suppose $G = K^d(G) X$ and $\text{Sol}(G) = 1$. By (a), $\text{Sol}(X) \triangleleft G$. Hence $\text{Sol}(X) = 1$. Let $X_1 = X \cap K(G)$, and $X_2 = C_X(X_1)$. Then $X = X_1 \times X_2$. Since $X$ and $X_1$ are DA-invariant, it follows that $X_2$ is DA-invariant. By Lemma 5.14, $X_2 \leq K_A(G)$.

(d) This is equivalent to (c).

(e) This is immediate from (b).

(f) By (c), $K(C_G(B)) = K(C_{K(G)}(B))$. Certainly, $K_B(G) = 1$.

So to complete (f) we may suppose by induction that $G = K(G)$, that $A = B$, and that $A$ is transitive on $\mathcal{L}(G)$. Let $J \in \mathcal{L}(G)$. Then $C_G(A) = J \ast A$. By Lemma 5.5, $C_G(A) \cong C_J(N_J(J))$. So we may suppose that $G$ is simple. The conclusion now follows from Hypothesis A.

(g) By (b), $K(C_G(B)) \leq K(G)$. By (f), $C_{K(G)}(B) = K(C_{K(G)}(B))$. This proves (g) and the theorem.

**Theorem 5.16.** Suppose $A$ is an elementary Abelian $p$-group of operators of the group $G$. Suppose $(G, p)$ satisfies Hypothesis B. Let $D = C_G(A)$. Let $\text{NS}(G)$ be the set of all subgroups of $G$ which are DA-invariant and near $A$-solvable. Let $G_{ns} = \langle \text{NS}(G) \rangle$. Then

(a) $G_{ns} \in \text{NS}(G)$, and

(b) $G_{ns}$ admits all DA-invariant $K$-subgroups of $G$.

**Proof.** (a) $DA$ permutes $\text{NS}(G_{ns})$ and therefore normalizes $(G_{ns})_{ns}$. Hence we may suppose $G = G_{ns}$. We may also suppose $G$ has no near $A$-solvable normal subgroups. Theorem 5.15(a) implies that $G = K_A(G)$. Hence $K(G)$, being near $A$-solvable, is trivial. Hence $G = 1$.

(b) By Theorem 5.15(a, d) we may suppose that $G = K^d(G) K_A(G)$. We may also suppose that $G$ has no nontrivial near $A$-solvable normal subgroup. Hence $K(K_A(G)) = 1$. Hence $K_A(G) = 1$. So $G_{ns} \leq K_A(G) = 1$, proving (b).

**Theorem 5.17.** Suppose $H$ is a group, $\text{Sol}(H) = 1$, and $H$ has a subgroup $B \cong Z_p \times Z_p$ acting regularly on $\mathcal{L}(H)$. Suppose $\theta$ is a $B$-signalizer functor on $H$ which satisfies:

$$C_{K(H)}(b) \leq \theta(C_H(b)) \quad \text{for all } b \in B^*$$

and

$$\theta(C_H(b)) = p(\theta(C_H(B))) \quad \text{for all } b \in B^*.$$
Let \( \tilde{N} = C_H(C_{K(H)}(N)) \) for each \( N \triangleleft K(H) \). Then \( \theta \) is complete. Moreover,
\[
\theta(HB) = \times \left\{ \theta(HB) \cap \tilde{J} \mid J \in \mathcal{L}(H) \right\} \\
= \times \left\{ \text{Proj}_J(\theta(C_{H}(B))) \mid J \in \mathcal{L}(H) \right\},
\]
where projections in \( \langle \tilde{J} \mid J \in \mathcal{L}(H) \rangle \) are taken with respect to \( \tilde{J} \mid J \in \mathcal{L}(H) \).

**Proof:** By Lemma 5.7, \( \langle \mathcal{U}_0(B) \rangle \leq \hat{K}(H) \). Hence we may suppose that \( H \cong \text{Aut}(J) \lhd B \) for any \( J \in \mathcal{L}(H) \). Let \( H_0 = \hat{K}(H) \). Then \( H = H_0B \), \( H_0 = \langle \tilde{J} \mid J \in \mathcal{L}(H) \rangle \), and \( B \) acts regularly on \( \langle \tilde{J} \mid J \in \mathcal{L}(H) \rangle \). In particular, we can take projections in \( H_0 \) with respect to \( \langle \tilde{J} \mid J \in \mathcal{L}(H) \rangle \).

Let \( W = \theta(C_H(B)) \) and \( W_1 = \langle \text{Proj}_J(W) \mid J \in \mathcal{L}(H) \rangle \). Then \( C_{W_1}(b) \cong pW \cong \theta(C_H(b)) \) for all \( b \in B^* \). So it suffices to show \( \theta(C_{H}(b)) \leq W_1 \) for all \( b \in B^* \).

Fix \( E \in \mathcal{S}_1(B) \). Let \( S = \{ \langle J^E \rangle \mid J \in \mathcal{L}(H) \} \), \( T = \{ \tilde{L} \mid L \in S \} \), and \( V = \theta(C_{G}(E)) \). By hypothesis, \( V = V_1 \times V_2 \times \cdots \times V_p \), where each \( V_i \cong W \) and \( C_{K(H)}(E) \leq V \). Thus
\[
\{ V \cap K(G) \mid 1 \leq i \leq p \} = \{ C_L(E) \mid L \in S \}.
\]

Suppose \( C_L(E) = V_1 \cap K(G) \). Then
\[
V_1 = \bigcap \{ C_{r}(V_j \cap K(G)) \mid j \neq i \} = \bigcap \{ C_{r}(C_{m}(E)) \mid L \neq M \in S \}
\]
\[
= \bigcap \{ C_{r}(M) \mid L \neq M \in S \}
\]
\[
\leq \bigcap \{ C_{G}(M) \mid L \neq M \in S \} = \mathcal{L}.
\]
So \( V = \times \{ \mathcal{V} \cap \mathcal{L} \mid L \in T \} \). Let \( E \times F = B \). Then for \( \mathcal{L} \in T \), \( (V \cap \mathcal{L}) ** F = C_{r}(F) = W \). Since \( F \) acts regularly on \( T \), Lemma 5.6 yields that \( V \cap \mathcal{L} \leq \text{Proj}_L(W) \leq W_1 \) for all \( \mathcal{L} \in T \). Hence \( V \leq W_1 \). Since \( E \in \mathcal{S}_1(B) \) was arbitrarily chosen, the theorem is complete.

