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Stability and Bifurcation of Steady-State Solutions for Predator–Prey Equations

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INTRODUCTION

In this paper we study the nonnegative time-independent ($\equiv$ equilibrium $\equiv$ steady-state) solutions of the parabolic system

$$\begin{cases} u_t = u_{xx} + f(u) - uv, \\ v_t = dv_{xx} + v[-v + m(u-\gamma)], \end{cases} \tag{0.1$_\text{d}$}$$

for $-L < x < L$, $t > 0$, with homogeneous Dirichlet boundary conditions

$$\begin{cases} u(\pm L, t) = v(\pm L, t) = 0, t > 0, & \text{for } d > 0, \\ u(\pm L, t) = 0, t > 0, & \text{for } d = 0. \end{cases} \tag{0.2}$$

We assume that $d \geq 0$, $m > 0$ and that $0 < \gamma < 1$. We shall take f to be a quadratic or cubic polynomial and give most attention to the cases, $f(u) = au(1-u)$ ("logistic") and $f(u) = au(u-b)(1-u)$, $0 < b < 1$ ("asocial"). In these cases the system models a two-species predator–prey interaction in which both species undergo simple diffusion in a one-dimensional medium (u = density of prey, v = density of predator). We will discuss this aspect of the equations at the end of this section.

The above system is an example of a reaction–diffusion system. Such equations have received a great deal of attention in the past few years (see the recent review by Fife [9]), motivated in part by their widespread occurrence in models of chemical and biological phenomena and in part by the richness of the structure of their solution sets. The combination of

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diffusion with nonlinear interaction produces features that are at first glance completely unexpected from the vantage point of either mechanism alone. Therefore, before these equations can be fully exploited in scientific models it seems necessary to establish a new "intuition" based upon a few simple guiding principles. As a first step in this process we think it necessary that some well-chosen examples be thoroughly understood, and it is in this spirit that the present work was undertaken.

Our goal was to find all equilibrium solutions for the system and to determine their stability. In pursuing this goal we have found solutions with novel features, and have uncovered nonstandard bifurcation phenomena. All of this depends in an essential way upon the parameters d, the relative rate of diffusion of u and v, and L, the length of the spatial region. We are able to give a complete analysis for most values of $d \geq 0$, $L > 0$ but only a partial analysis when (L, d) is in a certain subset of the parameter space. For these exceptional values of (L, d) we have made a large number of computer simulations, on the basis of which, we conjecture the number of solutions and their stability.

Outline of Results

Part I of this paper, which consists of Sections 1, 2, and 3, is devoted to the case of $f(u) = au(1 - u)$. Our results are best described while referring to Fig. 0. This shows the positive quadrant of the L–d parameter space divided into three regions defined by two critical values, L_f (depending only upon f) and L_γ (depending only upon f and γ), and by a certain monotone increasing function, $\bar{d}(L)$. In region A the only equilibrium solution is the constant $(u, v) \equiv (0,0)$, which is asymptotically stable and, in fact, is the uniform limit, as $t \to \infty$, of all nonnegative solutions of $(0.1)_d$–(0.2) (i.e., it is an attractor for all nonnegative solutions). As L increases across L_f, a new, nonconstant equilibrium solution, $(u_0(x), 0)$, bifurcates from $(0,0)$. For all values of (d, L) in region B the solution $(0,0)$ is unstable while $(u_0, 0)$ is stable and attracts all nonnegative solutions. There are no other steady-state solutions in this region. If we cross into region C at a point $(\bar{L}, \bar{d})$ with $\bar{d} = \bar{d}(\bar{L}) > 0$, then a third solution (u_1, v_1) bifurcates from $(u_0, 0)$ (i.e., secondary bifurcation occurs). In the intersection of a neighborhood of the graph of $d = \bar{d}(L)$ with C there are only three equilibria: (u_1, v_1) is stable while $(u_0, 0)$ and $(0,0)$ are both unstable. If we cross into region C along the L-axis with $d = 0$ then the situation is much more complex. As L increases across L_γ an interval of the (continuous) residual spectrum of the linearization becomes positive. This is reflected in the appearance of a continuum of new steady states bifurcating from $(u_0, 0)$. One of these new equilibria, (u_1, v_1), is continuous but v_1 has *compact support* in the interior of $|x| < L$. The other equilibria are steady-state solutions in a generalized sense and in fact their second derivatives are discontinuous. We were unable to obtain rigorous results for (L, d) far away from the boundary of C. It is here that we conjecture that there are no equilibria other than the three which are present near the curve $d = \bar{d}(L)$. This is supported by extensive computer simulation.

In Section 4 we discuss the case where $f(u) = au(u-b)(1-u)$, $0 < b < \gamma < 1$. This case differs from the previous case in two respects. First, the solution $(0, 0)$ remains stable for all values of (L, d) but is a global attractor only in a region analogous to A (Fig. 0 can still be used in this case but of course the values of L_f and L_γ are not the same as in the quadratic case). Second, as L increases across L_f there is the sudden appearance of two new solutions, $(u_1, 0)$ and $(u_2, 0)$, with opposite stabilities, and in this case the new solutions do not bifurcate from $(0, 0)$. We are able to give a comparison argument showing that the larger of the two, $(u_2, 0)$, is an attractor and that the smaller $(u_1, 0)$ is unstable, for values of (L, d) in region B. As we cross into region C the same situation prevails as in the quadratic case.

In Section 5 we very briefly indicate the situation in the case where $f(u) = (u-a)(u-b)(1-u)$, $a < b < 0 < 1$. Our purpose here is only to indicate the remarkable richness of the set of solutions. For example, if we hold $d = 0$ and let L increase from 0 to ∞ then we obtain a succession of bifurcations which can yield the following sequences of numbers of solutions: $(1, 2, 4, 6, 4, 2)$ or $(1, 3, 5, 6, 4, 2)$ or $(1, 3, 1, 2)$, among others. Which sequence we obtain depends only upon the position of γ.

In Section 6 we indicate how some of our results can be extended to more than one space variable.

A word about methods. Although we use spectral analysis and standard bifurcation theorems, the bulk of our results are obtained using phase-plane analysis and comparison theorems. In doing this we rely heavily on the very complete results concerning a single semilinear parabolic equation found in [19, 20]. This is especially true in our use of comparison theorems, where we employ comparison functions that are themselves solutions of scalar equations. In this way we have succeeded in giving direct comparison proofs of convergence to nonconstant equilibrium solutions of the above *system*.

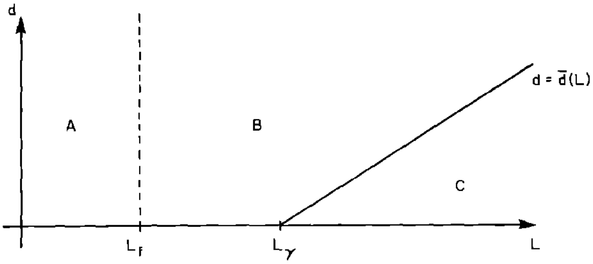

FIGURE 0

Predator–Prey Systems

In an earlier publication, [7], which contains direct references to the literature of mathematical ecology, we introduced the generalized Rosenzweig–MacArthur equations

$$u_t = d_1 u_{xx} + f(u) - \phi(u, v), \qquad v_t = d_2 v_{xx} + g(v) + m\phi(u, v).$$

If $\phi_u > 0$ and $\phi_v > 0$ then this describes a predator–prey interaction, where both the predator, whose spatial density is v, and the prey, whose density is u, undergo simple diffusion in a one-dimensional medium. We assume that $f(0) = g(0) = \phi(u, 0) = \phi(0, v) = 0$ and that, even in the absence of the predator, there is a limitation to the growth of prey, indicated by the fact that f is negative for sufficiently large u. We choose units of u to force $f(1) = 0$. The most common choice for $\phi(u, v)$ is simply cuv while for $g(v)$ it is $-\mu v$. For reasons outlined in [7] we choose $g(v) = -\mu v - \epsilon v^2$. Thus, we have the equation

$$\begin{aligned} u_t &= d_1 u_{xx} + f(u) - cuv, \\ v_t &= d_2 v_{xx} - \mu v - \epsilon v^2 + muv. \end{aligned}$$

At the expense of changing c we can choose units for v that have the effect of setting $\epsilon = 1$. If we also change the length scale, $x \to x/\sqrt{d_1}$, and let $\gamma = \mu/m$ we obtain

$$\begin{aligned} u_t &= u_{xx} + f(u) - cuv, \\ v_t &= \left(\frac{d_2}{d_1}\right) v_{xx} + v[-v + m(u - \gamma)]. \end{aligned}$$

Now the qualitative picture is found to be largely insensitive to the value of c so we have set $c = 1$ and denoted d_2/d_1 by d to obtain our Eqs. $(0.1)_{\rm d}$.

