JOURNAL OF ALGEBRA 78, 215–238 (1982)

Nonsolvable Signalizer Functors on Finite Groups

PATRICK P. MCBRIDE

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

Communicated by W. Feit

Received November 4, 1981

1. INTRODUCTION

Recently Gorenstein and Lyons obtained the first nonsolvable signalizer functor theorems [9]. They pinpointed certain "unbalancing" problems. This paper grew from an attempt to manage such problems. Theorem A is the result. Theorem B and Corollary C give some measure of the practical scope of Theorem A.

Suppose p is a prime, A is an elementary Abelian p-subgroup of a finite group G, and θ is an A-signalizer functor on G. The unbalancing difficulties, referred to above, occur only if there are "certain" nonidentity subgroups X of G, such that $C_X(A)$ is solvable. Using methods of Glauberman [5, Lemma 2.11 and Theorem 4.5] we are able to reduce the problem: either the "unbalancing" problems vanish or $\theta(C_G(A))$ is solvable. The latter case is treated in [13]. The rest of the work is treated here. This work pivots on Theorems 5.16 and 6.5, results which closely resemble [5, Lemma 2.11 and Theorem 4.5].

2. NOTATION, DEFINITIONS, AND CONVENTIONS

Conventions. All groups treated in this paper are finite. All simple groups are nonabelian. We shall reserve p and r for primes.

Suppose A, B are groups and B acts on A. Then AB is the usual product if A and B are subgroups of a common group; otherwise AB is the semidirect product of A by B.

Suppose a group G is the direct product of subgroups $A_1, A_2, \ldots, A_n$. Let X be a subset of G. Then $\mathrm{Proj}_{A_i}(X)$ is the usual projection map of X on A_i. We often write $\mathrm{Proj}_A(X)$ when $G = A \times C_G(A)$. Then projections are taken with respect to the pair $(A, C_G(A))$. If X is contained in a subgroup N, we may

* Partially supported by NSF Grant MCS 76-06626.

0021-8693/82/090215-24$02.00/0

apply the above conventions to N. We do so by stating that projections are being taken in N.

Notations and Definitions. Our notation for groups of Lie type agrees with [2]. Most of the specialized notation is taken from [5, 7, 13]. For the convenience of the reader we shall repeat many of these. What is not explained can be found in [2, 8], or is hopefully self-explanatory.

(1) Let S be a finite set. When the members of S are sets, $\bigcap S$ is the intersection of the members of S. When the members of S are groups, $\times S$ is the direct product of the members of S. When the members of S are real numbers, $\sum S$ is the sum of the members of S.

(2) A *section* of a group G is a quotient group K/L of a subgroup K of G by a normal subgroup L of K.

(3) A simple group G is *outer p-cyclic* means that the outer automorphism group of G, $\mathrm{Out}(G)$, has cyclic Sylow p-subgroups.

(4) The group G is *near p-solvable* means that G is a p'-group, and any simple section of G is isomorphic to $A_1(q)$, $A_1(3^p)$, ${}^2B_2(q)$, or ${}^2A_2(q^2)$, where $q = 2^p$.

(5) A *localized subgroup* of a group G is any subgroup which normalizes a nonidentity solvable subgroup of G.

(6) *Hypothesis* A (applied to a pair (G, p)).

(A.1) p is a prime and G is a simple p'-group.

(A.2) G is outer p-cyclic.

(A.3) G is near p-solvable, or the following three conditions apply to any automorphism f of G of order p.

(A.3.1) Let $C = C_G(f)$. Then C is not a localized subgroup of G.

(A.3.2) $F^*(C)$ is simple.

(A.3.3) Any p'-automorphism of G which centralizes C is trivial.

(7) *Hypothesis* B (applied to a pair (G, p)). p is a prime. G is a p'-group. Hypothesis A applies to (K, p) for all simple sections K of G.

(8) The group G *is near A-solvable* means that A is an elementary p-group, (G, p) satisfies Hypothesis B, and $C_G(A)$ is solvable.

(9) The statement "*θ is an A-signalizer functor on G*" means that A is an Abelian p-subgroup of the group G for some prime p, and that for each $a \in A^{\#}$ there is defined an A-invariant p'-subgroup $\theta(C_G(a))$ of $C_G(a)$ such that

$$\theta(C_G(a)) \cap C_G(b) \leqslant \theta(C_G(b)) \qquad \text{for all } a, b \in A^{\#}. \qquad (*)$$

The property (*) is called *balance.*

In definitions (10) through (18), let θ, G, A, and p be as in Definition 9.

(10) *Hypothesis* (C) (applied to θ). The pairs $(\theta(C_G(a)), p)$ satisfy Hypothesis B for all $a \in A^{\#}$.

(11) The *associated set of A-signalizers* is the set of all A-invariant p'-subgroups X of G such that $C_X(a) \leqslant \theta(C_G(a))$ for all $a \in A^{\#}$, and such that (X, p) satisfies Hypothesis B. It is denoted $\mathcal{N}_\theta(A)$. The set of all maximal elements of $\mathcal{N}_\theta(A)$ under inclusion is denoted by $\mathcal{N}_\theta^*(A)$.

(12) We say that θ is *complete* if G contains a unique maximal element of $\mathcal{N}_\theta(A)$ under inclusion. This element is then denoted by $\theta(G)$.

(13) We say that θ is *locally complete* if, for every nonidentity element X of $\mathcal{N}_\theta(A)$, $N_G(X)$ contains a group $\theta(N_G(X))$ which is the unique maximal element among all elements of $\mathcal{N}_\theta(A)$ contained in $N_G(X)$. In this case, we put $\theta(C_G(X)) = \theta(N_G(X)) \cap C_G(X)$.

(14) For every nonidentity subgroup B of A, let

$$\theta(C_G(B)) = \bigcap \{\theta(C_G(b)) \mid b \in B^{\#}\}.$$

(15) The set of all elements of $\mathcal{N}_\theta(A)$ which are $\theta(C_G(A))$-invariant is denoted $\hat{\mathcal{N}}_\theta(A)$.

(16) The set of all elements of $\mathcal{N}_\theta(A)$ which contain $\theta(C_G(A))$ is denoted $\tilde{\mathcal{N}}_\theta(A)$.

(17) $\pi(\theta) = \bigcup \{\pi(\theta(C_G(a))) \mid a \in A^{\#}\}$ and $|\theta| = \sum_{a \in A^{\#}} |\theta(C_G(a))|$.

(18) For any $r \in \pi(\theta)$, let $\mathcal{N}_\theta(A; r)$ be the set of all r-groups in $\mathcal{N}_\theta(A)$, and let $\mathcal{N}_\theta^*(A; r)$ be the set of maximal elements of $\mathcal{N}_\theta(A; r)$. The elements of $\mathcal{N}_\theta^*(A; r)$ are called $S_r(A)$-subgroups of G.

(19) The solvable radical of a group G is the maximal solvable normal subgroup of G. It is denoted $\mathrm{Sol}(G)$.

(20) The set of subnormal simple subgroups of a group G is denoted $\mathscr{S}(G)$. Let $\bar{G} = G/\mathrm{Sol}(G)$. Then $\mathscr{M}(G)$ is the set of all subgroups X of G, which contain $\mathrm{Sol}(G)$, and which satisfy $\bar{X} \in \mathscr{S}(\bar{G})$.

(21) A group is *semi-simple* means that it is the direct product of its normal simple subgroups. This use is not in accord with [8, p. 501]. A group is *perfect* if it is its own derived group. A group is an *E-group* if it is perfect, and modulo its center is semi-simple. A group is a K-group if modulo its solvable radical it is semi-simple. Let G be a group. The *Fitting subgroup* of G is denoted $F(G)$. The unique maximal normal E-subgroup of G is denoted $E(G)$. The *generalized Fitting subgroup* of G equals $E(G)\,F(G)$. It is denoted $F^*(G)$. The unique maximal normal K-subgroup of G is denoted $K(G)$. We define $\hat{K}(G) = (\bigcap \{N_G(M) \mid M \in \mathscr{M}(G)\})\,\mathrm{Sol}(G)$.

(22) Suppose A is an Abelian p-group acting on the p'-group G. For

each subgroup X of G, *the smear of X by A* is the subgroup $\langle X^A\rangle \cap C_G(A)$. It is denoted $X ** A$. $\mathscr{L}^A(G) = \{L \in \mathscr{L}(G) | L ** A$ is nonsolvable$\}$. $\mathscr{M}^A(G) = \{M \in \mathscr{M}(G) | M ** A$ is nonsolvable$\}$. $K^A(G) = \langle \mathscr{M}^A(G)\rangle$. Finally, $K_A(G) = C_G(K^A(G)/\mathrm{Sol}(G))$.

(23) We are interested in structures which are like wreathed structures. Suppose G is a group. The expression $G = H\underline{w}(A, N, C)$ means: A is an Abelian subgroup of G, H is a subgroup of G, $G = \langle H, A\rangle$, $\langle H^G\rangle = \times H^G$, $N = N_A(H)$, and $C = C_A(H)$.

(24) Suppose the group G is the direct product of its subgroups $G_1, G_2,..., G_n$. A *diagonal subgroup* of G, with respect to $\{G_1, G_2,..., G_n\}$, is any subgroup X such that $\mathrm{Proj}_{G_i}: X \to G_i$ is an isomorphism.

(25) A *direct factor* of the group G is any subgroup K of G such that $K \times L = G$ for some subgroup L of G. We say G is *indecomposable* if its only direct factors are G and 1. We denote the set of all indecomposable direct factors of G by $\mathrm{Ind}(G)$.

3. Statement of Main Results

THEOREM A. *Suppose p is a prime, A is an Abelian subgroup of a group G, $m(A) \geqslant 3$, and θ is an A-signalizer functor on G which satisfies Hypothesis* C. *Then θ is complete.*

THEOREM B. *Suppose p is a prime, G is a simple p'-group, and at least one of the following conditions apply to G:*

(a) $\mathrm{Out}(G)$ *is prime to p,*

(*b*) *G is a Chevalley or a twisted Chevalley group, or*

(c) *G has an Abelian Sylow 2-subgroup.*

Then it follows that (G,p) satisfies Hypothesis A.