6. The Minimal Counterexample

Henceforth we shall assume that Theorem A is false and that \( G \) is a counterexample of least possible order. Subject to this restriction we assume that \( |\theta| \) is minimal. When convenient we shall write \( H_B \) for \( \theta(C_{G}(B)) \) for each nonidentity subgroup \( B \) of \( A \), and \( H_a \) for \( H_{(a)} \) for each \( a \in A^* \). We shall also write \( D \) for \( H_A \).

Following Theorem 5.16, for each \( X \in \mathcal{U}_0(A) \), we define \( NS(X) \) to be the
set of $D\!A$-invariant near $A$-solvable subgroups of $X$, and $X_{ns} = \langle NS(X) \rangle$.

Now define $\theta_{ns}(C_G(a)) = (\theta(C_G(a)))_{ns}$ for each $a \in A^\#$.

The goal of this section is to obtain sufficient structure of $\theta$ to determine the structure of $G$. For the convenience of the reader, we capsule this information in our first theorem.

**Theorem 6.1.** The following hold.

(a) $A$ is elementary Abelian of order $p^3$.

(b) One of the following sets of conditions hold. Either (b1) or (b2) holds.

(b1) The following three conditions hold.

(b1.1) $D$ is simple

(b1.2) Let $F \in \mathcal{S}_1(A)$. Then $H_F, A = K_{w}(A, F, F)$ for some $K \cong D$.

(b1.3) $H_a \in \mathcal{W}_{\theta}^*(A)$ for all $a \in A^\#$.

(b2) The following five conditions hold.

(b2.1) There is a distinguished $E \in \mathcal{S}_1(A)$ and a simple group $K$.

(b2.2) $F^*(D)$ is simple.

(b2.3) $H_E, A = L_{w}(A, E, E)$ for some $L \cong D$.

(b2.4) Let $E \neq F \in \mathcal{S}_1(A)$. Then $H_F = L_{w}(A, EF, F)$ for some $L \cong K$.

(b2.5) $H_a \in \mathcal{W}_{\theta}^*(A)$ if $a \in A - E$.

(c) $G = \langle \mathcal{W}_{\theta}(A) \rangle A$.

(d) $Z(\langle \mathcal{W}_{\theta}(A) \rangle) = 1$.

**Lemma 6.2.** (a) $A$ is elementary abelian of order $p^3$.

(b) There is an $a \in A^\#$ for which $\theta(C_G(a))$ is not near $A$-solvable.

(c) $\theta$ is locally complete.

(d) $G = A \langle \mathcal{W}_{\theta}(A) \rangle$.

**Proof:** (a), (d). These follow from the conditions of the counterexample.

(c) See [7, Lemma 5.11].

(b) This follows from Lemma 5.13 and [13, Main Theorem].

**Lemma 6.3.** Let $X \in \mathcal{W}_{\theta}(A)$. Then

(a) There is an $a \in A^\#$ such that $K(H_a) \not\subseteq X$.

(b) There is a $B \in \mathcal{S}_2(A)$ such that $K(H_B) \not\subseteq X$.

**Proof:** Let $a \in A^\#$. By Lemma 5.11,

$$K(H_a) = K((K(H_B) \mid a \in B \in \mathcal{S}_2(A))) \not\subseteq (K(H_F) \mid F \in \mathcal{S}_2(A)).$$

Hence it suffices to show that (a) is true.
Suppose that (a) is false. Choose \( X \in \mathcal{H}(A) \) such that \( K(H_a) \leq X \) for all \( a \in A^* \). Let \( B \in \mathcal{E}_2(A) \). By Lemmas 5.9 and 5.11,

\[
K(X) = K(\langle K(C_x(b)) | b \in B^* \rangle) = K(\langle K(H_a) | b \in B^* \rangle)
\]

admits \( H_B \). This is contrary to Theorem 6.2(c), which proves the lemma.

**Lemma 6.4.** \( Z(\langle \mathcal{H}(A) \rangle) = 1 \).

**Proof.** See [13, Theorem 5.1(d)].

**Theorem 6.5.**

(a) \( \theta_{n_2} \) is a complete \( A \)-signalizer functor on \( G \).

(b) \( \theta_{n_2}(C_G(a)) \) admits any \( DA \)-invariant \( K \)-subgroup of \( \theta(C_G(a)) \).

**Proof.** (a) This follows from Theorem 5.16(a) and Lemma 6.2(b).

(b) This follows from Theorem 5.16(b).

**Theorem 6.6.** \( \theta_{n_2}(G) = 1 \). In particular, \( K_A(X) = 1 \) whenever \( X \in \mathcal{H}(A) \).

**Proof.** Let \( W = \theta_{n_2}(G) \). Choose a \( B \in \mathcal{E}_2(A) \). By Lemma 5.1 and Theorem 6.5,

\[
K(H_a) \leq N_G(\langle \theta_{n_2}(C_G(b)) | b \in B^* \rangle) = N_G(\langle C_w(b) | b \in B^* \rangle) = N_G(W).
\]

Now Lemmas 6.2(c) and 6.3(b) imply that \( W = 1 \).

Suppose \( X \in \mathcal{H}(A) \). Then \( K_A(X) \cap K(X) \leq \theta_{n_2}(G) = 1 \). So \( \text{Sol}(X) = 1 \) and \( K(X) = K^A(X) \). Hence \( K_A(X) = C_A(K(G)) = 1 \), as required.

**Lemma 6.7.** \( \tilde{K} \circ \theta = \theta \).

**Proof.** Theorem 5.15(g) and Theorem 6.6 imply that \( \tilde{K} \circ \theta \) is an \( A \)-signalizer functor on \( G \). Lemma 6.3 implies that \( \theta = \tilde{K} \circ \theta \) as required.