The function f determines the population growth of the prey when there is no predator. Except for Section 5, our discussion is confined to $f(u) = au(1 - u)$ which gives use to "logistic" growth and to $f(u) = au(u - b)(1 - u)$, $0 < b < 1$, which describes populations which have been termed "asocial" in [2].

We imagine our one-dimensional medium to be of finite length and, at each end, to be in contact, through a permeable membrane, with hostile reservoirs in which neither prey nor predator can survive, i.e., $u \equiv v \equiv 0$. We treat the limiting case of infinite permeability by imposing Dirichlet conditions (0.2). However, our methods yield essentially the same (qualitative) results for the more general assumption that the flux out from the region is proportional to the amount by which the concentration exceeds that of the reservoir, i.e.,

$$u(\pm L, t) \pm \beta u_x(\pm L, t) = 0.$$

Related Work

Dirichlet conditions for a class of predator–prey equations were considered in [12] but there the interest is upon positive concentrations of predator or prey at the boundary. There is very little overlap with our results and the methods are quite different. Solutions which have compact support in the interior of the spatial region were also obtained in [14] but this paper is devoted to the case of Neuman boundary conditions and to the case where the $d_1 \gg d_2$. Again, our methods are quite different.

I. Quadratic Nonlinearity

1. Construction of the Steady-State Solutions

In this part, we shall consider Eqs. $(0.1)_{\mathrm{d}}$, (0.2), where f is a quadratic function of the form $f(u) = au(1 - u)$, and a is a positive constant. Here m and γ denote constants satisfying only the conditions $m > 0$, $1 > \gamma > 0$. It is straightforward to check that with these conditions, there are arbitrarily large "invariant rectangles" in the region $u \geq 0$, $v \geq 0$, and thus, from the results in [3], nonnegative solutions of $(0.1)_{\mathrm{d}}$, (0.2) exist for all $t > 0$, provided that the initial data are uniformly bounded; we shall always assume this to be the case.

From Theorem 3.1 of [7] it follows that the region

$$\sum = \{(u, v) \colon 0 \leq u \leq 1, 0 \leq v \leq m(1 - \gamma)\} \tag{1.1}$$

attracts all nonnegative solutions of $(0.1)_{\mathrm{d}}$, (0.2); hence in particular, it must contain all of the steady-state solutions, i.e., all solutions of the equations

$$u'' + f(u) - uv = 0, \qquad dv'' + v[-v + m(u - \gamma)] = 0, \qquad (1.2)_{\mathrm{d}}$$

on the interval $-L < x < L$, satisfying the boundary conditions

$$u(\pm L) = 0 \text{ and } v(\pm L) = 0. \qquad \text{(If } d = 0 \text{ we omit the condition on } v.) \tag{1.3}$$

This yields a-priori bounds on all solutions of $(1.2)_{\mathrm{d}}$, (1.3). We shall find this useful in what follows.

Now it is obvious that any function of the form $(u, v) = (u_0, 0)$, where u_0 satisfies

$$u'' + f(u) = 0, \qquad -L < x < L; \qquad u(\pm L) = 0, \tag{1.4}$$

is a steady-state solution. The phase plane portrait of (1.4) is depicted in Fig. 1, where the nonconstant solutions are curves in $u \geq 0$ lying in the "teardrop," which "begin" and "end" on the line $u = 0$, and take "time" $2L$ to make the journey.

We call attention to two distinguished values of L which we denote by L_f and L_γ. We define L_f to be the smallest $L > 0$ for which a nonconstant solution of (1.4) exists. More precisely, we have the following simple lemma.

LEMMA 1.1. *Let* $L_f = \pi/2\sqrt{f'(0)} \equiv \pi/2\sqrt{a}$.

(i) $L < L_f$ *implies* $u \equiv 0$ *is the only nonnegative solution of* (1.4).

(ii) $a < \lambda$ *iff* $L < L_f$; *here* $\lambda = \pi^2/4L^2$, *the first eigenvalue of* $-d^2/dx^2$ *on* $-L < x < L$, *with homogeneous Dirichlet boundary conditions.*

Proof. The second statement is a trivial consequence of the definitions; thus, to show (i) we can assume $a < \lambda$. Now $0 = uu'' + uf(u) < uu'' + au^2$, and integrating from $x = -L$ to $x = L$, we get

$$0 \leq -\int_{-L}^{L} (u')^2 + a\int_{-L}^{L} u^2 \leq -\lambda\int_{-L}^{L} u^2 + \int_{-L}^{L} au^2 = (a - \lambda)\int_{-L}^{L} u^2,$$

where we have used a standard inequality (cf. [13]). This shows that $u \equiv 0$, if $L < L_f$.

From the results in [20], it follows that for each $L \geq L_f$, there is precisely one nonconstant (nonnegative) solution u_0 of (1.4), and it is "nondegenerate." By this we mean that the operator $d^2/dx^2 + f'(u_0(x))$ on $-L < x < L$, with homogeneous Dirichlet boundary conditions, is invertible; i.e., 0 is not in its spectrum.

The second distinguished value of L, L_γ, is defined to be the interval length L associated with the unique solution of (1.4) satisfying $u(0) = \gamma$. In

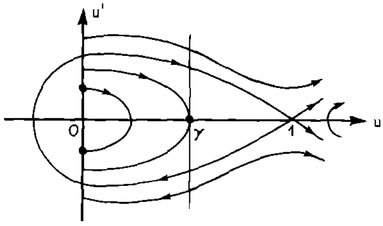

FIGURE 1

other words, it is the L associated with the unique orbit which passes through the point $(u, u') = (\gamma, 0)$ (refer to Fig. 1). We recall that $0 < \gamma < 1$ so that

$$L_f < L_\gamma. \tag{1.5}$$

In the next sections we shall study the stability of, and bifurcation from, the solution $(u_0, 0)$; in preparation for this we define a new function,

$$\begin{aligned} h(u) &= f(u) && \text{if } u \leq \gamma \\ &= f(u) - mu(u-\gamma) && \text{if } u \geq \gamma. \end{aligned}$$

It is clear that h is Lipschitz-continuous so that the problem

$$u'' + h(u) = 0, \qquad -L < x < L; \qquad u(\pm L) = 0 \tag{1.6}$$

admits C^2 solutions. In fact, we see that $h(u) < 0$ for $u < 0$, $h(0) = 0$, $h(b) = 0$, where $\gamma < b = (a + m\gamma)/(a + m) < 1$, $h(u) > 0$ for $0 < u < b$ and $h(u) < 0$ for $u > b$. From this we see that the phase plane for (1.6) (Fig. 2) is qualitatively similar to that for (1.4) (Fig. 1). If we define L_h in a manner analogous to that used to define L_f then because $h = f$ for $u \leq \gamma$ we see that $L_h = L_f$.

LEMMA 1.2. *For $0 < L \leq L_f$, the only nonnegative solution of* (1.6) *is $u \equiv 0$. For each $L > L_f$ there is a unique nonconstant, nonnegative solution of* (1.6).