COROLLARY C. *Suppose $p = 2$ or 3, A is an Abelian p-subgroup of the finite group G, $m(A) \geqslant 3$, and θ is an A-signalizer functor on G. Then θ is complete.*

4. Proof of Theorem B and Corollary C

We list the Lie notation used in this section. For greater detail see [2].

DEFINITION. Let K be a finite field. We write $A(K)$ for any of the groups $A_n(K)$, $B_n(K)$, $C_n(K)$, $D_n(K)$, $G_2(K)$, $F_4(K)$, or $E_n(K)$. In this section we shall reserve $G(K)$ to mean $\Lambda(K)$ or some twisted version ${}^{i}\Lambda(K)$ of $\Lambda(K)$. The root system and fundamental root system corresponding to Λ are given respectively by Φ and Π.

Let Z be the integers. Then $\hat{H}$ is the set of automorphisms of $\Lambda(K)$ of the form $h(\chi)$, $\chi \in \mathrm{Hom}(Z\Phi, K)$, defined by $h(\chi)\colon x_r(s) \to x_r(s\chi(r))$ for $r \in \Phi$. The group of field automorphisms of $\Lambda(K)$ is denoted $\mathscr{F}$. Let A_1 be the inner automorphism group of $G(K)$, A_2 the automorphism group induced by $N_{\hat{H}}(G(K))$ on $G(K)$, $A_3 = \mathscr{F}$, and $A_4 =$ the automorphism group generated by the graph automorphism of $G(K)$. By [14, 15, 17], $\mathrm{Aut}(G(K)) = A_1A_2A_3A_4$, $A_2 \cong N_{\hat{H}}(G(K))$, and $A_3 \cong N_{\mathscr{F}}(G(K))$. Hence we shall identify A_2 and A_3 with $N_{\hat{H}}(G(K))$ and $N_{\mathscr{F}}(G(K))$, respectively. Also when convenient we may identify $\mathrm{Aut}({}^{i}\Lambda(K))$ with a subgroup of $\mathrm{Aut}(\Lambda(K))$ and $G(K)$ with A_1.

Let U (resp. V) be the positive (resp. negative) unipotent subgroups of $\Lambda(K)$.

LEMMA 4.1. *Suppose $G(K)$ is a p'-group. Then A_3 contains a Sylow p-subgroup P of* $\mathrm{Aut}(G)$. *Moreover P is cyclic.*

Proof. $\pi(A_1) = \pi(G)$, $\pi(A_2) = \pi(K^{\#}) \subseteq \pi(G)$, and $\pi(A_4) \subseteq \pi(G)$. Hence by Sylow's theorems, A_3 contains a Sylow p-subgroup of $\mathrm{Aut}(G)$. Since A_3 is cyclic, the result follows.

LEMMA 4.2. *Let K have characteristic r. Suppose T is a subgroup of U, such that for all $s \in \Pi$, T contains an element $\prod_{t \in \Phi^+} x_t(b_t)$, for which $b_s \neq 0$. Then U is the unique Sylow r-subgroup of $\Lambda(K)$ which contains T.*

Proof. The proof of [1, Lemma 1.1] is based on these conditions and shows $N(T) \leqslant N(U)$. Since the conditions are inherited by $N_U(T)$, the result follows by induction on $|U\colon T|$.

LEMMA 4.3. *Suppose $G(K)$ is a p'-group and f is an automorphism of $G(K)$ of order p. Let $C = C_{G(K)}(f)$ and $D = C_{\mathrm{Aut}(G(K))}(C)$. Then $D = \langle f \rangle$.*

Proof. Let r be the characteristic of K. By Lemma 4.1 we may suppose f is a field automorphism. Then by Lemma 4.2, U is the unique Sylow r-subgroup of $\Lambda(K)$ containing $U \cap C$. Since $U \cap C$ and $V \cap C$ are conjugate, it follows that V is the unique Sylow r-subgroup of $\Lambda(K)$ containing $V \cap C$. Hence $D \subseteq N(U) \cap N(V) \cap \mathrm{Aut}(G(K)) = A_2A_3A_4$. Since A_2A_3 normalizes each root group it follows that $D \subseteq A_2A_3$. Now straightforward calculations assisted by [2, Theorem 5.3.3(ii), Proposition 13.6.1] yield the result.

LEMMA 4.4. *Suppose $G \cong A_1(q)$, $A_1(3^p)$, ${}^2A_2(q)$, or ${}^2B_2(q)$, where $q = 2^p$. Suppose in addition that G is a p'-group. Then G is near p-solvable.*

Proof. When $G \cong {}^2B_2(q)$, the result is given by [17, Theorem 9]. Otherwise the result follows from [4, Sects. 8.4 and 8.5].

Proof of Theorem B. (a) In this case all the conditions are vacuously satisfied.

(b) Suppose $G(K)$ is a p'-group. We must show $(G(K), p)$ satisfies Hypothesis A. By (a), we may suppose that $p \in \pi(\mathrm{Aut}(G(K))$. By Lemma 4.1, $G(K)$ is outer p-cyclic. Let f be an automorphism of $G(K)$ of order p and let $C = C_{G(K)}(f)$. By Lemma 4.3, any p'-automorphism of $G(K)$ which centralizes C is trivial.

By Lemma 4.4, we may suppose $G \not\cong A_1(q)$, $A_1(3^p)$, ${}^2A_2(q^2)$, or ${}^2B_2(q)$ for $q = 2^p$. By [1], C is a maximal subgroup of $G(K)$. Hence it suffices to show $F^*(C)$ is simple. By [2, Theorems 21.1.2, 14.4.1, comments on p. 175, and the note on p. 268] it suffices to show ${}^2G_2(3)$ and ${}^2F_4(2)$ have trivial center. The argument on [2, p. 173] carries over to the above two situations. This completes (b).

(c) Let G be a p'-simple group with Abelian Sylow 2-subgroup. We must show (G, p) satisfies Hypothesis A. By parts (a), (b), and [10, 20], we may suppose G has an elementary Abelian Sylow 2-subgroup P of order 8, that $C_G(j) \cong Z_2 \times A_1(q)$ where $q = 3^n$ for some odd integer n at least 3, and that G has an automorphism of order p. Such groups have been studied extensively [11, 15, 19, 21]. Let $N = N_G(P)$, A_1 be the group of inner automorphisms of G, and B_2 the group of automorphisms centralizing N. By [20, p. 335], there follows

$$\mathrm{Aut}(N) = \mathrm{Inn}(N) \cong N. \tag{4.1}$$

So by the Frattini argument

$$\mathrm{Aut}(G) = A_1 B_2. \tag{4.2}$$

G does not have a strongly embedded subgroup, and N is transitive on $P^{\#}$, whence $G = \langle N, C_G(j) \rangle$ for any $j \in P^{\#}$. Hence

$$B_2 \text{ acts faithfully on } (C_G(j))' \cong A_1(3^n) \text{ for any } j \in P^{\#}. \tag{4.3}$$

Now suppose f is an automorphism of G of order p. By (4.2), we may suppose $f \in B_2$. Hence by (4.3), G is outer p-cyclic. Let $C = C_G(f)$. Any automorphism k of G which centralizes C must centralize N. Hence $k \in B_2$. By (4.3) and (b), it follows that $k \in \langle f \rangle$.

Let $j \in P^{\#}$. Then $C_N(j)$ normalizes no nontrivial subgroup of odd order of $C_G(j)$. Hence N normalizes no nontrivial subgroup of odd order. Since N is transitive on $P^{\#}$ we obtain

$$\text{Suppose } N \leqslant H \leqslant G. \text{ Then } H = N \text{ or } F^*(H) \text{ is simple.} \tag{4.4}$$

To complete the proof it suffices by (4.4) to show that $N \neq C$. The order of N is 168. Let e be an element of N of order 3, and t an involution of N centralizing e. By [11, 21], e is contained in a unique Sylow 3-subgroup R of G. So t and f normalize R. Now $e \in C_R(t) \leqslant R'$. Hence it suffices to show $C_{R/\Phi(R)}(f) \neq 1$. However, $N_G(R)$ is transitive on $(R/\Phi(R))^{\#}$ whence $\langle e \rangle < C_R(f)$. Hence $N \neq C$. This completes the proof of Theorem B.

Proof of Corollary C (assuming Theorem A). θ satisfies Hypothesis B by [3] if $p = 2$, or by Theorem B part (b) and [6] or [18] if $p = 3$. Theorem A then yields the corollary.

5. Preliminary Lemmas

LEMMA 5.1. *Suppose the Abelian p-group A acts on the p'-group X. Then* $X = \langle C_X(A_0) | A/A_0$ *is cyclic*$\rangle$.

Proof. See [7, Lemma 2.1].

LEMMA 5.2 (Glauberman). *Suppose the π-group A acts on the π'-group K. Suppose K is generated by A-invariant pairwise permuting subgroups* $K_1, K_2, \ldots, K_n$. *Then* $C_K(A) = C_{K_1}(A)\, C_{K_2}(A) \cdots C_{K_n}(A)$.

Proof. See [9, Lemma 2.1].

LEMMA 5.3. *Suppose θ is an A-signalizer functor on a group G,* $P \in И_\theta(A; r)$ *and B is a noncyclic subgroup of A. Then the following statements are equivalent*:

(1) $P \in И^*_\theta(A; r)$
(2) $C_P(b)$ *is an* $S_r(A)$*-subgroup of* $\theta(C_G(b))$ *for all* $b \in B^{\#}$.

Proof. See [7, Lemma 3.2].

LEMMA 5.4. *Let G be a group and* $\bar{G} = G/\mathrm{Sol}(G)$. *Then the functors* F^*, K, E, *and* Sol *satisfy*:

(a) $\mathrm{Sol}(\bar{G}) = \bar{1}$,
(b) $C_G(F^*(G)) \subseteq F^*(G)$, *and*
(c) $K(G) = K(\bar{G}) = E(\bar{G}) = F^*(\bar{G})$ *is semi-simple.*

Proof. See [13, Lemma 2.4].