**Lemma 6.8.** \( F^*(D) \) is simple.

**Proof.** By Lemma 6.2(b), \( D \neq 1 \). So Theorem 6.6 implies that \( \mathcal{L}(D) \) is nonempty. Let \( J \in \mathcal{L}(D) \). Define \( \theta_J(C_G(a)) = \theta(C_G(a)) \cap C_G(J) \). Clearly \( \theta_J \) is an \( A \)-signalizer functor of order less than \( \theta \). Hence \( \theta_J \) is complete. Let \( W = \theta_J(G) \). Suppose

Whenever \( B \in \mathcal{E}_2(A) \), \( L \in \mathcal{L}(H_B) \), and \( L \nleq W \), it follows that \( [W, L] = 1 \).

(6.1)

Then by Lemmas 6.2(c), 6.3(b), and Theorem 6.6 it follows that \( W = 1 \). So \( F^*(D) \) is simple.
We shall prove (6.1). Let $B \leq g_2(A)$ and $L \leq \mathcal{L}(H_B)$. Suppose $[W, L] \neq 1$. Then by Lemma 5.1, there is a $b \in B^*$ for which $[C_w(b), L] \neq 1$. Let $H = H_b$ and $H_j = \theta_j(C_G(b))$. Thus $[H_j, L] \neq 1$. By Theorem 5.15(f), $L = F^*(M \ast B)$ for some $M \in \mathcal{L}(H)$. Since $H_j$ is $B$-invariant and $L \leq \langle M^B \rangle$, it follows that $[H_j, M] \neq 1$. Since $H_j$ is $DA$-invariant, Lemma 5.14 implies that $L = F^*(M \ast A) \leq H_j \leq W$. This proves (6.1) and completes the lemma.

**Definitions.** For each nonidentity subgroup $B$ of $A$ define $B_c = C_A(K(H_B))$, and $B_N = A \cap \hat{K}(H_B A)$. Let $\mathcal{S}_i = \{ F \in \mathcal{S}_1(A) \| \mathcal{L}(H_F) = p^i \}$. 

**Lemma 6.9.** Let $B$ be a nonidentity subgroup of $A$. Let $E, F \in \mathcal{S}_1(A)$. Then all of the following hold.

- (a) $B_c = C_A(H_B) = C_A(L)$ for any $L \in \mathcal{L}(H_B)$.
- (b) $B_N = N_A(L)$ for any $L \in \mathcal{L}(H_B)$.
- (c) $|\mathcal{L}(H_B)| = |A/B_N|$.
- (d) $A/B_N$ acts regularly on $\mathcal{L}(H_B)$.
- (e) Suppose $F \leq E_N$ and $E \leq F_N$. Then $|\mathcal{L}(H_E)| = |\mathcal{L}(H_F)|$.
- (f) Suppose $F \leq E_N$ and $E \not\leq F_N$. Then $|\mathcal{L}(H_E)| = p |\mathcal{L}(H_F)|$.
- (g) Suppose $F \leq E_N$ and $E \not\leq F_N$. Then $|\mathcal{L}(H_E)| = |\mathcal{L}(H_F)|$.
- (h) $B_N/B_c$ is cyclic.

**Proof.** (a), (b), (c), (d). Clearly $B_c = C_A(H_B)$. By Lemma 6.8, $A$ acts transitively on $\mathcal{L}(H_B)$. Let $V$ be any subgroup of $A$. Since $A$ is Abelian, the members of $\mathcal{L}(H_B)$ centralized by $V$ is a union of $A$ orbits. Similarly, the members of $\mathcal{L}(H_B)$ normalized by $V$ is a union of $A$ orbits. Hence (a), (b), (c), (d) easily follow.

- (h) This follows from (a), (b) and the outer $p$-cyclic property of members of $\mathcal{L}(H_B)$.

- (e), (f), (g). Let $k = |\mathcal{L}(H_F)|$, $rk = |\mathcal{L}(H_E)|$, and $sk = |\mathcal{L}(H_F)|$. By Theorem 6.6 and Theorem 5.15(g), $s = 1$ if $E \leq F_N$, and $s = |E| = p$ if $E \not\leq F_N$. The symmetric statements for $r$ obtained by interchanging $E$ and $F$ yield (e), (f), and (g).

**Lemma 6.10.** $\mathcal{S}_2 \neq \emptyset$.

**Proof.** Suppose $\mathcal{S}_2 = \emptyset$. Then $\mathcal{S}_1(A) = \mathcal{S}_0 \cup \mathcal{S}_1$. Suppose in addition that $\mathcal{S}_1 = \emptyset$. Choose $a \in A^*$ with $H_a$ of maximal possible order. Let $B = \langle a \rangle C$. By Lemma 6.9(a), $H_a \leq H_b$ for all $b \in B^*$. Hence $H_a = H_b$ for all $b \in B^*$. By Lemma 6.9(h), $m(B) > 2$. Hence $\langle H_a(A) \rangle = H_a$. This is false; so $\mathcal{S}_1 \neq \emptyset$.

Let $F \in \mathcal{S}_1$ and $B = F_N$. By Lemma 6.9(d), $m(B) = 2$. Let $F \neq E \in \mathcal{S}_1(B)$. Since $\mathcal{S}_2 = \emptyset$, Lemma 6.9(f), implies that $F \leq E_N$. By Lemma 6.9(g),
Let \( E \in \mathcal{S}_1 \). Hence \( \mathcal{S}_1(B) \subseteq \mathcal{S}_1 \) and \( B = E_N \) for all \( E \in \mathcal{S}_1(B) \). Let \( L \in \mathcal{S}_1 \) and \( E \in \mathcal{S}_1(L_N \cap B) \). Then \( L_N = E_N = B \). Hence \( \mathcal{S}_1(B) = \mathcal{S}_1 \).