Proof. Except for the uniqueness statement, this is a consequence of the fact that $h(u) = f(u)$ for $u < \gamma$ and that $L_f < L_\gamma$. The uniqueness is proved in the Appendix by extending the results of [20] to nonsmooth functions.

We are now ready to construct a new solution of $(1.2)_d$, (1.3) when $d = 0$ and $L > L_\gamma$. Rewriting these steady-state equations we have

$$u'' + f(u) - uv = 0, \qquad v[-v + m(u-\gamma)] = 0 \quad \text{for } -L < x < L \tag{1.7}$$

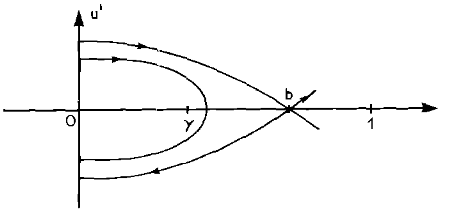

FIGURE 2

with the boundary conditions

$$u(\pm L) = 0. \tag{1.8}$$

For $L > L_f$ let $u_1(x)$ be the unique nonnegative, nonconstant solution of (1.6). Define $v_1(x)$ by

$$\begin{aligned} v_1(x) &= 0 && \text{if } u_1(x) < \gamma \\ &= m(u_1(x) - \gamma) && \text{if } u_1(x) \geq \gamma. \end{aligned} \tag{1.9}$$

It is then clear that (1.7) is satisfied by $(u_1(x), v_1(x))$ at each x in $(-L, +L)$ and of course (1.8) is also satisfied. Note that (u_1, v_1) is distinct from $(u_0, 0)$ only when $L > L_\gamma$. In fact, as L increases through L_γ the solution (u_1, v_1) bifurcates out of $(u_0, 0)$. In this case the structure of (u_1, v_1) is quite interesting since v_1 has support $[-\alpha(L), \alpha(L)] \subset\subset (-L, +L)$, where $\alpha(L)$ is defined by $u_1(x) \geq \gamma$ if and only if $|x| \leq \alpha(L)$ (Fig. 3).

Remarks. It is interesting that as $L \to \infty$, the quantity $L - \alpha(L)$ tends, in a monotone decreasing fashion, to a nonzero limit. To see this, note that $L - \alpha(L)$ is the "time" an orbit of (1.6) takes to go from the line $u = 0$ to the line $u = \gamma$ (see Fig. 2). Analytically,

$$L - \alpha(L) = \int_0^\gamma \frac{du}{\sqrt{p^2 - 2H(u)}},$$

where $p = u'(-L)$ and $H'(u) = h(u)$. The right-hand side is a decreasing function of p and p is an increasing function of L. This shows that $L - \alpha(L)$ is a decreasing function of L. The positive lower bound, δ, is the value when $p = U'(-L)$, where U is the separatrix, i.e., the solution such that $U(x) \to b$ as $x \to +\infty$. Note that $u_1'(-L) < U'(-L)$ and $u_1'(-L) \to U'(-L)$ as $L \to \infty$.

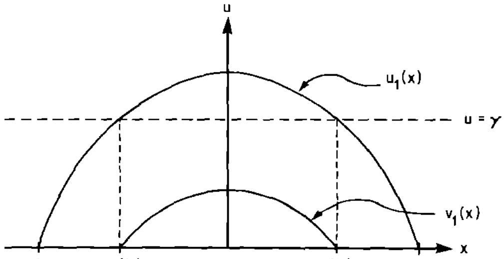

FIGURE 3

If we interpret the quantities u and v as concentrations of prey and predator, then the solution (u_1, v_1) corresponds to a situation in which the predator survives only in the interior of the spatial domain, and its "dead zone" is at least of thickness δ.

From an examination of the phase portrait in Fig. 2, it is clear that as $L \to \infty$, $(u_1(x), v_1(x)) \to (b, c)$, $c = m(b - \gamma)$, for every x. The point (b, c) is the unique critical point of the ordinary differential equations

$$\begin{aligned} \dot{u} &= f(u) - uv, \\ \dot{v} &= v[-v + m(u - \gamma)], \end{aligned} \tag{1.10}$$

which describe the "kinetics" of the interaction. It is easy to check that this critical point is the only attractor for the system (1.10), which fact dovetails nicely with the observation that for very long intervals the quantities (u, v) tend to (b, c) in the interior.

We summarize the constructions of this section in the following theorem.

THEOREM 1.3. *Under the assumption that* $0 < \gamma < 1$ *and* $d = 0$, *the following statements concerning the solutions of* (1.7)–(1.8) *are valid*:

(i) *If* $L \leq L_f$ *then* $(0, 0)$ *is the only solution.*

(ii) *If* $L_f < L \leq L_\gamma$ *then there are two nonnegative solutions*: $(0, 0)$ *and* $(u_0, 0)$.

(iii) *If* $L_\gamma < L$ *then there are three nonnegative solutions*: $(0,0)$, $(u_0, 0)$ *and* (u_1, v_1).

Remarks. (1) If $d = 0$ then there are no other solutions such that u is C^2 and v is continuous.

(2) The functions $(0, 0)$ and $(u_0, 0)$ are also solutions of (1.2) for all $d > 0$.

2. STABILITY OF STEADY-STATE SOLUTIONS

In this section we shall be concerned with the stability and instability of the solutions we constructed in Section 1. We shall be particularly concerned with obtaining information on the domains of attraction of our steady-state solutions; in other words, we will identify solutions which lie in the "stable manifold" of our rest points. To this end, we first study the case of a single equation. Some of the results we obtain could be obtained by suitably modifying the techniques in [0], but we prefer to give alternate simple proofs using the gradient-like nature of the equations.

A. Stability via Comparison Theorems

Consider the scalar equation

$$U_t = U_{xx} + f(U), \qquad -L < x < L, t > 0, \tag{2.1}$$

with boundary conditions

$$U(\pm L, t) = 0, \qquad t \geq 0. \tag{2.2}$$

Here, as above, $f(U) = aU(1 - U)$, $a > 0$. Let L_f be as in Section 1, i.e., (2.1), (2.2) admit a nonconstant steady-state solution $U_0(x)$, if and only if $L \geq L_f$.

PROPOSITION 2.1. *Let $U(x, t)$ be the solution of* (2.1), (2.2) *with initial data $U(x, 0)$, where $U(x, 0) \geq 0$, $-L < x < L$, and $U(x, 0) \not\equiv 0$.*

A. *If $L < L_f$ then $U(x, t) \to 0$, uniformly in x, as $t \to \infty$.*

B. *If $L \geq L_f$, then $U(x, t) \to U_0(x)$, uniformly in x as $t \to \infty$.*

Proof. Using standard comparison theorems [16] we see that $U(x, t) \geq 0$ for all x, $-L < x < L$ and for all $t \geq 0$. Moreover, an easy calculation shows that the equation is gradient-like with respect to the functional

$$\Phi(t) = \int_{-L}^{L} \left[\frac{1}{2} U_x^2 - F(U)\right] dx,$$

where $F' = f$. This means that $\Phi'(t) \leq 0$ when U is a solution and that $\Phi'(t) = 0$ only when U is a rest point of (2.1), (2.2), i.e., a steady-state solution. Therefore (cf. [15]), all solutions of (2.1) converge (uniformly in x) to a rest point. Thus, if $L < L_f$ the solution must converge to zero since that is the only available rest point. On the other hand, if $L > L_f$ then the strong maximum principle ([11] or [16]) shows that $U(x, t) > 0$ for $(x, t) \in (-L, +L) \times (0, \infty)$ and that $(\pm 1)U_x(\pm L, t) < 0$ for $t > 0$. Now let $\omega(x)$ denote a principal eigenfunction defined by

$$-\omega'' = \lambda\omega, \qquad -L < x < +L, \lambda = \frac{\pi^2}{4L^2},$$
$$\omega(\pm L) = 0.$$

It is well known ([8] or [13]) that we may choose $\omega(x) > 0$ for $-L < x < L$. Fix t_0 and choose $\delta > 0$ so that

$$\phi(x) \equiv \delta\omega(x) < U(x, t_0), \qquad -L < x < L.$$

According to Lemma 1.1, we can choose lowercase δ so that

$$a - \lambda \geq a\phi(x)$$

and thus

$$\begin{aligned} -\phi'' - f(\phi) &= \lambda\phi - a\phi(1 - \phi) \\ &= \phi(\lambda - a + a\phi) \leq 0 = U_t - U_{xx} - f(U). \end{aligned}$$

Since $\phi(\pm L) = U(\pm L, t) = 0$ it follows from the maximum principle that $\phi(x) \leq U(x, t)$ for (x, t) in $(-L, L) \times (t_0, \infty)$. (This comparison result can be found in [0].) *Since* $\phi(x) > 0$, this shows that $U(x, t)$ cannot converge to zero and therefore must converge to U_0, the only other rest point. This completes the proof.