LEMMA 5.5. *Suppose the Abelian group A acts on the group*

$G = G_1 \times G_2 \times \cdots \times G_n$. *Suppose* A *acts on* $\{G_1, G_2, ..., G_n\}$ *via the induced action of* A *on subgroups. Then*

$$\mathrm{Proj}_{G_i}(C_G(A)) = C_{G_i}(N_A(G_i))$$

where projections are taken with respect to $\{G_1, G_2, ..., G_n\}$.

Proof. See [13, Lemma 2.9].

LEMMA 5.6. *Suppose the group* A *acts on the group* G, G *is a direct product of a set* S *of subgroups of* G *on which* A *acts semi-regularly. Suppose* W *is a subgroup of* $C_G(A)$, *and* $T \leqslant K \in S$ *satisfies* $T ** A \leqslant W$. *Then* $T \leqslant \mathrm{Proj}_K(W)$ *when projections are taken with respect to* S.

Proof. Let $t \in T$ and $y = \prod_{a \in A} t^a$. The elements of t^A commute pairwise; so y is well defined and centralized by A. So $t = \mathrm{Proj}_K(y) \in \mathrm{Proj}_K(W)$ as required.

LEMMA 5.7. *Suppose the group* G *acts faithfully on the set* Ω, G *has a Sylow p-subgroup* S *acting transitively on* Ω, *and* $O^p(G) = O_{p'}(G)$. *Then* $G = S$.

Proof. [See 13, Lemma 2.6].

LEMMA 5.8. *Suppose* M *is a group of operators of the semi-simple group* K. *Then* $[K, M] =$ *the product of components of* K *not centralized by* M.

Proof. Suppose that K has a component L centralized by M. Then $[K, M] = [C_K(L) \times L, M] \leqslant C_K(L) < K$.

Now let $K_1 = [K, M]$ and $K_2 = C_K(K_1)$. Then both K_1 and K_2 are normal in KM, and $K = K_1 \times K_2$. Hence $[K_2, M] \leqslant K_1 \cap K_2 = 1$. So $K_1 = [K, M] = [K_1 \times K_2, M] = [K_1, M]$. The previous paragraph implies that K_1 has no component centralized by M.

LEMMA 5.9. *Suppose* G *is a group and* $K(G) \leqslant X \leqslant G$. *Then* $K(G) = K(X)$.

Proof. See [13, Lemma 2.15].

LEMMA 5.10. *Suppose the group* G *is semisimple. Let* $\mathscr{L} = \mathscr{L}(G)$. *Suppose* H *is a subgroup of* G *such that* $\mathrm{Proj}_L(H) = L$ *for all* $L \in \mathscr{L}$. *For each nonempty subset* T *of* $\mathscr{L}$ *let* $G_T = \langle T \rangle$ *and* $H_T = H \cap G_T$. *Then*

(a) $\mathscr{L}$ *is the disjoint union of subsets* $\mathscr{L}_i$, $1 \leqslant i \leqslant k$,
(b) H *is the direct product of* $H_{\mathscr{L}_i}$, $1 \leqslant i \leqslant k$, *and*
(c) $H_{\mathscr{L}_i}$ *is a diagonal subgroup of* $G_{\mathscr{L}_i}$.

Proof. Let T be a nonempty subset of $\mathscr{L}$ of least possible order subject to $G_T \cap H \neq 1$. If $T = \mathscr{L}$, then H is already a diagonal subgroup of G and we are done. Suppose then that T is a proper subset of $\mathscr{L}$. Let $H^* = \mathrm{Proj}_{G_T}(H)$. Now $H_T \geqslant [H, H_T] = [H^*, H_T]$. So $H_T \lhd H^*$. Let $L \in T$. Then $1 \neq \mathrm{Proj}_L(H_T) \lhd \mathrm{Proj}_L(H) = L$. Hence H_T is a diagonal of G_T. So $H_T \leqslant H^* \leqslant N_{G_T}(H_T) = H_T$. Hence $H = H_T \times C_H(G_T)$. The result now follows by induction on $|\mathscr{L}|$.

LEMMA 5.11. *Suppose the elementary Abelian p-group A acts on the p'-group G, $m(A) \geqslant 2$, and each member of $\mathscr{L}(G/\mathrm{Sol}(G))$ is outer p-cyclic. Let $L = \langle K(C_G(a)) \mid a \in A^{\#} \rangle$. Then $K(L) = K(G)$.*

Proof. By Lemma 5.9, it suffices to show that $L \geqslant K(G)$. We may now make the following sequence of reductions; first $G = K(G)$, then $\mathrm{Sol}(G) = 1$, then A is of order p^2, then $C_A(G) = 1$, and finally A acts transitively on $\mathscr{L}(G)$. By the outer p-cyclic property and Lemma 5.1, we may suppose $B = A \cap \hat{K}(GA) \cong Z_p$. Let $K \in \mathscr{L}(G)$. Then $1 \neq K(C_K(B)) = K(C_G(B)) \cap K \leqslant L$. So $L \cap K \neq 1$. Let $B \neq E \in \mathscr{E}_1(A)$. Then $C_G(E) \leqslant L$. By Lemma 5.10, $L = G$.

LEMMA 5.12. *Suppose H is a group of operators on the group $G = \times\, \Omega$. Suppose the action of H on G induces a semiregular action of H on Ω. Then $Z(C_G(H)) = C_{Z(G)}(H)$.*

Proof. Let $C = C_G(H)$. Take projections in G with respect to Ω. Since H acts semiregularly on Ω, it follows that $\mathrm{Proj}_K(C) = K$ for all $K \in \Omega$. Hence

$$C_G(C) = \times\, \{C_K(C) \mid K \in \Omega\} = \times\, \{C_K(\mathrm{Proj}_K(C)) \mid K \in \Omega\}$$
$$\times\, \{C_K(K) \mid K \in \Omega\} = Z(G).$$

Hence $Z(C) = C \cap Z(G)$ as required.

LEMMA 5.13. *Suppose A is a p-group and G is near A-solvable. Then G is near p-solvable.*

Proof. A simple section of G is isomorphic to a simple section of some chief factor of GA in G. Hence we may assume G is nonsolvable and is minimal normal in GA. Then G is near $(A \cap \hat{K}(GA))$-solvable. Hence we may suppose that G is simple. Since $C_G(A)$ is a localized subgroup of G and $A/C_A(G) \cong Z_p$, it follows by Hypothesis A.3. that G is near p-solvable.

LEMMA 5.14. *Suppose A is an elementary Abelian p-group of operators on the group G. Suppose (G, p) satisfies Hypothesis* B. *Let $D = C_G(A)$. Suppose X is a DA-invariant subgroup of G, and* $\mathrm{Sol}(G) = 1$. *Let $J \in \mathscr{L}^A(G)$. Then $F^*(J ** A) \leqslant X$ or J is centralized by X.*

Proof. Let $L = \langle J^A \rangle$, $X_1 = X \cap K(G)$, and $X_2 = \mathrm{Proj}_L(X_1)$, where projections are being taken in $K(G)$. Then X_2 is $C_L(A)$-invariant. First suppose $X_2 \cap (J ** A) = 1$. Thus $C_{X_2}(A) = 1$. By [12], X_2 is solvable. By Lemma 5.5 and Hypothesis A, $\mathrm{Proj}_J(J ** A) = C_J(N_A(J))$ is not localized. Hence, $\mathrm{Proj}_J(X_2) = 1$. Hence $X_2 = 1$. Hence $[J ** A, X] \leqslant X_1 \leqslant C_G(L)$. By the 3-subgroup lemma, $[F^*(J** A), X] = [F^*(J ** A), F^*(J ** A), X] = 1$. Since $L =$ the product of components of G not centralized by $F^*(J ** A)$, it follows that L admits X. In particular $[J ** A, X] \leqslant X_1 \cap L = 1$. By Lemma 5.7, X normalizes J. Hence by Lemma 5.5, X centralizes $\mathrm{Proj}_J(J ** A) = C_J(N_A(J))$. By Hypothesis A.3.3, X centralizes J.

Next suppose $X_2 \cap (J ** A) \neq 1$. By Lemma 5.5, $J ** A \cong C_J(N_A(J))$. Hence by Hypothesis A.3.2, $F^*(J ** A)$ is simple and is the unique minimal normal subgroup of $J ** A$. In particular, $F^*(J ** A) \leqslant X_2$. So

$$F^*(J ** A) = [F^*(J ** A), F^*(J ** A)] \leqslant [J ** A, X_2] = [J ** A, X_1] \leqslant [D, X] \leqslant X.$$

This completes the proof of the lemma.

THEOREM 5.15. *Suppose A is an elementary Abelian p-group of operators of the group G. Suppose (G, p) satisfies Hypothesis* B. *Let $D = C_G(A)$. Suppose X is a DA-invariant subgroup of G. Then all of the following hold.*

(a) *$X \leqslant K_A(G)$ if X is near A-solvable.*

(b) *Suppose $J \in \mathscr{M}^A(G)$ and $X = \hat{K}(X)$. Then J admits X.*

(c) *Suppose $X = K(X)$. Then X normalizes $K^A(G)$ and induces inner automorphisms on $K^A(G)/\mathrm{Sol}(G)$.*

(d) *Suppose $X = K(X)$. Then $X \leqslant K^A(G) K_A(G)$.*

(e) *Suppose $K_A(G) = 1$ and $X = \hat{K}(X)$. Then $X \leqslant \hat{K}(G)$.*

(f) *Suppose $K_A(G) = 1$ and $B \leqslant A$. Then $K(C_G(B)) = K(C_{K(G)}(B))$. Moreover $\mathscr{L}(C_G(B)) = \{F^*(J ** B) \mid J \in \mathscr{L}(G)\}$.*

(g) *Suppose $K_A(G) = 1$ and $B \leqslant A$. Then $\hat{K}(C_G(B)) = C_{\hat{K}(G)}(B)$.*

Proof. (a) Without loss of generality assume $\mathrm{Sol}(G) = 1$. If $K^A(G) = 1$, there is nothing to prove. Suppose $K^A(G) \neq 1$. Let $J \in \mathscr{L}^A(G)$. Then $F^*(J** A) \not\leqslant X$. Hence by Lemma 5.14, $[J, X] = 1$. So $X \leqslant K_A(G)$.