Next choose \( t \in A - B \) subject to \( H_t \) having maximal possible order. Let \( R = \langle t \rangle_C \). By Lemma 6.9(h), \( A/R \) is cyclic. Choose \( t \in T \in \mathcal{S}_2(R) \). Let \( E = T \cap B \). By Lemma 6.9(a), \( H_r = H_t \) for all \( r \in T - E \), and \( H_t \leq H_E \). Hence \( \langle \mathcal{U}_\theta(A) \rangle = \langle H_r \mid r \in T^* \rangle = H_E \), a contradiction.

**Lemma 6.11.** One of the following hold.

(a) \( \mathcal{S}_2 = \mathcal{S}_1(A) \), or

(b) \( |\mathcal{S}_2| = 1 \), \( \mathcal{S}_1 \cup \mathcal{S}_2 = \mathcal{S}_1(A) \), and \( F_N = F(\mathcal{S}_2) \) for all \( F \in \mathcal{S}_1(A) \).

**Proof.** Since \( A \) has order \( p^3 \), and \( \mathcal{S}_2 \neq \emptyset \) by Lemma 6.10, it follows from Lemma 6.9(d, e, f) that \( \mathcal{S}_1 \cup \mathcal{S}_2 = \mathcal{S}_1(A) \). We may suppose \( \mathcal{S}_2 \neq \mathcal{S}_1(A) \). Let \( B = \langle \mathcal{S}_2 \rangle \). We must show that \( B \) is cyclic. Choose \( E \in \mathcal{S}_1 \) with \( E \leq B \) if possible. For each \( F \in \mathcal{S}_2 \), \( F_N = F \); so by Lemma 6.9(e), \( F \leq E_N \). Hence \( B \leq E_N \) and \( B \) is cyclic.

**Lemma 6.12.** Suppose \( X \) is a subgroup of \( G \) generated by some elements of \( \mathcal{U}_\theta(A) \). Then either

(a) \( X \) contains every element of \( \mathcal{U}_\theta(A) \) and \( X \in \mathcal{U}_\theta(A) \), or

(b) \( X \in \mathcal{U}_\theta(A) \) and for any \( B \in \mathcal{S}_2(A) \) there is an \( a \in B^* \) such that \( H_a \leq X \).

**Proof.** This is an easy variation of [5, Lemma 5.4].

**Theorem 6.13.** Suppose \( \mathcal{S}_1 = \emptyset \). Then Theorem 6.1 holds.

**Proof.** By Lemma 6.2(a, d), and Lemma 6.4, it suffices to show conclusion (b1) holds.

Let \( E, F \in \mathcal{S}_1(A) \) be distinct. \( E_N = E \); so \( F \) acts regularly on \( \mathcal{L}(H_E) \). Hence \( K(H_E) \cap C(F) = K(K(H_E) \cap C(F)) \). Hence by Theorem 5.15(f), \( K(H_E) \cap C(F) = K(H_E) \cap C(F) \). By symmetry, \( K(H_E) \cap C(F) = K(H_E) \cap C(F) \). Hence \( K \circ \theta \) is an \( A \)-signalizer functor on \( G \). By Lemma 6.3(a), \( K(H_a) = H_a \) for all \( a \in A^* \). Let \( a \in A^* \) and \( L \in \mathcal{S}(H_a) \). Since \( A/\langle a \rangle \) acts regularly on \( \mathcal{L}(H_a) \), it follows that \( D = L \ast A \ast L \). Hence (b1.1) and (b1.2) hold.

Suppose \( H_a \leq X \leq \mathcal{U}_\theta(A) \). By Lemma 5.1, there is a \( b \in A^* \) such that \( C_{H_a}(b) \neq C_{H_b}(b) = H_{a,b} \). By Lemma 5.10, \( H_{a,b} \) is a maximal \( A \)-invariant subgroup of \( H_a \). Hence \( H_b = C_{H_b}(b) \). Choose \( B \in \mathcal{S}_2(A) \) with \( b \in B \) but \( a \not\in B \). Then \( H_d = (H_d \cap H_b) \leq X \) for any \( d \in B^* \). This is false. Hence conclusion (b1.3) holds and the theorem is complete.
THEOREM 6.14. Suppose $\mathcal{S}_1 \neq \emptyset$. Then $\mathcal{S}_2 = \{E\}$ for a unique $E \in \mathcal{S}_1(A)$. Moreover the following conditions hold:

(a) For each $a \in A - E$,

$$H_aA = Xw(A, E\langle a \rangle, \langle a \rangle)$$

for some simple group $X$ whose isomorphic class is independent of $a$.

(b) $H_eA = Yw(A, E, E), Y \cong D,$ and $F^*(D)$ is simple.

Proof. The first statement holds by Lemma 6.11. For each complement $B$ of $E$ in $A$ define

$$\theta_B^*(C_G(a)) = K(H_b)$$

if $a \in A - E$

$$= \langle K(H_b) \cap H_E | b \in B^* \rangle$$

if $a \in E^*$.  

The gist of the proof is to show $\theta_B^*$ is an $A$-signalizer functor on $G$ with additional suitable properties.

Again for $B$ a complement of $E$ in $A$, define $\theta_B^*(C_G(b)) = K(H_b)$ for $b \in B^*$. By Lemma 6.11(b), $\langle e \rangle$ acts regularly on $\mathcal{L}(H_b)$ whenever $\langle e, f \rangle = B$. By Lemma 5.15(f) and Theorem 6.6, $K(H_e) \cap C_G(f) = K(C_G(\langle e, f \rangle))$. Hence $\theta_B^*$ is a $B$-signalizer functor on $G$. Now define $\tilde{\theta}_B^*(C_{BH_E}(b)) = \theta_B^*(C_G(b)) \cap H_E$. Then $\tilde{\theta}_B^*$ is clearly a $B$-signalizer functor on $BH_E$. Since $E_N = E$, it follows that $B$ acts regularly on $\mathcal{L}(H_E)$. Hence by Theorem 5.15(f) and Theorem 6.6, $C_{K(H_E)}(b) \leq \tilde{\theta}_B^*(C_{BH_E}(b))$ for all $b \in B^*$. Also

$$\tilde{\theta}_B^*(C_{BH_E}(b)) = C_{K(H_b)}(E) = \times \{C_L(E)| L \in \mathcal{L}(H_b)\}$$

$$\cong p((K(H_b) \cap C_G(B)) \cap C(E))$$

$$= p(K(H_b) \cap C(E))$$

$$= \tilde{\theta}_B^*(C_{BH_E}(B)).$$

We have established all the conditions of Theorem 5.17 with $(H, \theta, B)$ replaced by $(BH_E, \tilde{\theta}_B^*, B)$. For each $L \in \mathcal{L}(H_E)$, let $\tilde{L} = C_{H_E}(C_{K(H_E)}(L))$. By Theorem 5.17, we obtain

$$\tilde{\theta}_B^*(BH_E) = \times \{Proj_L(K(H_b) \cap H_E)| L \in \mathcal{L}(H_E)\}$$