We now apply this proposition to obtain stability results for steady-state solutions of the system $(0.1)_d$, (0.2), i.e., for solutions of $(1.2)_d$, (1.3). Our first result is a simple consequence of Proposition 2.1 and the comparison technique of [0].

THEOREM 2.2. *If* $L < L_f$ *then every nonnegative solution of* $(0.1)_d$, (0.2) *converges to* $(0, 0)$ *as* $t \to \infty$. *The convergence is uniform in* x.

Proof. Let $u(x, 0) = u_0(x) \geq 0$ and let U be the solution of (2.1), (2.2) with $U(x, 0) = u_0(x)$. Then

$$u_t - u_{xx} - f(u) = -uv \leq 0 = U_t - U_{xx} - f(U).$$

Since $u = U$ for $t = 0$ and for $x = \pm L$, it follows again from the comparison principle that $u(x, t) \leq U(x, t)$. From Proposition 2.1 we obtain that $u(t, x) \to 0$ as $t \to \infty$ uniformly in x. Therefore, there is a $T > 0$ such that $t \geq T$ implies $u(x, t) < \gamma$ for $|x| \leq L$. Consider first the case when $d = 0$. If $t \geq T$, we have

$$v_t = -v^2 + mv(u - \gamma) < -v^2,$$

from which it follows that $v(t, x) \to 0$ as $t \to \infty$ uniformly in x. Now consider the case of $d > 0$. Let V be the solution of

$$V_t = dV_{xx} - V^2, \qquad -L < x < L, t > T,$$

with boundary condition $V(\pm L, t) = 0$ and initial condition $V(x, T) = v(x, T)$. We have

$$v_t - dv_{xx} + v^2 = vm(u - \gamma) < 0 = V_t - dV_{xx} + V^2$$

so that $v(x, t) \leq V(x, t)$ for $t \geq T$ and $|x| \leq L$. The proof will be complete if we show that $V \to 0$ as $t \to \infty$. To see this, note that since $V \geq 0$ we have,

$$VV_t = dVV_{xx} - V^3 \leq dVV_{xx}$$

so that

$$\begin{aligned} \frac{d}{dt}\int_{-L}^{+L} V^2 dx \leq 2d\int_{-L}^{+L} VV_{xx} dx &= -2d\int_{-L}^{+L} V_x^2 dx \\ &\leq -2d\lambda\int_{-L}^{+L} V^2 dx, \end{aligned}$$

where λ is as in Lemma 1.1 and we have used the classical variational property of the first eigenvalue ([13]). It therefore follows that

$$\int_{-L}^{+L} |V(x,t)|^2 dx = 0(e^{-2d\lambda t}).$$

From this we obtain ([6] or [17]) that $V \to 0$ uniformly in x as $t \to \infty$.

Now in the case of $L > L_f$ we have seen that $(1.2)_d$, (1.3) admits the solution $(u_0(x), 0)$, where u_0 is the solution of (2.1)–(2.2). We now show that this solution attracts all solutions of the original system $(0.1)_d$, (0.2) with non-negative initial values (except $(0,0)$ of course). As a consequence it follows that $(0,0)$ becomes unstable as L crosses L_f; i.e., we have an "exchange of stability" between $(0,0)$ and $(u_0, 0)$.

THEOREM 2.3. *Let $(u(x,t), v(x,t))$ be any nonnegative solution of $(0.1)_d$–(0.2) with $u(x,0) \not\equiv 0$ on $(-L, L)$. If $L_f < L < L_\gamma$ then $(u(\cdot,t), v(\cdot,t))$ converges uniformly to $(u_0, 0)$ as $t \to \infty$.*

Proof. Let $U(x,t)$ be the solution of

$$U_t = U_{xx} + f(U), \qquad -L < x < L; \qquad U(\pm L, t) = 0,$$

such that $U(x,0) = u(x,0)$. Then, as in the proof of Theorem 2.2,

$$u_t - u_{xx} - f(u) = -uv \leq 0 = U_t - U_{xx} - f(U)$$

so that $u(x,t) \leq U(x,t)$ for $t \geq 0$ and $|x| < L$. From Proposition 2.1 it then follows that

$$\limsup_{t\to\infty} u(x,t) \leq u_0(x) < \gamma \tag{2.3}$$

uniformly in $(-L, L)$, where the second inequality holds since $L < L_\gamma$.

From (2.3) we see that there is a $T > 0$ such that $u(x,t) \leq \gamma$ for $t \geq T$ and $|x| \leq L$. We now show

$$\lim_{t\to\infty} v(x,t) = 0, \text{ uniformly in } -L \leq x \leq L. \tag{2.4}$$

First consider the case where $d = 0$. For $t \geq T$ we have

$$v_t = -v^2 + mv(u-\gamma) \leq -v^2$$

so that $v(x,t) \leq v(x,T)\,[1 + tv(x,T)]^{-1}$ for $t \geq T$. Thus (2.4) is valid in this case. Now assume $d > 0$. Then,

$$v_t - dv_{xx} + v^2 = mv(u-\gamma) \leq 0 = V_t - dV_{xx} + V^2, \qquad t \geq T,$$

where we assume $V(x, T) = v(x, T)$ and that $V(\pm L, t) = 0, t \geq 0$. As in the proof of Theorem 2.2 we obtain that $V(x, t) \to 0$ uniformly and thus (2.4) is verified in this case also.

Now for $0 < \epsilon < 1$, consider the problem

$$w'' + f(w) - \epsilon a w = 0, \qquad -L < x < L; w(\pm L) = 0. \tag{2.5}$$

Since $f_\epsilon(w) \equiv f(w) - \epsilon a w = aw(1 - \epsilon - w)$ we see that the phase portrait of the differential equation in (2.5) is exactly that pictured in Fig. 1 except that the critical point $(1, 0)$ must be changed to $(1 - \epsilon, 0)$. We thus see that analogous to L_f, there is a well-defined L_{f_ϵ} which converges to L_f as $\epsilon \to 0$, so that for sufficiently small ϵ we have $L_{f_\epsilon} < L$. On our interval, $-L < x < L$, we have a unique positive solution of (2.5), w_ϵ, and we see that $w_\epsilon \to u_0$ uniformly on $(-L, L)$ as $\epsilon \to 0$. Thus for any small $\delta > 0$, there is an $\epsilon > 0$ such that

$$w_\epsilon(x) \geq u_0(x) - \delta \qquad \text{for } |x| \leq L.$$

For this value of ϵ we see from (2.4) that there is a $T_\epsilon > 0$ such that $v(x, t) \leq \epsilon a$ for $t \geq T_\epsilon$ and $|x| \leq L$. Now define W^ϵ to be the solution of

$$\begin{gathered} W_t = W_{xx} + f(W) - \epsilon a W, \qquad t > T_\epsilon, |x| < L, \\ W(\pm L, t) = 0, \\ W(x, T_\epsilon) = u(x, T_\epsilon), \qquad |x| \leq L. \end{gathered}$$

For $t \geq T_\epsilon$ we have

$$u_t - u_{xx} - f(u) + \epsilon a u = u(a\epsilon - v) \geq 0 = W_t^\epsilon - W_{xx}^\epsilon - f(W^\epsilon) + \epsilon a W^\epsilon$$

so that, as before, $W^\epsilon(x, t) \leq u(x, t)$ for $t \geq T_\epsilon$. Since $u(x, T_\epsilon) \not\equiv 0$ we may apply Proposition 2.1 (or, to be precise, its analog for f replaced by f_ϵ) to conclude that $W^\epsilon(x, t) \to w_\epsilon(x)$ uniformly as $t \to \infty$ and hence that

$$\liminf_{t \to \infty} u(x, t) \geq w_\epsilon(x) \geq u_0(x) - \delta$$

uniformly for $|x| \leq L$. Since δ was arbitrary, this result, together with (2.3), shows that $u(\cdot, t)$ converges uniformly to u_0 and thus the proof of Theorem 2.3 is complete.