(b) We may suppose $\mathrm{Sol}(G) = 1$. Let $J \in \mathscr{L}^A(G)$. If $F^*(J ** A) \not\leqslant X$, then $[X, J] = 1$ by Lemma 5.14. Suppose then $F^*(J ** A) \leqslant X$. Let $L = \langle J^A \rangle$ and $X_1 = K(X \cap L)$. By Lemma 5.14, $F^*(J ** A) \leqslant X_1$. By (a), $\mathrm{Sol}(X_1) \leqslant K_A(L) = 1$. Also by (a), $\mathrm{Sol}(X) \leqslant K_A(G) \leqslant C_G(L)$. Hence $X_1 = (X_1 \times \mathrm{Sol}(X))^\infty$ admits X. Since L is the product of components of G

not centralized by X_1, it follows that L admits X. By Lemma 5.7, J admits X.

(c) By (b), X normalizes $K^A(G)$. Hence we may suppose $G = K^A(G)X$ and $\mathrm{Sol}(G) = 1$. By (a), $\mathrm{Sol}(X) \lhd G$. Hence $\mathrm{Sol}(X) = 1$. Let $X_1 = X \cap K(G)$, and $X_2 = C_X(X_1)$. Then $X = X_1 \times X_2$. Since X and X_1 are DA-invariant, it follows that X_2 is DA-invariant. By Lemma 5.14, $X_2 \leqslant K_A(G)$.

(d) This is equivalent to (c).

(e) This is immediate from (b).

(f) By (c), $K(C_G(B)) = K(C_{K(G)}(B))$. Certainly, $K_B(G) = 1$.

So to complete (f) we may suppose by induction that $G = K(G)$, that $A = B$, and that A is transitive on $\mathscr{L}(G)$. Let $J \in \mathscr{L}(G)$. Then $C_G(A) = J ** A$. By Lemma 5.5, $C_G(A) \cong C_J(N_A(J))$. So we may suppose that G is simple. The conclusion now follows from Hypothesis A.

(g) By (b), $\hat{K}(C_G(B) \leqslant \hat{K}(G)$. By (f), $C_{\hat{K}(G)}(B) = \hat{K}(C_{\hat{K}(G)}(B))$. This proves (g) and the theorem.

THEOREM 5.16. *Suppose A is an elementary Abelian p-group of operators of the group G. Suppose (G,p) satisfies Hypothesis* B. *Let $D = C_G(A)$. Let $NS(G)$ be the set of all subgroups of G which are DA-invariant and near A-solvable. Let $G_{ns} = \langle NS(G)\rangle$. Then*

(a) *$G_{ns} \in NS(G)$, and*

(b) *G_{ns} admits all DA-invariant K-subgroups of G.*

Proof. (a) DA permutes $NS(G_{ns})$ and therefore normalizes $(G_{ns})_{ns}$. Hence we may suppose $G = G_{ns}$. We may also suppose G has no near A-solvable normal subgroups. Theorem 5.15(a) implies that $G = K_A(G)$. Hence $K(G)$, being near A-solvable, is trivial. Hence $G = 1$.

(b) By Theorem 5.15(a, d) we may suppose that $G = K^A(G)K_A(G)$. We may also suppose that G has no nontrivial near A-solvable normal subgroup. Hence $K(K_A(G)) = 1$. Hence $K_A(G) = 1$. So $G_{ns} \leqslant K_A(G) = 1$, proving (b).

THEOREM 5.17. *Suppose H is a group,* $\mathrm{Sol}(H) = 1$, *and H has a subgroup $B \cong Z_p \times Z_p$ acting regularly on $\mathscr{L}(H)$. Suppose θ is a B-signalizer functor on H which satisfies*:

$$C_{K(H)}(b) \leqslant \theta(C_H(b)) \qquad \textit{for all } b \in B^{\#}$$

and

$$\theta(C_H(b)) \cong p(\theta(C_H(B))) \qquad \textit{for all } b \in B^{\#}.$$

Let $\tilde{N} = C_H(C_{K(H)}(N))$ *for each* $N \lhd K(H)$. *Then* θ *is complete. Moreover,*

$$\theta(HB) = \times\ \{\theta(HB) \cap \tilde{J} \mid J \in \mathscr{L}(H)\}$$
$$= \times\ \{\mathrm{Proj}_{\tilde{J}}(\theta(C_H(B))) \mid J \in \mathscr{L}(H)\},$$

where projections in $\langle \tilde{J} \mid J \in \mathscr{L}(H)\rangle$ *are taken with respect to* $\{\tilde{J} \mid J \in \mathscr{L}(H)\}$.

Proof. By Lemma 5.7, $\langle И_\theta(B)\rangle \leqslant \hat{K}(H)$. Hence we may suppose that $H \cong \mathrm{Aut}(J) \wr B$ for any $J \in \mathscr{L}(H)$. Let $H_0 = \hat{K}(H)$. Then $H = H_0 B$, $H_0 = \{\tilde{J} \mid J \in \mathscr{L}(H)\}$, and B acts regularly on $\{\tilde{J} \mid J \in \mathscr{L}(H)\}$. In particular, we can take projections in H_0 with respect to $\{\tilde{J} \mid J \in \mathscr{L}(H)\}$.

Let $W = \theta(C_H(B))$ and $W_1 = \langle \mathrm{Proj}_{\tilde{J}}(W) \mid J \in \mathscr{L}(H)\rangle$. Then $C_{W_1}(b) \cong pW \cong \theta(C_H(b))$ for all $b \in B^{\#}$. So it suffices to show $\theta(C_H(b)) \leqslant W_1$ for all $b \in B^{\#}$.

Fix $E \in \mathscr{E}_1(B)$. Let $S = \{\langle J^E\rangle \mid J \in \mathscr{L}(H)\}$, $T = \{\tilde{L} \mid L \in S\}$, and $V = \theta(C_G(E))$. By hypothesis, $V = V_1 \times V_2 \times \cdots \times V_p$, where each $V_i \cong W$ and $C_{K(H)}(E) \leqslant V$. Thus

$$\{V_i \cap K(G) \mid 1 \leqslant i \leqslant p\} = \{C_L(E) \mid L \in S\}.$$

Suppose $C_L(E) = V_i \cap K(G)$. Then

$$V_i = \bigcap \{C_V(V_j \cap K(G)) \mid j \neq i\} = \bigcap \{C_V(C_M(E)) \mid L \neq M \in S\}$$
$$= \bigcap \{C_V(M) \mid L \neq M \in S\}$$
$$\leqslant \bigcap \{C_G(M) \mid L \neq M \in S\} = \tilde{L}.$$

So $V = \times\ \{V \cap \tilde{L} \mid L \in T\}$. Let $E \times F = B$. Then for $\tilde{L} \in T$, $(V \cap \tilde{L}) ** F = C_V(F) = W$. Since F acts regularly on T, Lemma 5.6 yields that $V \cap \tilde{L} \leqslant \mathrm{Proj}_L(W) \leqslant W_1$ for all $\tilde{L} \in T$. Hence $V \leqslant W_1$. Since $E \in \mathscr{E}_1(B)$ was arbitrarily chosen, the theorem is complete.

6. The Minimal Counterexample

Henceforth we shall assume that Theorem A is false and that G is a counterexample of least possible order. Subject to this restriction we assume that $|\theta|$ is minimal. When convenient we shall write H_B for $\theta(C_G(B))$ for each nonidentity subgroup B of A, and H_a for $H_{\langle a\rangle}$ for each $a \in A^{\#}$. We shall also write D for H_A.

Following Theorem 5.16, for each $X \in \tilde{И}_\theta(A)$, we define $NS(X)$ to be the set of DA-invariant near A-solvable subgroups of X, and $X_{ns} = \langle NS(X) \rangle$. Now define $\theta_{ns}(C_G(a)) = (\theta(C_G(a)))_{ns}$ for each $a \in A^{\#}$.

The goal of this section is to obtain sufficient structure of θ to determine the structure of G. For the convenience of the reader, we capsule this information in our first theorem.

THEOREM 6.1. *The following hold.*

(a) *A is elementary Abelian of order p^3.*

(b) *One of the following sets of conditions hold. Either* (b1) *or* (b2) *holds.*

(b1) *The following three conditions hold.*
(b1.1) *D is simple*
(b1.2) *Let $F \in \mathscr{E}_1(A)$. Then $H_F A = K\underline{w}(A, F, F)$ for some $K \cong D$.*
(b1.3) *$H_a \in \mathit{И}^*_\theta(A)$ for all $a \in A^{\#}$.*
(b2) *The following five conditions hold.*
(b2.1) *There is a distinguished $E \in \mathscr{E}_1(A)$ and a simple group K.*
(b2.2) *$F^*(D)$ is simple.*
(b2.3) *$H_E A = L\underline{w}(A, E, E)$ for some $L \cong D$.*
(b2.4) *Let $E \neq F \in \mathscr{E}_1(A)$. Then $H_F = L\underline{w}(A, EF, F)$* for some *$L \cong K$.*
(b2.5) *$H_a \in \mathit{И}^*_\theta(A)$ if $a \in A - E$.*

(c) *$G = \langle \mathit{И}_\theta(A) \rangle A$.*

(d) *$Z(\langle \mathit{И}_\theta(A) \rangle) = 1$.*

LEMMA 6.2. (a) *A is elementary abelian of order p^3.*

(b) *There is an $a \in A^{\#}$ for which $\theta(C_G(a))$ is not near A-solvable.*

(c) *θ is locally complete.*

(d) *$G = A\langle \mathit{И}_\theta(A) \rangle$.*

Proof. (a), (d). These follow from the conditions of the counterexample.

(c) See [7, Lemma 5.1].

(b) This follows from Lemma 5.13 and [13, Main Theorem].

LEMMA 6.3. *Let $X \in \mathit{И}_\theta(A)$. Then*

(a) *There is an $a \in A^{\#}$ such that $K(H_a) \nleqslant X$.*

(b) *There is a $B \in \mathscr{E}_2(A)$ such that $K(H_B) \nleqslant X$.*

Proof. Let $a \in A^{\#}$. By Lemma 5.11,

$$K(H_a) = K(\langle K(H_B) \mid a \in B \in \mathscr{E}_2(A) \rangle) \leqslant \langle K(H_F) \mid F \in \mathscr{E}_2(A) \rangle.$$

Hence it suffices to show that (a) is true.