(6.2)

and

$$\theta_B^*(C_G(E)) \cap C_G(b) = \theta_B^*(C_G(b)) \cap C_G(E) \quad \text{for} \quad b \in B^*.$$  

(6.3)

The functor $\theta_B^*$ is independent of the complement $B$ of $E$ in $A$ on the subgroups $C_G(b)$ for $b \in A - E$. We next want to show that it is also
independent on $C_\alpha(E)$. Suppose then that $T$ is a complement for $E$ in $A$ distinct from $B$. Let $F = T \cap B$. Then $F \in \mathcal{S}(A)$. By (6.2),

$$\tilde{\theta}_B(BH_E) = \times \{\text{Proj}_L(K(H_B) \cap H_E) \mid L \in \mathcal{L}(H_E)\}$$

$$= \times \{\text{Proj}_L(K(H_B) \cap H_E) \mid L \in \mathcal{L}(H_E)\}$$

$$= \times \{\text{Proj}_L(K(H_B) \cap H_E) \mid L \in \mathcal{L}(H_E)\}$$

$$= \tilde{\theta}_T(TH_E).$$

Hence $\theta_B^*$ is independent of the complement $B$ of $E$ in $A$. Therefore by (6.3) there follows

$$\theta_B^*(C_\alpha(E)) \cap C_\alpha(a) = C_\alpha(E) \cap \theta_B^*(C_\alpha(a)) \quad \text{for all } a \in A - E. \quad (6.4)$$

Next we show $\theta^* = \theta_B^*$ is balanced. Let $a, b \in A^*$ and $T = \langle a, b \rangle$. We have already shown $\theta^*(C_\alpha(a)) \cap C_\alpha(b) \leq \theta^*(C_\alpha(b))$ if $E \leq T$. Certainly $\theta^*(C_\alpha(a)) \cap C_\alpha(b) \leq \theta^*(C_\alpha(b))$ if $T$ is cyclic. Suppose then $E < T$ and $a, b \in A - E$. By (6.4)

$$\theta^*(C_\alpha(a)) \cap C_\alpha(b) = \theta^*(C_\alpha(a)) \cap C_\alpha(T)$$

$$= \theta^*(C_\alpha(E)) \cap C_\alpha(T)$$

$$= C_\alpha(T) \cap \theta^*(C_\alpha(b)) \leq \theta^*(C_\alpha(b)).$$

Hence $\theta^*$ is an $A$-signalizer functor on $G$. By Lemma 6.3, $\theta^* = \theta$.

Clearly $A$ is transitive on $\{L \mid L \in \mathcal{L}(H_E)\}$. Hence by (6.2), $H_E A = YW(A, E, E)$ for some $Y \cong D$. By Lemma 6.8, $F^*(D)$ is simple. This proves (b). Certainly, $H_a A = X_a Y(A, \langle a \rangle, E, \langle a \rangle)$ for some simple group $X_a$, whenever $a \in A - E$. It remains to show that the isomorphic type of $X_a$ is independent of $a \in A - E$. Define an equivalence relation $\sim$ on $\mathcal{S}$ by $T \sim F$ if and only if $X_T \cong X_F$. Certainly the elements of $\mathcal{S}$ are equivalent if $B$ is any complement for $E$ in $A$. All hyperplanes of $A$ have a nontrivial intersection. Hence $\mathcal{S}$ is an equivalence class, as required.

**Proof of Theorem 6.1.** By Theorem 6.13 we may suppose $\mathcal{S} \neq \emptyset$. By Lemma 6.2(a, d) and Lemma 6.4, it suffices to show conclusion (b2) holds. By Theorem 6.14 it remains to show $H_a \in \mathcal{W}(A)$ whenever $a \in A - E$. Suppose $a \in A - E$ and $H_a < X \in \mathcal{W}(A)$. Extend $\langle a \rangle$ to a complement $B$ of $E$ in $A$. By Lemma 5.1, $H_B < C_X(b)$ for some $b \in B - \langle a \rangle$. By Lemma 5.10, $C_X(b) = H_x$. Hence $K(H_E) = \langle K(H_E) \cap C(a), K(H_E) \cap C(b) \rangle \leq X$. Hence for any $f \in B^*$, $H_f = \langle H_B, K(H_E) \cap H_f \rangle \leq X$, a contradiction. This completes the proof of Theorem 6.1.
7. $S_{\tau}(A)$-Subgroups

We say $\theta$ is type (A) if $\theta$ satisfies conclusion (b1) of Theorem 6.1. We say $\theta$ is type (B) if $\theta$ satisfies conclusion (b2) of Theorem 6.1. When $\theta$ is type B we reserve $E$ for the unique element of $\mathcal{S}_\tau$. For the remainder of the paper we will fix the following notation. Suppose $B$ is a nonidentity subgroup of $A$ and $S$ is an $S_{\tau}(A)$-subgroup of $G$. Then

$$\text{Ind}(S, B) = \{S \cap L | L \in \text{Ind}(H_B)\}.$$  

We shall also reserve $S$ for some $S_{\tau}(A)$-subgroup of $G$, and $Z$ for $Z(S)$.

**Lemma 7.1.** Suppose $B \in \mathcal{S}_2(A)$ and $E \leq B$ if $\theta$ is type (B). Then

(a) $Z(C_S(a)) \cap C(B) = Z(C_S(B))$ for $a \in B^*$, and

(b) $Z(C_S(B)) = C_{Z(S)}(B)$.