An examination of the proof shows that it yields the following corollary.

COROLLARY 2.4. *Let u and v be as in Theorem 2.3. For all $L > L_f$ we have*

$$\text{(i)} \ \limsup_{t \to \infty} u(x, t) \leq u_0(x) \qquad \text{uniformly on } [-L, L].$$

In the case $d = 0$ we also have

$$\text{(ii)} \quad \lim_{t \to \infty} v(t, x) = 0 \qquad \text{uniformly on } \{x \colon u_0(x) < \gamma\}.$$

B. Stability via Spectral Analysis

Theorem 2.3 shows that all nontrivial, nonnegative solutions of our system are in the domain of attraction of $(u_0, 0)$. Since we considered only nonnegative solutions, this does not, strictly speaking, prove the asymptotic stability of $(u_0, 0)$. The following discussion, which is based upon an analysis of the spectrum of the linearization of our problem, will provide a complete proof of stability. The two approaches are complementary, however, since this second approach yields no information concerning the domain of attraction.

We linearize our problem by considering the operator

$$A(d, L) \colon \begin{pmatrix} w \\ z \end{pmatrix} \to \begin{pmatrix} w'' + f'(u_0)w - u_0 z \\ dz'' + m(u_0 - \gamma)z \end{pmatrix} \tag{2.6}$$

from $C_0^{2+\alpha}(-L, +L) \times C_0^{2+\alpha}(-L, +L)$ into $C^\alpha \times C^\alpha$ for some $\alpha > 0$. The subscript zero on the domain spaces indicates that the functions z and w vanish at $x = \pm L$. (When $d = 0$, A operates from $C_0^{2+\alpha} \times C^\alpha$ into $C^\alpha \times C^\alpha$.) Our work will be based upon the following theorem:

THEOREM 2.5. *Let ν be the supremum of the real part of the spectrum of $A(d, L)$. If $\nu < 0$ then $(u_0, 0)$ is asymptotically stable in the topology of $C^\alpha(-L, +L)$. If $\nu > 0$ then $(u_0, 0)$ is unstable.*

This theorem, which asserts the validity of the "principle of linearized stability" in this context, is a special case of a result in [15] to which we refer for the proof.

In order to apply this theorem we must have more information concerning the spectrum of $A(d, L)$. It turns out that we are able to calculate it exactly. In preparation for this we recall the disjoint decomposition of the spectrum,

$$\Sigma = \Sigma_p \cup \Sigma_c \cup \Sigma_r,$$

where Σ_p denotes the point spectrum, Σ_c denotes the continuous spectrum and Σ_r denotes the residual spectrum. (For these notions we refer to [21]). We also need the following lemma.

LEMMA 2.6. A. *For all $L > L_f$, the operator*

$$G(L) \colon w \to w'' + f'(u_0)w$$

defined on $C_0^{2+\alpha}(-L, +L)$ *has pure point spectrum* $\mu_1 > \mu_2 > \dots$, *i.e., the continuous and residual spectra are empty. The eigenvalues,* μ_j, *are real, negative, and have no finite point of accumulation.*

B. *For all* $L > L_f$ *and* $d > 0$, *the operator*

$$H(d, L)\colon z \to dz'' + m(u_0 - \gamma)z$$

defined on $C_0^{2+\alpha}(-L, +L)$ *has pure point spectrum,* $\nu_1 > \nu_2 > \nu_3 > \dots$. *The eigenvalues* ν_j *are real, have no finite point of accumulation and*

$$\nu_1 \leq m(u_0(0) - \gamma). \tag{2.7}$$

If $L > L_\gamma$ *then* $\nu_1 > 0$ *for all sufficiently small values of* $d > 0$.

Proof. That these operators have pure point spectra is classical, (see [13] for the L^2 theory; [11] for the Schauder theory), as is the fact that the eigenvalues are real and have no finite point of accumulation. From Proposition 2.1 it follows that u_0 is an attractor for Eq. (2.1) so that $\mu_1 \leq 0$. But from [19] it follows that zero is not in the spectrum of G so that $\mu_1 < 0$ and the proof of Part A is complete.

Inequality (2.7) follows from the classical theory ([13]) since $m(u_0(0) - \gamma)$ is the maximum value of $m(u_0(x) - \gamma)$. To see that $\nu_1 > 0$ for $L > L_\gamma$ and d small we use the following variational characterization (cf. [13, Chap. IV])

$$\nu_1 = \sup_{\phi \in H_0^1(-L, +L)} \left\{ \frac{-d\int_{-L}^{+L} (\phi')^2 dx + \int_{-L}^{+L} q\phi^2 dx}{\int_{-L}^{+L} \phi^2} \right\}, \tag{2.8}$$

where we have denoted $m(u_0(x) - \gamma)$ by $q(x)$. For $L > L_\gamma$ we know that $u_0(0)$, the maximum value of u_0, is greater than γ so that $q(x) \geq 0$ for x in some interval, $-\mu \leq x \leq \mu$. If we choose a fixed function ϕ which is supported in $(-\mu, +\mu)$, then it is clear that the numerator in (2.8) will be positive for all sufficiently small $d > 0$. This completes the proof of the lemma.

We can now determine the spectrum of $A(d, L)$.

THEOREM 2.7. A. *If* $L > L_f$ *and* $d > 0$ *then* Σ_c *and* Σ_r *are empty and* $\Sigma = \Sigma_p = \{\mu_1, \mu_2, \dots \mu\} \cup \{\nu_1, \nu_2, \dots\}$.

B. *If* $L > L_f$ *and* $d = 0$ *then* Σ_c *is empty, and*

$$\sum_p = \{\mu_1, \mu_2, \dots\},$$

$$\sum_r = [-m\gamma, m(u_0(0) - \gamma)] \setminus \sum_p.$$

Remark. If we were considering A as an operator in L^2 rather than C^α then the roles of Σ_c and Σ_r in Part B would be reversed. That is, Σ_r would be empty while Σ_c would be $[-m\gamma, m(u_0(0)-\gamma)]\setminus\Sigma_p$. This will be shown in the remark following the proof.

Proof. First consider the case $d>0$. If λ is an eigenvalue of $A \equiv A(d, L)$ then

$$A\binom{w}{z} = \lambda\binom{w}{z};\ \binom{w}{z} \not\equiv \binom{0}{0}. \tag{2.9}$$

From the second component of this equation we see that if $z \not\equiv 0$ then λ is an eigenvalue of $H(d, L)$, i.e., $\lambda = \nu_j$ for some $j \geq 1$. On the other hand if $z \equiv 0$ then $w'' + f'(u_0)w = \lambda w$ so that $\lambda = \mu_j$ for some $j \geq 1$. Thus, Σ_p is contained in $\{\mu_1, \mu_2, \ldots\} \cup \{\nu_1, \nu_1, \ldots\}$. Conversely, if $\lambda = \mu_j$ for any j then $(w, 0)$ is an eigenfunction of A associated with λ. But if $\lambda = \nu_j$ for some j while $\lambda \neq \mu_i$ for any i, then, choosing z to be an eigenfunction of $H(d, L)$, we can clearly solve

$$w'' + f'(u_0)w - \lambda w = u_0 z,\ w(\pm L) = 0$$

for w. The element (w, z) is then seen to be an eigenfunction of A. Thus $\Sigma_p = \{\mu_1, \mu_2, \ldots\} \cup \{\nu_1, \nu_2, \ldots\}$.