Suppose that (a) is false. Choose $X \in И_\theta(A)$ such that $K(H_a) \leqslant X$ for all $a \in A^\#$. Let $B \in \mathscr{E}_2(A)$. By Lemmas 5.9 and 5.11,

$$K(X) = K(\langle K(C_X(b)) \mid b \in B^\# \rangle) = K(\langle K(H_b) \mid b \in B^\# \rangle)$$

admits H_B. This is contrary to Theorem 6.2(c), which proves the lemma.

LEMMA 6.4. $Z(\langle И_\theta(A) \rangle) = 1$.

Proof. See [13, Theorem 5.1(d)].

THEOREM 6.5. (a) θ_{ns} *is a complete* A*-signalizer functor on* G.

(b) $\theta_{ns}(C_G(a))$ *admits any* DA*-invariant* K*-subgroup of* $\theta(C_G(a))$.

Proof. (a) This follows from Theorem 5.16(a) and Lemma 6.2(b).

(b) This follows from Theorem 5.16(b).

THEOREM 6.6. $\theta_{ns}(G) = 1$. *In particular,* $K_A(X) = 1$ *whenever* $X \in \tilde{И}_\theta(A)$.

Proof. Let $W = \theta_{ns}(G)$. Choose a $B \in \mathscr{E}_2(A)$. By Lemma 5.1 and Theorem 6.5,

$$K(H_B) \leqslant N_G(\langle \theta_{ns}(C_G(b)) \mid b \in B^\# \rangle) = N_G(\langle C_W(b) \mid b \in B^\# \rangle) = N_G(W).$$

Now Lemmas 6.2(c) and 6.3(b) imply that $W = 1$.

Suppose $X \in \tilde{И}_\theta(A)$. Then $K_A(X) \cap K(X) \leqslant \theta_{ns}(G) = 1$. So $\mathrm{Sol}(X) = 1$ and $K(X) = K^A(X)$. Hence $K_A(X) = C_X(K(G)) = 1$, as required.

LEMMA 6.7. $\hat{K} \circ \theta = \theta$.

Proof. Theorem 5.15(g) and Theorem 6.6 imply that $\hat{K} \circ \theta$ is an A-signalizer functor on G. Lemma 6.3 implies that $\theta = \hat{K} \circ \theta$ as required.

LEMMA 6.8. $F^*(D)$ *is simple.*

Proof. By Lemma 6.2(b), $D \neq 1$. So Theorem 6.6 implies that $\mathscr{L}(D)$ is nonempty. Let $J \in \mathscr{L}(D)$. Define $\theta_J(C_G(a)) = \theta(C_G(a)) \cap C_G(J)$. Clearly θ_J is an A-signalizer functor of order less than θ. Hence θ_J is complete. Let $W = \theta_J(G)$. Suppose

$$\text{Whenever } B \in \mathscr{E}_2(A),\ L \in \mathscr{L}(H_B), \text{ and } L \nleqslant W, \text{ it follows that } [W, L] = 1. \tag{6.1}$$

Then by Lemmas 6.2(c), 6.3(b), and Theorem 6.6 it follows that $W = 1$. So $F^*(D)$ is simple.

We shall prove (6.1). Let $B \in \mathscr{E}_2(A)$ and $L \in \mathscr{L}(H_B)$. Suppose $[W, L] \neq 1$. Then by Lemma 5.1, there is a $b \in B^{\#}$ for which $[C_W(b), L] \neq 1$. Let $H = H_b$ and $H_J = \theta_J(C_G(b))$. Thus $[H_J, L] \neq 1$. By Theorem 5.15(f), $L = F^*(M ** B)$ for some $M \in \mathscr{L}(H)$. Since H_J is B-invariant and $L \leqslant \langle M^B \rangle$, it follows that $[H_J, M] \neq 1$. Since H_J is DA-invariant, Lemma 5.14 implies that $L = F^*(M ** A) \leqslant H_J \leqslant W$. This proves (6.1) and completes the lemma.

DEFINITIONS. For each nonidentity subgroup B of A define $B_C = C_A(K(H_B))$, and $B_N = A \cap \hat{K}(H_B A)$. Let $\mathscr{S}_i = \{F \in \mathscr{E}_1(A) \| \mathscr{L}(H_F)| = p^i\}$.

LEMMA 6.9. *Let B be a nonidentity subgroup of A. Let $E, F \in \mathscr{E}_1(A)$. Then all of the following hold.*

(a) $B_C = C_A(H_B) = C_A(L)$ *for any* $L \in \mathscr{L}(H_B)$.
(b) $B_N = N_A(L)$ *for any* $L \in \mathscr{L}(H_B)$.
(c) $|\mathscr{L}(H_B)| = |A/B_N|$.
(d) A/B_N *acts regularly on* $\mathscr{L}(H_B)$.
(e) *Suppose* $F \nleqslant E_N$ *and* $E \nleqslant F_N$. *Then* $|\mathscr{L}(H_E)| = |\mathscr{L}(H_F)|$.
(f) *Suppose* $F \nleqslant E_N$ *and* $E \leqslant F_N$. *Then* $|\mathscr{L}(H_E)| = p\,|\mathscr{L}(H_F)|$.
(g) *Suppose* $F \leqslant E_N$ *and* $E \leqslant F_N$. *Then* $|\mathscr{L}(H_E)| = |\mathscr{L}(H_F)|$.
(h) B_N/B_C *is cyclic.*

Proof. (a), (b), (c), (d). Clearly $B_C = C_A(H_B)$. By Lemma 6.8, A acts transitively on $\mathscr{L}(H_B)$. Let V be any subgroup of A. Since A is Abelian, the members of $\mathscr{L}(H_B)$ centralized by V is a union of A orbits. Similarly, the members of $\mathscr{L}(H_B)$ normalized by V is a union of A orbits. Hence (a), (b), (c), (d) easily follow.

(h) This follows from (a), (b) and the outer p-cyclic property of members of $\mathscr{L}(H_B)$.

(e), (f), (g). Let $k = |\mathscr{L}(H_{EF})|$, $rk = |\mathscr{L}(H_E)|$, and $sk = |\mathscr{L}(H_F)|$. By Theorem 6.6 and Theorem 5.15(g), $s = 1$ if $E \leqslant F_N$, and $s = |E| = p$ if $E \nleqslant F_N$. The symmetric statements for r obtained by interchanging E and F yield (e), (f), and (g).

LEMMA 6.10. $\mathscr{S}_2 \neq \varnothing$.

Proof. Suppose $\mathscr{S}_2 = \varnothing$. Then $\mathscr{E}_1(A) = \mathscr{S}_0 \cup \mathscr{S}_1$. Suppose in addition that $\mathscr{S}_1 = \varnothing$. Choose $a \in A^{\#}$ with H_a of maximal possible order. Let $B = \langle a \rangle_C$. By Lemma 6.9(a), $H_a \leqslant H_b$ for all $b \in B^{\#}$. Hence $H_a = H_b$ for all $b \in B^{\#}$. By Lemma 6.9(h), $m(B) \geqslant 2$. Hence $\langle И_\theta(A) \rangle = H_a$. This is false; so $\mathscr{S}_1 \neq \varnothing$.

Let $F \in \mathscr{S}_1$ and $B = F_N$. By Lemma 6.9(d), $m(B) = 2$. Let $F \neq E \in \mathscr{E}_1(B)$. Since $\mathscr{S}_2 = \varnothing$, Lemma 6.9(f), implies that $F \leqslant E_N$. By Lemma 6.9(g),

$E \in \mathscr{S}_1$. Hence $\mathscr{E}_1(B) \subseteq \mathscr{S}_1$, and $B = E_N$ for all $E \in \mathscr{E}_1(B)$. Let $L \in \mathscr{S}_1$ and $E \in \mathscr{E}_1(L_N \cap B)$. Then $L_N = E_N = B$. Hence $\mathscr{E}_1(B) = \mathscr{S}_1$.

Next choose $t \in A - B$ subject to H_t having maximal possible order. Let $R = \langle t \rangle_C$. By Lemma 6.9(h), A/R is cyclic. Choose $t \in T \in \mathscr{E}_2(R)$. Let $E = T \cap B$. By Lemma 6.9(a), $H_r = H_t$ for all $r \in T - E$, and $H_t \leqslant H_E$. Hence $\langle И_\theta(A) \rangle = \langle H_r | r \in T^\# \rangle = H_E$, a contradiction.

LEMMA 6.11. *One of the following hold.*

(a) $\mathscr{S}_2 = \mathscr{E}_1(A)$, *or*

(b) $|\mathscr{S}_2| = 1$, $\mathscr{S}_1 \cup \mathscr{S}_2 = \mathscr{E}_1(A)$, *and* $F_N = F\langle \mathscr{S}_2 \rangle$ *for all* $F \in \mathscr{E}_1(A)$.

Proof. Since A has order p^3, and $\mathscr{S}_2 \neq \varnothing$ by Lemma 6.10, it follows from Lemma 6.9(d, e, f) that $\mathscr{S}_1 \cup \mathscr{S}_2 = \mathscr{E}_1(A)$. We may suppose $\mathscr{S}_2 \neq \mathscr{E}_1(A)$. Let $B = \langle \mathscr{S}_2 \rangle$. We must show that B is cyclic. Choose $E \in \mathscr{S}_1$ with $E \nleqslant B$ if possible. For each $F \in \mathscr{S}_2, F_N = F$; so by Lemma 6.9(e), $F \leqslant E_N$. Hence $BE \leqslant E_N \in \mathscr{E}_2(A)$. In particular, $B < A$; so $E \nleqslant B$ and B is cyclic.

LEMMA 6.12. *Suppose X is a subgroup of G generated by some elements of $И_\theta(A)$. Then either*

(a) *X contains every element of $И_\theta(A)$ and $X \notin И_\theta(A)$, or*

(b) *$X \in И_\theta(A)$ and for any $B \in \mathscr{E}_2(A)$ there is an $a \in B^\#$ such that $H_a \nleqslant X$.*

Proof. This is an easy variation of [5, Lemma 5.4].