**Proof.** (a) $B/\langle a \rangle$ acts semi-regularly on $\text{Ind}(S, B)$, whence (a) follows by Lemma 5.12.

(b) By (a), $Z(C_S(B)) \leq C_S(\langle C_S(a) | a \in B^* \rangle) = Z(S)$. This proves (b), and the lemma.

**Theorem 7.2.** Suppose $\theta$ is type (A). Then

$$Z(C_S(a)) = C_{Z(S)}(a) \quad \text{for all } a \in A^*.$$ 

**Proof.** By Lemma 7.1,

$$Z(C_S(a)) = \langle Z(C_S(a)) \cap C(B) | a \in B \in \mathcal{S}_2(A) \rangle \leq Z(S),$$

as required.

**Theorem 7.3.** Suppose $\theta$ is type (B). Then

$$Z(C_S(a)) = C_{Z(S)}(a) \quad \text{for all } a \in A - E.$$ 

**Proof.** Let $Z = Z(S)$ and $Z_B = Z(C_S(B))$ for all subgroups $B$ of $A$. Let $E \neq F \in \mathcal{S}_1(A)$. Let

$$Z_F^0 = \langle Z_F \cap C_S(B) | E \times B = A \text{ and } F < B \rangle$$

and

$$Z_F^1 = \bigcap \{[Z_F, B] | E \times B = A \text{ and } F < B \}.$$
By Lemma 7.1, $Z^0 \leq Z$. By [8, Theorem 5.2.3], $Z^1 \leq C_{Z^2}(E) \leq Z(C(F))$, where $T = C_S(E)$. However, $F$ acts semi-regularly on $\text{Ind}(S, E)$. Hence $Z(C(F)) \leq Z(T) = Z_F$. So

$$Z_F = (Z_F \cap Z)(Z_F \cap Z_E) \quad \text{for all } F \in \mathcal{F}_1(A). \quad (7.1)$$

Let $V = C_Z(A)$ and $W = \times \{\text{Proj}_l(V) | L \in \text{Ind}(S, E)\}$. Let $F, K \in \mathcal{F}_1(A)$ satisfy $E \times F \times K = A$. By Lemma 7.1 and (7.1),

$$Z_F \cap Z_E \cap C_S(K) = (Z_F \cap C_S(E)) \cap C_S(K)$$

$$= (Z_F \cap C_S(K)) \cap C_S(E)$$

$$= C_Z(FK) \cap C_S(E) = V.$$

Since $E$ normalizes each member of $\text{Ind}(S, F)$, it follows from (7.1) that $Z_E \cap Z_F = \times \{C_{Z(R)}(E) | R \in \text{Ind}(S, F)\}$. Since $K$ acts regularly on $\text{Ind}(S, F)$, and $(Z_F \cap Z_E) \cap C_S(K) = V$, there follows from Lemma 5.6

$$Z_E \cap Z_F = \times \{\text{Proj}_l(V) | R \in \text{Ind}(S, F)\} \cong pV. \quad (7.2)$$

Since $\text{Ind}(S, EF) = \{C_R(E) | R \in \text{Ind}(S, F)\}$, (7.2) implies that

$$Z_E \cap Z_F = \times \{\text{Proj}_l(V) | T \in \text{Ind}(S, EF)\} \leq W. \quad (7.3)$$

Since $A/E$ acts regularly on $\text{Ind}(S, E)$, it follows that $p^2V \cong W \cong p(C_w(F))$. Hence by (7.2) and (7.3) we obtain

$$Z_E \cap Z_F = C_w(F) = C_w(EF) \quad \text{whenever } E \neq F \in \mathcal{F}_1(A). \quad (7.4)$$

In particular, (7.4) implies

$$Z_E \cap Z_F = Z_E \cap Z_T \quad \text{whenever } EF = ET \text{ and } F, T \in \mathcal{F}_1(A). \quad (7.5)$$

By (7.5), $Z_E \cap Z_F \leq Z$ whenever $E \neq F \in \mathcal{F}_1(A)$. Now (7.1) completes the theorem.

**Lemma 7.4.** Let $S$ be an $S_r(A)$-subgroup of $G$. Let $Z = Z(S)$. Let $a \in A^*$. Suppose $\langle a \rangle \neq E$ if $\theta$ is type (B). Assume $r \in \pi(\theta)$. Then

(a) $r \in \pi(H_a)$,
(b) $Z \cap L \neq 1$ for any $L \in \text{Ind}(H_a)$, and
(c) $Z \cap H_a = \times \{Z \cap L | L \in \text{Ind}(H_a)\}$.

**Proof.** Choose a subgroup $B$ of $A$ which contains $a$ but not $E$. By Theorem 6.1, $\pi(H_b) = \pi(H_a)$ for all $b, c \in B^*$. Hence by Lemmas 5.1 and 5.3, $1 \neq C_3(a)$ is an $S_r(A)$-subgroup of $H_a$. In particular, (a) holds. The structure of Sylow $r$-subgroups of $H_a$ and Theorems 7.2 and 7.3 yield (b) and (c).
8. Conclusion of Proof.

We continue the conventions introduced at the beginning of part 7. In particular, \( r \in \pi(\theta) \), \( S \in \mathcal{N}_0^\theta(A : r) \), and \( Z = Z(S) \).

**Theorem 8.1.** \( \theta \) is type (B).

*Proof.* Suppose false. Then by Theorem 6.1, \( \theta \) is type (A). In particular, \( D \) is simple, and for each nonidentity subgroup \( T \) of \( A \), \( AH_T = Lw(A, T, T) \) for some \( L \cong D \).