To show that Σ_c and Σ_r are empty we will show that the resolvent set is $\mathbb{C} \setminus \Sigma_p$. Thus, for a complex $\lambda \notin \Sigma_p$ we must be able to solve

$$(A - \lambda I)\binom{w}{z} = \binom{g}{h} \tag{2.10}$$

for arbitrary g and h in C^α. But since $\lambda \neq \nu_j$, $j = 1, 2, \ldots,$ the second component,

$$dz'' + m(u_0 - \gamma)z = h,$$

can be solved uniquely for z with $\|z\|_{2+\alpha} = 0(\|h\|_\alpha)$. In the same way, since $\lambda \neq \mu_j$, $j \geq 1$, we can solve the first component,

$$w'' + f'(u_0)w - \lambda w = g + u_0 z,$$

for w. Moreover,

$$\|w\|_{2+\alpha} = O(\|g + u_0 z\|_\alpha) = O(\|g\|_\alpha + \|h\|_\alpha)$$

since u_0 is bounded. This shows that λ is in the resolvent set of A.

We now turn to the case $d = 0$. First, let λ be an eigenvalue of $A \equiv A(0, L)$. From the second component of (2.9),

$$m(u_0(x) - \gamma)z(x) = \lambda z(x),$$

it follows that $z(x) \equiv 0$ so that the first component becomes

$$w'' + f'(u_0)w = \lambda w.$$

Thus, $\lambda = u_j$ for some $j \geq 1$. Conversely, if w is an eigenfunction of $G(L)$ associated with some μ_j then it is clear that μ_j is an eigenvalue of A with associated eigenfunction $(w, 0)$. Thus, $\Sigma_p = \{\mu_1, \mu_2, \dots\}$.

Now let I denote the interval $[-m\gamma, m(u_0(0) - \gamma)]$. We first show that every complex number λ not in $\Sigma_p \cup I$ is an element of the resolvent set. To do this we must again solve the resolvent equation (2.10). Because I is precisely the range of $m(u_0(x) - \gamma)$, $|x| \leq L$, we may solve the second component of (2.10) for z in terms of h:

$$z(x) = h(x)[m(u_0(x) - \gamma) - \lambda]^{-1}.$$

From this we see that $\|z\|_\alpha = O(\|h\|_\alpha)$. But then the first component of (2.10) is a single elliptic equation,

$$w'' + f'(u_0)w - \lambda w = g + u_0 z.$$

Since $\lambda \neq \mu_j$ this is uniquely solvable for w and

$$\|w\|_{2+\alpha} = O(\|g + u_0 z\|_\alpha) = O(\|g\|_\alpha + \|h\|_\alpha)$$

Thus, the resolvent set contains $\mathbb{C} \setminus \Sigma_p \cup I$, i.e., $\Sigma \subset \Sigma_p \cup I$.

We complete the proof by showing that every point of I, unless it happens to belong to Σ_p, is a point in the residual spectrum, Σ_r. To this end let $\lambda \in I$. If $\lambda \notin \Sigma_p$ then $A - \lambda I$ is one-to-one but it is easy to see that its range is not dense. Namely, there must be a point $\bar{x}$, $|\bar{x}| \leq L$, such that $m(u_0(\bar{x}) - \gamma) = \lambda$ so that if

$$(A - \lambda I)\binom{w}{z} = \binom{g}{h}$$

then h must vanish at $\bar{x}$. Such functions (g, h) are not dense in $C^\alpha \times C^\alpha$ so $\lambda \in \Sigma_r$ and the proof of the theorem is complete.

Remark. Note that if we were considering $A \equiv A(0, L)$ as an operator in $L_2 \times L_2$ (domain $= H_0^1 \times L^2$) then the set $I \setminus \Sigma_p$ would not be residual spectrum because the range of $(A - \lambda I)$ is dense in this space. In this case $I \setminus \Sigma_p = \Sigma_c$ and Σ_r is empty. To show this, we must show $(A - \lambda I)^{-1}$ is unbounded as an operator in $L^2 \times L^2$. (It is also unbounded in $C^\alpha \times C^\alpha$.) To this end let λ be in $I \setminus \Sigma_p$ and let $\bar{x}$ be such that $q(\bar{x}) \equiv m(u_0(\bar{x}) - \gamma) - \lambda = 0$. Then, for every sufficiently large $M > 0$, we can find z_M in L^2 such that

$$\|z_M\| \geq cM\|z_M q\|,$$

where c is some constant and the norms are those of L^2. (For example, let the graph of z_M be a triangle of height M supported on $[\bar{x} - 1/M, \bar{x} + 1/M]$ and use the fact that $|q(x)| \leq \text{const.}|x - \bar{x}|$). Then consider the problem

$$w'' + f'(u_0)w - \lambda w = u_0 z_M, \; w(\pm L) = 0.$$

Since $\lambda \neq \mu_j$, there is a unique solution w_M in H_0^1. Therefore,

$$\begin{aligned} \|w_M\| + \|z_M\| > \|z_M\| \geq cM\|z_M q\| = cM\|z_M q\| \\ +M\|w_M'' + f'(u_0)w_M - u_0 z_M - \lambda w_M\|. \end{aligned}$$

This shows that $(A - \lambda I)^{-1}$ is unbounded, i.e., λ is in the continuous spectrum of A.

We can now prove the stability of $(u_0, 0)$.

THEOREM 2.8. *If $L_f < L < L_\gamma$, then $(u_0, 0)$ is asymptotically stable.*

Proof. Since $L < L_\gamma$ it follows that $u_0(0) < \gamma$. From Theorem 2.7 and Lemma 2.6 it then follows that the spectrum of $A(d, L)$ is contained in $(-\infty, 0)$. The result therefore follows from Theorem 2.5.

3. Loss of Stability and Secondary Bifurcation

We turn now to the case where $L > L_\gamma$. In this regime the solution $(u_0, 0)$ becomes unstable for values of d less than a critical value $\bar{d} = \bar{d}(L)$. As d decreases across this value there is a bifurcation to a solution (u_1, v_1), where $v_1 > 0$. When $d = 0$ the solution $(u_0, 0)$ becomes unstable as L increases across L_γ and does so in a very interesting manner. Namely, the point spectrum of the linearized operator A (see (2.6)) remains negative while the residual (continuous) spectrum becomes positive. This should indicate that the unstable manifold about $(u_0, 0)$ is infinite dimensional, and indeed, we are able to construct a continuum of (generalized) steady-state solutions bifurcating out of $(u_0, 0)$ as L increases past L_γ.

We begin by summarizing the instability results which follow directly from the spectral results of the preceeding section.

THEOREM 3.1. *If $L > L_\gamma$, then for all sufficiently small $d \geq 0$ the function $(u_0, 0)$ is unstable as an equilibrium solution of* $(0.1)_d$, $(0, 2)$. *In fact, there are two possibilities; namely*

(A) *If $d > 0$ is sufficiently small then the largest eigenvalue is positive.*

(B) *If $d = 0$ then the residual spectrum contains an interval of positive numbers.*

Proof. When $L > L_\gamma$ then $u_0(0) > \gamma$ so that this theorem is an immediate consequence of Theorem 2.5, Lemma 2.6, and Theorem 2.7.

The following theorem also shows that $(u_0, 0)$ is unstable but it yields more precise information, namely, that the unstable manifold contains all nonnegative solutions (u, v) for which $v(0,0) > 0$.