THEOREM 6.13. *Suppose $\mathscr{S}_1 = \varnothing$. Then Theorem* 6.1 *holds.*

Proof. By Lemma 6.2(a, d), and Lemma 6.4, it suffices to show conclusion (b.1) holds.

Let $E, F \in \mathscr{E}_1(A)$ be distinct. $E_N = E$; so F acts regularly on $\mathscr{L}(H_E)$. Hence $K(H_E) \cap C(F) = K(K(H_E) \cap C(F))$. Hence by Theorem 5.15(f), $K(H_E) \cap C(F) = K(H_E \cap C(F)) = K(H_{EF})$. By symmetry, $K(H_E) \cap C(F) = K(H_{EF}) = C(E) \cap K(H_F)$. Hence $K \circ \theta$ is an A-signalizer functor on G. By Lemma 6.3(a), $K(H_a) = H_a$ for all $a \in A^\#$. Let $a \in A^\#$ and $L \in \mathscr{L}(H_a)$. Since $A/\langle a \rangle$ acts regularly on $\mathscr{L}(H_a)$, it follows that $D = L ** A \cong L$. Hence (b1.1) and (b1.2) hold.

Suppose $H_a < X \in И_\theta(A)$. By Lemma 5.1, there is a $b \in A^\#$ such that $C_X(b) \neq C_{H_a}(b) = H_{\langle a,b \rangle}$. By Lemma 5.10, $H_{\langle a,b \rangle}$ is a maximal A-invariant subgroup of H_b. Hence $H_b = C_X(b)$. Choose $B \in \mathscr{E}_2(A)$ with $b \in B$ but $a \notin B$. Then $H_d = \langle H_d \cap H_b, H_d \cap H_a \rangle \leqslant X$ for any $d \in B^\#$. This is false. Hence conclusion (b1.3) holds and the theorem is complete.

THEOREM 6.14. *Suppose* $\mathscr{S}_1 \neq \varnothing$. *Then* $\mathscr{S}_2 = \{E\}$ *for a unique* $E \in \mathscr{E}_1(A)$. *Moreover the following conditions hold*:

(a) *For each* $a \in A - E$,

$$H_a A = X\underline{w}(A, E\langle a\rangle, \langle a\rangle)$$

for some simple group X whose isomorphic class is independent of a.

(b) $H_E A = Y\underline{w}(A, E, E)$, $Y \cong D$, *and* $F^*(D)$ *is simple.*

Proof. The first statement holds by Lemma 6.11. For each complement B of E in A define

$$\begin{aligned} \theta_B^*(C_G(a)) &= K(H_a) && \text{if } a \in A - E \\ &= \langle K(H_b) \cap H_E \mid b \in B^\# \rangle && \text{if } a \in E^\#. \end{aligned}$$

The gist of the proof is to show θ_B^* is an A-signalizer functor on G with additional suitable properties.

Again for B a complement of E in A, define $\theta_B(C_G(b)) = K(H_b)$ for $b \in B^\#$. By Lemma 6.11(b), $\langle e\rangle$ acts regularly on $\mathscr{L}(H_f)$ whenever $\langle e, f\rangle = B$. By Lemma 5.15(f) and Theorem 6.6, $K(H_e) \cap C_G(f) = K(C_G(\langle e, f\rangle))$. Hence θ_B is a B-signalizer functor on G. Now define $\tilde{\theta}_B(C_{BH_E}(b)) = \theta_B(C_G(b)) \cap H_E$. Then $\tilde{\theta}_B$ is clearly a B-signalizer functor on BH_E. Since $E_N = E$, it follows that B acts regularly on $\mathscr{L}(H_E)$. Hence by Theorem 5.15(f) and Theorem 6.6, $C_{K(H_E)}(b) \leqslant \tilde{\theta}_B(C_{H_E B}(b))$ for all $b \in B^\#$. Also

$$\begin{aligned} \tilde{\theta}_B(C_{BH_E}(b)) &= C_{K(H_b)}(E) = \times \{C_L(E) \mid L \in \mathscr{L}(H_b)\} \\ &\cong p(((K(H_b) \cap C_G(B)) \cap C(E)) \\ &= p(K(H_B) \cap C(E)) \\ &= p\tilde{\theta}_B(C_{BH_E}(B)). \end{aligned}$$

We have established all the conditions of Theorem 5.17 with (H, θ, B) replaced by $(BH_E, \tilde{\theta}_B, B)$. For each $L \in \mathscr{L}(H_E)$, let $\tilde{L} = C_{H_E}(C_{K(H_E)}(L))$. By Theorem 5.17, we obtain

$$\tilde{\theta}_B(BH_E) = \times \{\mathrm{Proj}_{\tilde{L}}(K(H_B) \cap H_E) \mid L \in \mathscr{L}(H_E)\} \tag{6.2}$$

and

$$\theta_B^*(C_G(E)) \cap C_G(b) = \theta_B^*(C_G(b)) \cap C_G(E) \qquad \text{for } b \in B^\#. \tag{6.3}$$

The functor θ_B^* is independent of the complement B of E in A on the subgroups $C_G(b)$ for $b \in A - E$. We next want to show that it is also independent on $C_G(E)$. Suppose then that T is a complement for E in A distinct from B. Let $F = T \cap B$. Then $F \in \mathscr{E}_1(A)$. By (6.2),

$$\begin{aligned} \tilde{\theta}_B(BH_E) &= \times \{\operatorname{Proj}_{\tilde{L}}(K(H_B) \cap H_E) \mid L \in \mathscr{L}(H_E)\} \\ &= \times \{\operatorname{Proj}_{\tilde{L}}(K(H_F) \cap D) \mid L \in \mathscr{L}(H_E)\} \\ &= \times \{\operatorname{Proj}_{\tilde{L}}(K(H_T) \cap H_E) \mid L \in \mathscr{L}(H_E)\} \\ &= \tilde{\theta}_T(TH_E). \end{aligned}$$

Hence θ_B^* is independent of the complement B of E in A. Therefore by (6.3) there follows

$$\theta_B^*(C_G(E)) \cap C_G(a) = C_G(E) \cap \theta_B^*(C_G(a)) \quad \text{for all } a \in A - E. \tag{6.4}$$

Next we show $\theta^* = \theta_B^*$ is balanced. Let $a, b \in A^\#$ and $T = \langle a, b \rangle$. We have already shown $\theta^*(C_G(a)) \cap C_G(b) \leqslant \theta^*(C_G(b))$ if $E \nleqslant T$. Certainly $\theta^*(C_G(a)) \cap C_G(b) \leqslant \theta^*(C_G(b))$ if T is cyclic. Suppose then $E < T$ and $a, b \in A - E$. By (6.4)

$$\begin{aligned} \theta^*(C_G(a)) \cap C_G(b) &= \theta^*(C_G(a)) \cap C_G(T) \\ &= \theta^*(C_G(E)) \cap C_G(T) \\ &= C_G(T) \cap \theta^*(C_G(b)) \leqslant \theta^*(C_G(b)). \end{aligned}$$

Hence θ^* is an A-signalizer functor on G. By Lemma 6.3, $\theta^* = \theta$.

Clearly A is transitive on $\{\tilde{L} \mid L \in \mathscr{L}(H_E)\}$. Hence by (6.2), $H_E A = Y\underline{w}(A, E, E)$ for some $Y \cong D$. By Lemma 6.8, $F^*(D)$ is simple. This proves (b). Certainly, $H_a A = X_a \underline{w}(A, \langle a \rangle E, \langle a \rangle)$ for some simple group X_a, whenever $a \in A - E$. It remains to show that the isomorphic type of X_a is independent of $a \in A - E$. Define an equivalence relation $\sim$ on $\mathscr{S}_1$ by $T \sim F$ if and only if $X_T \cong X_F$. Certainly the elements of $\mathscr{E}_1(B)$ are equivalent if B is any complement for E in A. All hyperplanes of A have a nontrivial intersection. Hence $\mathscr{S}_1$ is an equivalence class, as required.

Proof of Theorem 6.1. By Theorem 6.13 we may suppose $\mathscr{S}_1 \neq \varnothing$. By Lemma 6.2(a, d) and Lemma 6.4, it suffices to show conclusion (b2) holds. By Theorem 6.14 it remains to show $H_a \in И_\theta^*(A)$ whenever $a \in A - E$. Suppose $a \in A - E$ and $H_a < X \in И_\theta(A)$. Extend $\langle a \rangle$ to a complement B of E in A. By Lemma 5.1, $H_B < C_X(b)$ for some $b \in B - \langle a \rangle$. By Lemma 5.10, $C_X(b) = H_b$. Hence $K(H_E) = \langle K(H_E) \cap C(a), K(H_E) \cap C(b) \rangle \leqslant X$. Hence for any $f \in B^\#$, $H_f = \langle H_B, K(H_E) \cap H_f \rangle \leqslant X$, a contradiction. This completes the proof of Theorem 6.1.

7. $S_r(A)$-SUBGROUPS

We say θ *is type* (A) if θ satisfies conclusion (b1) of Theorem 6.1. We say θ *is type* (B) if θ satisfies conclusion (b2) of Theorem 6.1. When θ is type B we reserve E for the unique element of $\mathscr{S}_2$. For the remainder of the paper we will fix the following notation. Suppose B is a nonidentity subgroup of A and S is an $S_r(A)$-subgroup of G. Then

$$\mathrm{Ind}(S, B) = \{S \cap L \mid L \in \mathrm{Ind}(H_B)\}.$$

We shall also reserve S for some $S_r(A)$-subgroup of G, and Z for $Z(S)$.

LEMMA 7.1. *Suppose $B \in \mathscr{E}_2(A)$ and $E \nleqslant B$ if θ is type* (B). *Then*

(a) $Z(C_S(a)) \cap C(B) = Z(C_S(B))$ *for $a \in B^{\#}$, and*
(b) $Z(C_S(B)) = C_{Z(S)}(B)$.

Proof. (a) $B/\langle a\rangle$ acts semi-regularly on $\mathrm{Ind}(S, B)$, whence (a) follows by Lemma 5.12.