Fix a hyperplane \( B \) of \( A \). For each \( L \in \mathcal{L}(H_B) \), let \( Z_L = \bigcap \{ C_2(K) \mid L \neq K \in \mathcal{L}(H_B) \} \), \( M_L = \langle L, Z_L \rangle \), and \( M = \langle M_L \mid L \in \mathcal{L}(H_B) \rangle \). By Lemma 7.4(c), \( Z \cap H_a \leq M \) for all \( a \in B^* \). By Lemma 7.4(b), \( H_a = \langle H_B, Z \cap H_a \rangle \leq M \) for all \( a \in B^* \). Hence by Theorem 6.1(b1.3) and (c) there follows

\[
M = \langle \mathcal{N}_\emptyset(A) \rangle. \tag{8.1}
\]

Since \( Z \) is Abelian, \( [M_L, M_K] = 1 \) whenever \( L \neq K \). Hence Theorem 6.1(d) yields

\[
M = \times \{ M_L \mid L \in \mathcal{L}(H_B) \}. \tag{8.2}
\]

Since \( A \) acts transitively on \( \mathcal{L}(H_B) \) there follows,

\( A \) acts transitively on \( \{ M_L \mid L \in \mathcal{L}(H_B) \} \) and \( B = N_A(M_L) \) for \( L \in \mathcal{L}(H_B) \). \tag{8.3}

By definition we also have

\[
H_B = \times \{ H_B \cap M_L \mid L \in \mathcal{L}(H_B) \}. \tag{8.4}
\]

Now let \( B_1, B_2, B_3 \) be 3 hyperplanes of \( A \) such that \( \{ B_i \cap B_j \mid 1 \leq i < j \leq 3 \} \) are cyclic subgroups of \( A \) which generate \( A \). Let \( \{ M_{ij} \mid 1 \leq j \leq p \} = \{ M_L \mid L \in \mathcal{L}(H_B) \} \) for \( i = 1, 2, \) or 3. Let \( M_{i,j,k} = M_i^j \cap M_j^k \cap M_k^i \). Since \( M \) is generated by perfect subgroups, (8.2) yields that

\[
M = [M, M, M] = \left[ X M_i^j, X M_j^k, X M_k^i \right] \leq \langle [M_i^1, M_j^2, M_k^3] \mid 1 \leq i, j, k \leq p \rangle \leq \langle M_{i,j,k} \mid 1 \leq i, j, k \leq p \rangle.
\]

By (8.2), \( [M_{i,j,k}, M_{u,v,w}] = 1 \) if \( (i, j, k) \neq (u, v, w) \). Hence Theorem 6.1(d) yields

\[
M = \times \{ M_{i,j,k} \mid 1 \leq i, j, k \leq p \}. \tag{8.5}
\]
The choice of $B_1, B_2, B_3$, together with (8.2) yields

$$A \text{ acts regularly on } \{M_{i,j,k} \mid 1 \leq i, j, k \leq p\}. \quad (8.6)$$

Now let $W_{i,j,k} = \text{Proj}_{M_{i,j,k}}(D)$, and $W = \langle W_{i,j,k} \mid 1 \leq i, j, k \leq p \rangle$. By (8.6), we obtain

$$W \cong p^3D. \quad (8.7)$$

By (8.4), we obtain

$$W \geq \langle H_{B_1}, H_{B_2}, H_{B_3} \rangle. \quad (8.8)$$

By Lemma 5.10 and (8.8), $H_{B_1 \cap B_2} = \langle H_{B_1}, H_{B_2} \rangle \leq W$. Hence by (8.8) and Theorem 6.1(b1.3), $W = M$. By (8.6), (8.7), $p^2D \cong H_a \leq C_w(a) \cong p^2D$ for all $a \in A^\#$. Hence $C_w(a) = H_a$ for all $a \in A^\#$. However $(W, p)$ satisfies Hypothesis B. This contradiction yields the result.

**Lemma 8.2.** $F^*(H_{E}) \leq \langle D, Z \rangle$.

**Proof.** Let $W = \langle D, Z \rangle$ and $W_b = W \cap H_b$ for each $b \in A^\#$. Let $b \in A - E$, and $J \in \mathcal{W}(H_b)$. By Lemma 7.4(b), $1 \not= Z \cap J \leq W_b \cap J \leq \text{Proj}_J(W_b)$ where projections are being taken in $H_b$ with respect to $\mathcal{W}(H_b)$. By Lemma 5.5, $C_J(E) = \text{Proj}_J(D) \leq \text{Proj}_J(W_b)$. Hence by Hypothesis (A.3.1), $W_b \cap J$ is nonsolvable. By [8, Theorem 10.2.1], $C_{W_b \cap J}(E) \neq 1$. By Hypothesis (A.3.2), $F^*(C_J(E))$ is the unique minimal normal subgroup of $C_J(E)$, whence $F^*(C_J(E)) \leq W_b$. So $F^*(H_{E}) \cap C_G(b) = F^*(H(E, b)) \leq \langle D, Z \rangle$ for all $b \in A - E$. Now Lemma 5.1 yields the lemma.

**Lemma 8.3.** Suppose $B$ is a hyperplane of $A$ which contains $E$. Let $L \in \text{Ind}(H_B)$. Define $\hat{L}$ to be the product of components of $H_E$ not centralized by $L$. Then $\hat{L} \leq \langle Z, L \rangle$.

**Proof.** Let $Z_0 = Z(C_0(E))$, $V = ZZ_0$, $V_L = \bigcap \{C_V(K) \mid L \neq K \in \text{Ind}(H_B)\}$, $W_L = \langle V_L, L \rangle$, and $W = \langle W_L \mid L \in \text{Ind}(H_B) \rangle$. Since $V$ is Abelian, it follows that $[W_L, W_K] = 1$ if $L \neq K$. In particular,

$$W_L \triangleleft W \quad \text{for any } L \in \text{Ind}(H_B). \quad (8.9)$$

By Lemma 7.4(c), $C_E(a) \leq \langle V_L \mid L \in \text{Ind}(H_B) \rangle \leq W$ if $a \in A - E$. By Lemma 5.3, $C_E(E) \leq Z_0 \leq W$. Hence by Lemma 5.1, $Z \leq W$. Lemma 8.2 yields

$$F^*(H_E) \leq \langle D, Z \rangle \leq \langle H_B, Z \rangle \leq W. \quad (8.10)$$

Let $\hat{L} = \langle L, Z \rangle'$. By (8.9), $\hat{L} \leq \langle W_L, Z \rangle' \leq W_L$. Since $[\hat{L}, K] \leq$
\[ [W_L, W_K] = 1 \text{ for distinct } L, K \in \text{Ind}(H_g), \text{ and } \tilde{L} \text{ admits } \langle L, Z \rangle, \text{ there follows} \]
\[ \tilde{L} \triangleleft \langle Z, H_g \rangle. \quad (8.11) \]

By (8.10) and (8.11), \( \tilde{L} = \langle F^*(L)^{F(H)} \rangle \leq \langle F^*(L)^{\langle H_u, Z \rangle} \rangle \leq \tilde{L} \leq \langle L, Z \rangle \) as required.