THEOREM 3.2. *If $L > L_\gamma$ then $(u_0, 0)$ is unstable as a solution of* $(0.1)_{\rm d}$, (0.2) *for all sufficiently small $d \geq 0$. In particular, for no nonnegative solution $(u(x,t), v(x,t))$ can the function v stay uniformly close to zero as $t \to \infty$.*

Proof. We first consider the case $d = 0$. For all sufficiently small $\epsilon > 0$ let U^ϵ be the unique positive solution of

$$U_{xx} + f(U) - \epsilon U = 0, \qquad U(\pm L) = 0.$$

Since $f(U) - \epsilon U = aU(1 - \epsilon/a - U)$, the existence and uniqueness of U^ϵ follow from an analog of Lemma 1.2 just as in the proof of Theorem 2.3. From continuity, it is clear that $U^\epsilon \to u_0$ uniformly as $\epsilon \to 0$. Again, as we argued in the proof of Theorem 2.3, an analog of Proposition 2.1 shows that U^ϵ is the uniform limit, as $t \to \infty$, of every nonnegative solution of

$$W_t = W_{xx} + f(W) - \epsilon W, \qquad W(\pm L, t) = 0 \tag{3.1}$$

provided $W \not\equiv 0$.

Now choose ϵ so small that

$$U^\epsilon(0) - \gamma > \epsilon(1 + 4/m). \tag{3.2}$$

This is possible since $U^\epsilon(0) \to u_0(0)$ as $\epsilon \to 0$ and $u_0(0) > \gamma$ because $L > L_\gamma$. If $(u_0, 0)$ were stable then all solutions which at $t = 0$ were sufficiently close to $(u_0, 0)$, would stay within an ϵ-neighborhood of $(u_0, 0)$. On the contrary, however, we show that the assumptions

$$0 \leq v(x,t) \leq \epsilon, \qquad |x| \leq L, t > T, \tag{3.3}$$

and $v(0,0) > 0$, leads to a contradiction. To see this, let W^ϵ be the solution of (3.1) in $t > T$ such that $W(x, T) = u(x, T)$, $|x| < L$. Then for $t > T$ we have

$$u_t - u_{xx} - f(u) + \epsilon u = \epsilon u - uv \geq 0 = W_t^\epsilon - W_{xx}^\epsilon - f(W) + \epsilon W^\epsilon$$

so that $u(x,t) \geq W^\epsilon(x,t)$ for $t \geq T$, $|x| \leq L$. Since $W^\epsilon \to U^\epsilon$ uniformly as $t \to \infty$ we see that there is a $T_\epsilon > T$ such that

$$u(x,t) \geq U^\epsilon(x) - \epsilon, \qquad t \geq T_\epsilon, |x| \leq L. \tag{3.4}$$

Since $d = 0$ we have

$$v_t = v[-v + m(u - \gamma)]$$

so that, by the uniqueness theorem for ODE's if $v(0, t) > 0$ for $t = 0$ then $v(0, t) > 0$ for all $t > 0$. If we let $V(t) = v(0, t)$ then from (3.4) and (3.2) we have

$$\begin{aligned}\dot{V} &= V[-V + m(u(0, t) - \gamma)]\\ &\geq V[-V + m(U^\epsilon(0) - \epsilon - \gamma)]\\ &\geq V[-V + 4\epsilon]\end{aligned}$$

for all $t \geq T$. But because of (3.3) this would imply that

$$1 \leq \frac{\dot{V}}{V(4\epsilon - V)} = \frac{1}{4\epsilon}\left(\frac{\dot{V}}{V} + \frac{\dot{V}}{4\epsilon - V}\right),$$

from which we get

$$v(0, t) \geq \frac{4\epsilon K}{K + \exp(-4\epsilon(t - T_\epsilon))}, \qquad t \geq T_\epsilon,$$

where $K = V(T_\epsilon)/(4\epsilon - V(T_\epsilon))$. Since the right-hand side approaches 4ϵ as $t \to \infty$, this clearly contradicts our assumption (3.3).

Now we consider the case $d > 0$. From (3.2) we may choose δ so small that

$$U^\epsilon(x) - \gamma > \epsilon\left(1 + \frac{2}{m}\right) \qquad \text{for } |x| \leq \delta. \tag{3.5}$$

We shall again show that the assumption (3.3) leads to a contradiction. We define W^ϵ as in the case $d = 0$, and we again have (3.4).

Define V to be the solution of

$$V_t = dV_{xx} + V(-V + 2\epsilon) = 0, \qquad t > T_\epsilon, |x| \leq \delta, \tag{3.6}$$

satisfying

$$V(\pm\delta, t) = 0, \qquad V(x, T_\epsilon) \leq v(x, T_\epsilon) \qquad \text{for } |x| < \delta.$$

For $t > T_\epsilon$ and $|x| < \delta$ we see from (3.4) and (3.5) that

$$\begin{aligned} v_t - dv_{xx} - v(-v + 2\epsilon) &= mv(u - \gamma) - 2\epsilon v\\ &\geq mv\left[U^\epsilon - \gamma - \epsilon - \frac{2\epsilon}{m}\right]\\ &> 0 = V_t - dV_{xx} - V(-V + 2\epsilon).\end{aligned}$$

Because $d > 0$, the maximum principle assures us that $v > 0$ for all x, $|x| < L$, and in particular for $x = \pm\delta$. Thus by the basic comparison result of [0] we have

$$v(x, t) > V(x, t), \qquad |x| \leq \delta, t > T_\epsilon. \tag{3.7}$$

We now examine the asymptotic behavior of V. The steady-state equation corresponding to (3.6) is

$$dV'' + V(2\epsilon - V) = 0, \qquad V(\pm\delta) = 0. \tag{3.8}$$

The phase plane for this equation is as portrayed in Fig. 4 and in fact the change of scales $x \to x/\sqrt{d}$, $V \to V/2\epsilon$ transforms this equation into our original (1.4) with $a = 2\epsilon$. Thus we see that for all sufficiently small $d > 0$ there is a unique solution V_0 of (3.8) such that $V_0(x) > 0$ for $|x| < \delta$. Moreover, V_0 is the uniform limit, as $t \to \infty$, of $V(x, t)$. If we further choose d small enough to force $V_0(0) > \epsilon$ (cf. Fig. 4) then (3.7) yields $v(0, t) > \epsilon$ for all sufficiently large t. This is the desired contradiction, which completes the proof of Theorem 3.2.

Corresponding to the loss of stability of $(u_0, 0)$ we expect a bifurcation of new solutions from $(u_0, 0)$ and it is to this question that we now turn. We first discuss the case of $d = 0$.

A. Bifurcation when $d = 0$

We have shown in Section 1 that, if $d = 0$ and $L > L_\gamma$, there is a new equilibrium solution (u_1, v_1). This new solution is a bifurcation from $(u_0, 0)$ since $u_1 \to u_0$ and $v_1 \to 0$ as $L \to L_\gamma$. However, from Theorem 3.1 we know that for all $L > L_\gamma$ there is a continuum of spectral points that are positive. From this we might expect a continuum of new equilibrium solutions bifurcating from $(u_0, 0)$. This is indeed the case if we generalize the notion of solution of $(1.2)_d$–(1.3) to allow discontinuous functions v when $d = 0$. This is reasonable since, when $d = 0$, the second equation of $(0.1)_d$ is simply a family of ordinary differential equations parametrized by x. Thus, by a *generalized solution* of $(1.2)_0$–(1.3) we shall mean a pair of functions (u, v) such that v is bounded and piecewise continuous while u is C^1 with a

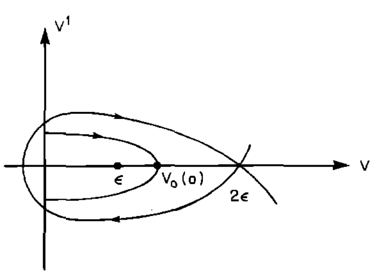

FIGURE 4

piecewise continuous second derivative, and $(1.2)_0$–(1.3) is satisfied except at points of discontinuity.