(b) By (a), $Z(C_S(B)) \leqslant C_S(\langle C_S(a) \mid a \in B^{\#}\rangle) = Z(S)$. This proves (b), and the lemma.

THEOREM 7.2. *Suppose θ is type* (A). *Then*

$$Z(C_S(a)) = C_{Z(S)}(a) \qquad \textit{for all } a \in A^{\#}.$$

Proof. By Lemma 7.1,

$$Z(C_S(a)) = \langle Z(C_S(a)) \cap C(B) \mid a \in B \in \mathscr{E}_2(A)\rangle \leqslant Z(S),$$

as required.

THEOREM 7.3. *Suppose θ is type* (B). *Then*

$$Z(C_S(a)) = C_{Z(S)}(a) \qquad \textit{for all } a \in A - E.$$

Proof. Let $Z = Z(S)$ and $Z_B = Z(C_S(B))$ for all subgroups B of A. Let $E \neq F \in \mathscr{E}_1(A)$. Let

$$Z_F^0 = \langle Z_F \cap C_S(B) \mid E \times B = A \text{ and } F < B\rangle$$

and

$$Z_F^1 = \bigcap \{[Z_F, B] \mid E \times B = A \text{ and } F < B\}.$$

By Lemma 7.1, $Z_F^0 \leqslant Z$. By [8, Theorem 5.2.3], $Z_F^1 \leqslant C_{Z_F}(E) \leqslant Z(C_T(F))$, where $T = C_S(E)$. However, F acts semi-regularly on $\mathrm{Ind}(S, E)$. Hence $Z(C_T(F)) \leqslant Z(T) = Z_E$. So

$$Z_F = (Z_F \cap Z)(Z_F \cap Z_E) \qquad \text{for all } F \in \mathscr{E}_1(A). \tag{7.1}$$

Let $V = C_Z(A)$ and $W = \times\,\{\mathrm{Proj}_L(V) | L \in \mathrm{Ind}(S, E)\}$. Let $F, K \in \mathscr{E}_1(A)$ satisfy $E \times F \times K = A$. By Lemma 7.1 and (7.1),

$$\begin{aligned}(Z_F \cap Z_E) \cap C_S(K) &= (Z_F \cap C_S(E)) \cap C_S(K)\\ &= (Z_F \cap C_S(K)) \cap C_S(E)\\ &= C_Z(FK) \cap C_S(E) = V.\end{aligned}$$

Since E normalizes each member of $\mathrm{Ind}(S, F)$, it follows from (7.1) that $Z_E \cap Z_F = \times\,\{C_{Z(R)}(E) | R \in \mathrm{Ind}(S, F)\}$. Since K acts regularly on $\mathrm{Ind}(S, F)$, and $(Z_F \cap Z_E) \cap C_S(K) = V$, there follows from Lemma 5.6

$$Z_E \cap Z_F = \times\,\{\mathrm{Proj}_R(V) | R \in \mathrm{Ind}(S, F)\} \cong pV. \tag{7.2}$$

Since $\mathrm{Ind}(S, EF) = \{C_R(E) | R \in \mathrm{Ind}(S, F)\}$, (7.2) implies that

$$Z_E \cap Z_F = \times\,\{\mathrm{Proj}_T(V) | T \in \mathrm{Ind}(S, EF)\} \leqslant W. \tag{7.3}$$

Since A/E acts regularly on $\mathrm{Ind}(S, E)$, it follows that $p^2 V \cong W \cong p(C_W(F))$. Hence by (7.2) and (7.3) we obtain

$$Z_E \cap Z_F = C_W(F) = C_W(EF) \qquad \text{whenever } E \neq F \in \mathscr{E}_1(A). \tag{7.4}$$

In particular, (7.4) implies

$$Z_E \cap Z_F = Z_E \cap Z_T \qquad \text{whenever } EF = ET \text{ and } F, T \in \mathscr{E}_1(A). \tag{7.5}$$

By (7.5), $Z_E \cap Z_F \leqslant Z$ whenever $E \neq F \in \mathscr{E}_1(A)$. Now (7.1) completes the theorem.

LEMMA 7.4. *Let S be an $S_r(A)$-subgroup of G. Let $Z = Z(S)$. Let $a \in A^{\#}$. Suppose $\langle a \rangle \neq E$ if θ is type* (B). *Assume $r \in \pi(\theta)$. Then*

(a) $r \in \pi(H_a)$,
(b) $Z \cap L \neq 1$ *for any* $L \in \mathrm{Ind}(H_a)$, *and*
(c) $Z \cap H_a = \times\,\{Z \cap L | L \in \mathrm{Ind}(H_a)\}$.

Proof. Choose a subgroup B of A which contains a but not E. By Theorem 6.1, $\pi(H_b) = \pi(H_c)$ for all $b, c \in B^{\#}$. Hence by Lemmas 5.1 and 5.3, $1 \neq C_S(a)$ is an $S_r(A)$-subgroup of H_a. In particular, (a) holds. The structure of Sylow r-subgroups of H_a and Theorems 7.2 and 7.3 yield (b) and (c).

8. Conclusion of Proof.

We continue the conventions introduced at the beginning of part 7. In particular, $r \in \pi(\theta)$, $S \in И^*_\theta(A: r)$, and $Z = Z(S)$.

THEOREM 8.1. *θ is type* (B).

Proof. Suppose false. Then by Theorem 6.1, θ is type (A). In particular, D is simple, and for each nonidentity subgroup T of A, $AH_T = L\underline{w}(A, T, T)$ for some $L \cong D$.

Fix a hyperplane B of A. For each $L \in \mathscr{L}(H_B)$, let $Z_L = \bigcap \{C_Z(K) | L \neq K \in \mathscr{L}(H_B)\}$, $M_L = \langle L, Z_L \rangle$, and $M = \langle M_L | L \in \mathscr{L}(H_B) \rangle$. By Lemma 7.4(c), $Z \cap H_a \leqslant M$ for all $a \in B^{\#}$. By Lemma 7.4(b), $H_a = \langle H_B, Z \cap H_a \rangle \leqslant M$ for all $a \in B^{\#}$. Hence by Theorem 6.1(b1.3) and (c) there follows

$$M = \langle И_\theta(A) \rangle. \tag{8.1}$$

Since Z is Abelian, $[M_L, M_K] = 1$ whenever $L \neq K$. Hence Theorem 6.1(d) yields

$$M = \times \{M_L | L \in \mathscr{L}(H_B)\}. \tag{8.2}$$

Since A acts transitively on $\mathscr{L}(H_B)$ there follows,

A acts transitively on $\{M_L | L \in \mathscr{L}(H_B)\}$ and $B = N_A(M_L)$ for $L \in \mathscr{L}(H_B)$.
(8.3)

By definition we also have

$$H_B = \times \{H_B \cap M_L | L \in \mathscr{L}(H_B)\}. \tag{8.4}$$

Now let B_1, B_2, B_3 be 3 hyperplanes of A such that $\{B_i \cap B_j | 1 \leqslant i < j \leqslant 3\}$ are cyclic subgroups of A which generate A. Let $\{M^i_j | 1 \leqslant j \leqslant p\} = \{M_L | L \in \mathscr{L}(H_{B_i})\}$ for $i = 1, 2,$ or 3. Let $M_{i,j,k} = M^1_i \cap M^2_j \cap M^3_k$. Since M is generated by perfect subgroups, (8.2) yields that

$$\begin{aligned} M = [M, M, M] &= \left[\mathop{\times}_i M^1_i, \mathop{\times}_j M^2_j, \mathop{\times}_k M^3_k\right] \\ &\leqslant \langle [M^1_i, M^2_j, M^3_k] | 1 \leqslant i, j, k \leqslant p \rangle \\ &\leqslant \langle M_{i,j,k} | 1 \leqslant i, j, k \leqslant p \rangle. \end{aligned}$$

By (8.2), $[M_{i,j,k}, M_{u,v,w}] = 1$ if $(i, j, k) \neq (u, v, w)$. Hence Theorem 6.1(d) yields

$$M = \times \{M_{i,j,k} | 1 \leqslant i, j, k \leqslant p\}. \tag{8.5}$$

The choice of B_1, B_2, B_3, together with (8.2) yields

$$\text{A acts regularly on } \{M_{i,j,k} \mid 1 \leqslant i,j,k \leqslant p\}. \tag{8.6}$$

Now let $W_{i,j,k} = \text{Proj}_{M_{i,j,k}}(D)$, and $W = \langle W_{i,j,k} \mid 1 \leqslant i,j,k \leqslant p\rangle$. By (8.6), we obtain

$$W \cong p^3 D. \tag{8.7}$$

By (8.4), we obtain

$$W \geqslant \langle H_{B_1}, H_{B_2}, H_{B_3}\rangle. \tag{8.8}$$

By Lemma 5.10 and (8.8), $H_{B_i \cap B_j} = \langle H_{B_i}, H_{B_j}\rangle \leqslant W$. Hence by (8.8) and Theorem 6.1(b1.3), $W = M$. By (8.6), (8.7), $p^2 D \cong H_a \leqslant C_W(a) \cong p^2 D$ for all $a \in A^{\#}$. Hence $C_W(a) = H_a$ for all $a \in A^{\#}$. However (W, p) satisfies Hypothesis B. This contradiction yields the result.

LEMMA 8.2. $F^*(H_E) \leqslant \langle D, Z\rangle$.

Proof. Let $W = \langle D, Z\rangle$ and $W_b = W \cap H_b$ for each $b \in A^{\#}$. Let $b \in A - E$, and $J \in \mathscr{L}(H_b)$. By Lemma 7.4(b), $1 \neq Z \cap J \leqslant W_b \cap J \lhd \text{Proj}_J(W_b)$ where projections are being taken in H_b with respect to $\mathscr{L}(H_b)$. By Lemma 5.5, $C_J(E) = \text{Proj}_J(D) \leqslant \text{Proj}_J(W_b)$. Hence by Hypothesis (A.3.1), $W_b \cap J$ is nonsolvable. By [8, Theorem 10.2.1], $C_{W_b \cap J}(E) \neq 1$. By Hypothesis (A.3.2), $F^*(C_J(E))$ is the unique minimal normal subgroup of $C_J(E)$, whence $F^*(C_J(E)) \leqslant W_b$. So $F^*(H_E) \cap C_G(b) = F^*(H_{\langle E,b\rangle}) \leqslant \langle D, Z\rangle$ for all $b \in A - E$. Now Lemma 5.1 yields the lemma.