**Lemma 8.4.** Suppose \( E \neq F \in \mathcal{E}_1(A) \), \( L \in \text{Ind}(H_{EF}) \), and \( K \in \text{Ind}(H_F) \). Suppose in addition that \( C_K(E) \neq L \). Let \( \tilde{L} \) be the product of components of \( H_E \) not centralized by \( L \). Then \[ [L, K] = 1. \]

**Proof:** Let \( L, K \in \text{Ind}(H_E) \) satisfy \( C_L(E) = L \). Then \( [L, S \cap K] \leq [L_1, K] = 1 \). Clearly, \( [Z, S \cap K] = 1 \). Hence \( [\langle L, Z \rangle, S \cap K] = 1 \). By Lemma 8.3, \( [\tilde{L}, S \cap K] = 1 \). Since \( K = \langle K \cap S \rangle \) \( S \) is some \( S_r(A) \)-subgroup, \( r \in \pi(\theta) \), it follows that \( [\tilde{L}, K] = 1 \).

**Theorem 8.5.** Let \( W = \langle U_\theta(A) \rangle \). Suppose \( E \neq F \in \mathcal{E}_1(A) \). Then for each \( K \in \text{Ind}(H_F) \), \( W \) has direct factors \( W_K \) which contain \( K \) and satisfy \[ W = \times \{ W_K | K \in \text{Ind}(H_F) \} \]. Moreover, \( A \) acts transitively on \( \{ W_K \} \).

**Proof:** For each \( K \in \text{Ind}(H_F) \), let \( K_0 = C_K(E) \), and \( \tilde{K} \) be the product of components of \( H_E \) not centralized by \( K_0 \). Now let \( W_K = \langle K, \tilde{K} \rangle \). By Lemma 8.4, \( [\tilde{L}, K] = 1 \) whenever \( L, K \), are distinct members of \( \text{Ind}(H_F) \). Moreover, \( \{ T \cap H_F | T \in \mathcal{L}(H_F) \} = \text{Ind}(H_{EF}) = \{ R ** F | R \in \text{Ind}(H_E) \} \), whence \( [\tilde{L}, \tilde{K}] = 1 \) if \( L \neq K \). Hence \( [W_L, W_K] = 1 \) if \( L \neq K \). Now \( \langle F^*(H_F), H_F \rangle \leq \langle W_L | L \in \text{Ind}(H_F) \rangle \). Hence Theorem 6.1(b2.5) and Lemma 6.12, yields \( W = \times \{ W_K | K \in \text{Ind}(H_F) \} \). Since \( A \) acts transitively on \( \text{Ind}(H_F) \) and \( EF = N_A(K) \) for each \( K \in \text{Ind}(H_F) \) the remaining statements also hold.

**Proof of Theorem A.** Let \( F_1, F_2 \in \mathcal{E}_1(A) \) satisfy \( EF_1F_2 = A \). Let \( \mathcal{L}_i = \mathcal{L}(H_{F_i}) \) for \( i = 1 \) or 2. Let \( W = \langle U_\theta(A) \rangle \). Following Theorem 8.5, for each \( K \in \mathcal{L}_i \) let \( W_K \) be direct factors of \( W \) which contain \( K \) and which satisfy

(a) \( W = \times \{ W_K | K \in \mathcal{L}_i \} \) for \( i = 1 \) or 2.

(b) \( A \) is transitive on \( \{ W_K | K \in \mathcal{L}_i \} \) and

\( EF_i = N_A(W_K) \) for any \( K \in \mathcal{L}_i \).

Let \( \Omega = \{ W_K \cap W_L | K \in \mathcal{L}_1, L \in \mathcal{L}_2 \} \). As in Theorem 8.1 we obtain

\( W = \times \Omega \), and

(8.12)

\( A \) acts transitively on \( \Omega \), and \( N_A(X) = E \) for any \( X \in \Omega \). \( \quad (8.13) \)

Let \( M = H_{F_1F_2} \), \( M_X = \text{Proj}_X(M) \) for \( X \in \Omega \), and \( \hat{M} = \times \{ M_X | X \in \Omega \} \). Let \( K \in \mathcal{L}_2 \). When \( (A, G, S, T, K, W) \) is replaced by \( (F_1, F_2, W, \mathcal{L}_1, K, W_K) \),...
Lemma 5.6 implies that $K \leq \hat{M}$. Hence $\langle H_F, H_F^1 \rangle \leq \hat{M}$. By Theorem 6.1(b,2.5) and Lemma 6.12, $\hat{M} = W$. By (8.12) and (8.13), $W \cong p^2M$. Let $a \in A - E$. By (8.13) and Theorem 6.1(b,2.4), $C_w(a) \cong pM \leq H_a$. Hence $H_a = C_w(a)$ for all $a \in A - E$. Since $W$ is a $p'$-group, $H_k \leq C_w(E) = \langle C_w(E) \cap C_w(a) | a \in A - E \rangle = \langle C_w(E) \cap H_a | a \in A - E \rangle \leq H_E$. Hence $C_w(b) = H_b$ for all $b \in A^*$. Since $(M, p)$ satisfies Hypothesis B, it follows that $(W, p)$ satisfies Hypothesis B. Hence $W \in \mathcal{I}_p(A)$, a contradiction. This completes the proof of Theorem A. Hence Corollary C also holds, thus completing the proof of all parts.

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