LEMMA 8.3. *Suppose B is a hyperplane of A which contains E. Let $L \in \text{Ind}(H_B)$. Define $\hat{L}$ to be the product of components of H_E not centralized by L. Then $\hat{L} \leqslant \langle Z, L\rangle$.*

Proof. Let $Z_0 = Z(C_S(E))$, $V = ZZ_0$, $V_L = \bigcap \{C_V(K) \mid L \neq K \in \text{Ind}(H_B)\}$, $W_L = \langle V_L, L\rangle$, and $W = \langle W_L \mid L \in \text{Ind}(H_B)\rangle$. Since V is Abelian, it follows that $[W_L, W_K] = 1$ if $L \neq K$. In particular,

$$W_L \lhd W \qquad \text{for any } L \in \text{Ind}(H_B). \tag{8.9}$$

By Lemma 7.4(c), $C_Z(a) \leqslant \langle V_L \mid L \in \text{Ind}(H_B)\rangle \leqslant W$ if $a \in A - E$. By Lemma 5.3, $C_Z(E) \leqslant Z_0 \leqslant W$. Hence by Lemma 5.1, $Z \leqslant W$. Lemma 8.2 yields

$$F^*(H_E) \leqslant \langle D, Z\rangle \leqslant \langle H_B, Z\rangle \leqslant W. \tag{8.10}$$

Let $\tilde{L} = \langle L, Z\rangle'$. By (8.9), $\tilde{L} \leqslant \langle W_L, Z\rangle' \leqslant W_L$. Since $[\tilde{L}, K] \leqslant$

$[W_L, W_K] = 1$ for distinct L, $K \in \operatorname{Ind}(H_B)$, and $\tilde{L}$ admits $\langle L, Z\rangle$, there follows

$$\tilde{L} \lhd \langle Z, H_B\rangle. \tag{8.11}$$

By (8.10) and (8.11), $\hat{L} = \langle F^*(L)^{F^*(H_E)}\rangle \leqslant \langle F^*(L)^{\langle H_B, Z\rangle}\rangle \leqslant \tilde{L} \leqslant \langle L, Z\rangle$ as required.

LEMMA 8.4. *Suppose $E \neq F \in \mathscr{E}_1(A)$, $L \in \operatorname{Ind}(H_{EF})$, and $K \in \operatorname{Ind}(H_F)$. Suppose in addition that $C_K(E) \neq L$. Let $\hat{L}$ be the product of components of H_E not centralized by L. Then $[\hat{L}, K] = 1$.*

Proof. Let $L_1 \in \operatorname{Ind}(H_F)$ satisfy $C_{L_1}(E) = L$. Then $[L, S \cap K] \leqslant [L_1, K] = 1$. Clearly, $[Z, S \cap K] = 1$. Hence $[\langle L, Z\rangle, S \cap K] = 1$. By Lemma 8.3, $[\hat{L}, S \cap K] = 1$. Since $K = \langle K \cap S \mid S$ is some $S_r(A)$-subgroup, $r \in \pi(\theta)\rangle$, it follows that $[\hat{L}, K] = 1$.

THEOREM 8.5. *Let $W = \langle И_\theta(A)\rangle$. Suppose $E \neq F \in \mathscr{E}_1(A)$. Then for each $K \in \operatorname{Ind}(H_F)$, W has direct factors W_K which contain K and satisfy $W = \times\{W_K | K \in \operatorname{Ind}(H_F)\}$.* Moreover, A acts transitively on $\{W_K\}$.

Proof. For each $K \in \operatorname{Ind}(H_F)$, let $K_0 = C_K(E)$, and $\hat{K}$ be the product of components of H_E not centralized by K_0. Now let $W_K = \langle K, \hat{K}\rangle$. By Lemma 8.4, $[\hat{L}, K] = 1$ whenever L, K, are distinct members of $\operatorname{Ind}(H_F)$. Moreover, $\{T \cap H_E \mid T \in \mathscr{L}(H_F)\} = \operatorname{Ind}(H_{EF}) = \{R ** F \mid R \in \operatorname{Ind}(H_E)\}$, whence $[\hat{L}, \hat{K}] = 1$ if $L \neq K$. Hence $[W_L, W_K] = 1$ if $L \neq K$. Now $\langle F^*(H_E), H_F\rangle \leqslant \langle W_L | L \in \operatorname{Ind}(H_F)\rangle$. Hence Theorem 6.1(b2.5) and Lemma 6.12, yields $W = \times\{W_K | K \in \operatorname{Ind}(H_F)\}$. Since A acts transitively on $\operatorname{Ind}(H_F)$ and $EF = N_A(K)$ for each $K \in \operatorname{Ind}(H_F)$ the remaining statements also hold.

Proof of Theorem A. Let $F_1, F_2 \in \mathscr{E}_1(A)$ satisfy $EF_1F_2 = A$. Let $\mathscr{L}_i = \mathscr{L}(H_{F_i})$ for $i = 1$ or 2. Let $W = \langle И_\theta(A)\rangle$. Following Theorem 8.5, for each $K \in \mathscr{L}_i$ let W_K be direct factors of W which contain K and which satisfy

(a) $W = \times\{W_K | K \in \mathscr{L}_i\}$ for $i = 1$ or 2.
(b) A is transitive on $\{W_K | K \in \mathscr{L}_i\}$ and

$EF_i = N_A(W_K)$ for any $K \in \mathscr{L}_i$.

Let $\Omega = \{W_K \cap W_L | K \in \mathscr{L}_1, L \in \mathscr{L}_2\}$. As in Theorem 8.1 we obtain

$$W = \times\,\Omega, \text{ and} \tag{8.12}$$

$$A \text{ acts transitively on } \Omega, \text{ and } N_A(X) = E \text{ for any } X \in \Omega. \tag{8.13}$$

Let $M = H_{F_1F_2}$, $M_X = \operatorname{Proj}_X(M)$ for $X \in \Omega$, and $\hat{M} = \times\{M_X | X \in \Omega\}$. Let $K \in \mathscr{L}_2$. When (A, G, S, T, K, W) is replaced by $(F_1F_2, W, \mathscr{L}_i, K, W_K)$, Lemma 5.6 implies that $K \leqslant \hat{M}$. Hence $\langle H_{F_1}, H_{F_2} \rangle \leqslant \hat{M}$. By Theorem 6.1(b.2.5) and Lemma 6.12, $\hat{M} = W$. By (8.12) and (8.13), $W \cong p^2M$. Let $a \in A - E$. By (8.13) and Theorem 6.1(b2.4), $C_W(a) \cong pM \cong H_a$. Hence $H_a = C_W(a)$ for all $a \in A - E$. Since W is a p'-group, $H_E \leqslant C_W(E) = \langle C_W(E) \cap C_W(a) | a \in A - E \rangle = \langle C_W(E) \cap H_a | a \in A - E \rangle \leqslant H_E$. Hence $C_W(b) = H_b$ for all $b \in A^{\#}$. Since (M, p) satisfies Hypothesis B, it follows that (W, p) satisfies Hypothesis B. Hence $W \in \mathcal{N}_\theta(A)$, a contradiction. This completes the proof of Theorem A. Hence Corollary C also holds, thus completing the proof of all parts.

References

1. N. Burgoyne, R. L. Greiss, Jr., and R. Lyons, Field automorphisms and maximal subgroups of finite Chevalley groups, unpublished.
2. R. Carter, "Simple Groups of Lie Type," Wiley, New York, 1972.
3. W. Feit and J. G. Thompson, Solvability of groups of odd order, *Pacific J. Math.* **13** (1963), 775–1029.
4. W.· Feit, "The Current Situation in the Theory of Finite Simple Groups," Yale Univ. Press, New Haven, 1971.
5. G. Glauberman, On solvable signalizer functors in finite groups, *Proc. London Math. Soc.* (3) **33** (1976), 1–27.
6. G. Glauberman, Factorizations in Local Subgroups of Finite Groups, U. of Minnesota, Duluth monograph (1976).
7. D. Goldschmidt, Solvable signalizer functors on finite groups, *J. Algebra* **21** (1972), 341–351.
8. D. Gorenstein, "Finite Groups," Harper and Row, New York, 1969.
9. D. Gorenstein and R. Lyons, Nonsolvable signalizer functors on finite groups, unpublished.
10. Z. Janko, A new finite simple group with Abelian 2-Sylow groups and its characterization, *J. Algebra* **3** (1966), 147–187.
11. Z. Janko and J. G. Thompson, On a class of simple groups of Ree, *J. Algebra* **4** (1966), 274–292.
12. P. Martineau, Elementary Abelian fixed point free automorphism groups, *Quart. J. of Math. (Oxford)* (2) **23** (1972), 205–212.
13. P. McBride, Near solvable signalizer functors on finite groups, unpublished.
14. R. Ree, A family of simple groups associated with the simple Lie algebra of type (F_4), *Amer. J. Math.* **83** (1961), 401–420.
15. R. Ree, A family of simple groups associated with the simple Lie algebra of type (G_2), *Amer. J. Math.* 432–462.
16. R. Steinberg, Automorphisms of finite linear groups, *Canad. J. Math.* **12** (1960), 606–615.
17. M. Zuzuki, On a class of doubly transitive groups, *Ann. of Math.* **75** (1962), 105–145.
18. J. G. Thompson, Simple groups of order prime to 3, I, II, unpublished.
19. J. Walter, Finite groups with Abelian Sylow 2-subgroups of order 8, *Invent. Math.* **2** (1967), 333–376.
20. J. Walter, The characterization of finite groups with Abelian Sylow 2-subgroups, *Ann. of Math.* **89** (1969), 405–514.
21. H. N. Ward, On Ree's series of simple groups, *Trans. Amer. Math. Soc.*, **121** (1966), 62–89.