THEOREM 3.3. *Let J be a finite union of open subintervals of* $(-L, +L)$. *Then there exists a generalized solution* (u_J, v_J) *of* $(1.2)_0$, (1.3), *with the following properties.*

(A) $u_1(x) \le u_J(x) \le u_0(x)$ for $|x| \le L$.

(B) $v_J = mI_J P(u_J - \gamma)$,

(C) $J' \subset J$ implies $u_{J'}(x) \ge u_J(x)$ for $|x| \le L$.

Remarks. (1) I_J is the indicator function of the set J, i.e., $I_J(x) = 1$ if $x \in J$ and $I_J(x) = 0$ otherwise.

(2) $P(y)$ is the positive part of y, i.e., $P(y) = \max\{0, y\}$.

(3) Since $L > L_\gamma$ it follows that $u_J \ge u_1 > \gamma$ on the interval $(-\alpha, +\alpha)$ cf. Fig. 3. Thus $v_J(x) > 0$ for x in $J \cap (-\alpha, \alpha)$.

(4) If J is an interval whose closure is contained in $(-\alpha, \alpha)$ then it will be clear from (B) and the proof below that v_J and u_J'' will suffer jump discontinuities at the end points of J but will be continuous elsewhere. This shows that we do indeed have a continuum of *distinct* steady-state solutions.

(5) The solutions (u_J, v_J) must be unstable in any reasonable norm, but a rigourous discussion of stability would be extremely complicated since these solutions are not isolated. Computer simulation indicates, however, that each such solution has a robust stable manifold. On the basis of these calculations we conjecture that (u_J, v_J) is an attractor for every solution of the system $(0.1)_0$–(0.2) with initial values (u_0, v_0) satisfying $u_0(x) \ge 0$, $v_0(x) \equiv 0$ for $x \notin J$, $v_0(x) > 0$ for $x \in J$.

(6) It will be clear from the construction that when $J = (-L, +L)$, $(u_J, v_J) = (u_1, v_1)$.

Proof of Theorem 3.3. The idea is to construct u as a solution of

$$u'' + f(u) - uI_J mP(u - \gamma) = 0, \qquad |x| < L;\ u(\pm L) = 0. \quad (3.9)$$

We do this by solving the "regularized" equation

$$u'' + f(u) - uI_J^\epsilon mP(u - \gamma) = 0, \qquad |x| < L;\ u(\pm L) = 0, \quad (3.10)$$

where $I_J^\epsilon = \omega_\epsilon * I_J$ and ω_ϵ is the Gaussian averaging kernel (mollifier) of support $|x| \le \epsilon$. Equation (3.10) is solved by the method of upper and lower solutions (cf. [18]), and we then pass to the limit as $\epsilon \downarrow 0$.

Let $A_J(u)$ denote the left-hand side of (3.10). Observe that

$$A_J(u_0) = -u_0 I_J^\epsilon mP(u_0 - \gamma) \le 0$$

so that u_0 is an upper solution of (3.10). If we let I denote the indicator function of $(-\alpha, +\alpha)$ (cf. Fig. 3) then we see that u_1 satisfies

$$u_1'' + f(u_1) - u_1 ImP(u_1 - \gamma) = 0$$

and that $P(u_1 - \gamma) = IP(u_1 - \gamma)$. Therefore,

$$\begin{aligned} A_J(u_1) &= Iu_1 mP(u_1 - \gamma) - I_J^{\epsilon} u_1 mP(u_1 - \gamma) \\ &= (1 - I_J^{\epsilon}) Iu_1 mP(u_1 - \gamma) \geq 0 \end{aligned}$$

so that u_1 is a lower solution of (3.10). It therefore follows from [18] that there is a $C^{2+\alpha}$ solution, u_ϵ, of the regularized problem (3.10), which satisfies

$$u_1(x) \leq u_\epsilon(x) \leq u_0(x), \qquad |x| \leq L. \tag{3.11}$$

From (3.10) we see that

$$u_\epsilon'(x) - u_\epsilon'(-L) = \int_{-L}^{x} [-f(u_\epsilon) + u_\epsilon I_J^{\epsilon} mP(u_\epsilon - \gamma)]\, dx \tag{3.12}$$

so that $\{u_\epsilon', \epsilon > 0\}$, as well as $\{u_\epsilon, \epsilon > 0\}$ are uniformly bounded on $[-L, +L]$. Finally, it also follows from (3.10) that the second derivatives, u_ϵ'', are also uniformly bounded in ϵ. Thus, standard compactness arguments yield the existence of a sequence of values $\{\epsilon_j, j = 1, 2, \ldots\}$ converging to zero and a C^1 function, u_J, having Lipschitz continuous second derivatives such that $u_{\epsilon_j} \to u_J$ and $u_{\epsilon_j}' \to u_J'$ uniformly in $|x| \leq L$. Passing to the limit in (3.12) we see that u_J satisfies (3.10) at every point except, perhaps, the end points of the constituent intervals of J, where u_J'' may suffer jump discontinuities.

If we now define v_J by $v_J = mI_J P(u_J - \gamma)$ then the pair (u_J, v_J) is a generalized solution of $(1.2)_0$–(1.3) and (A) and (B) are clearly satisfied.

To prove (C), let v_ϵ denote a solution of (3.10) with J replaced by J'. We then have

$$A_J(v) = (I_{J'}^{\epsilon} - I_J^{\epsilon}) mv_\epsilon P(v_\epsilon - \gamma) \leq 0$$

since $J' \subset J$. Thus v_ϵ is an upper solution of (3.10) for u_ϵ. Part (C) follows immediately from this fact and thus completes the proof of the theorem.

B. Bifurcation Analysis for $d > 0$

In the case of positive d we shall first make our result on the loss of stability more precise and then show bifurcation using the results of [8].

We begin by improving Lemma 2.6(B) concerning the operator $H(d, L)$: $z \to dz'' + m(u_0 - \gamma)z$. Recall that this operator has a real spectrum consisting only of eigenvalues. We let $\nu(d, L)$ denote the maximum eigenvalue.

LEMMA 3.3. *There is a smooth function, $\bar{d}$, defined on $[L_\gamma, \infty)$ such that $\bar{d}(L_\gamma) = 0$, $\bar{d}' > 0$ and $\nu(\bar{d}(L), L) = 0$ for all $L > L_\gamma$. Furthermore, $\nu(d, L) < 0$ for $d > \bar{d}(L)$ and $\nu(d, L) > 0$ for $d < \bar{d}(L)$.*

Proof. It is well known (cf. [13]) that ν is nondegenerate and, if $v = v(x; d, L)$ is the unique principal eigenfunction which is positive and has unit L^2 norm, that v and ν are smoothly dependent upon the parameters d and L. It is also known that

$$\frac{\partial \nu}{\partial d} \leq 0 \tag{3.13}$$

and, in fact, we show below that a somewhat stronger result is true. Now, if we let $q(x)$ denote $m(u_0(x) - \gamma)$ then since v satisfies the equation $dv'' + qv = \nu v$ it follows that

$$\begin{aligned} \nu &= -d\int_{-L}^{+L} (v')^2 dx - \int_{-L}^{+L} qv^2 dx \\ &\leq -dC + \max|q|, \end{aligned} \tag{3.14}$$

where we have used the fact that $\|v\|_{L^2} \leq C\|v'\|_{L^2}$. Since $|q|$ is bounded independently of d it follows that $\nu < 0$ for all sufficiently large values of d. On the other hand, from Lemma 2.6 we know that $\nu > 0$ for values of d sufficiently near zero. Thus there is a value $\bar{d}$ for which $\nu(\bar{d}, L) = 0$. If we now let $w = \partial v/\partial d$ we have

$$dw'' + v'' + qw = \nu w + \frac{\partial \nu}{\partial d} v.$$

Since $w(\pm L) = 0$,

$$\begin{aligned} \frac{\partial \nu}{\partial d} &= \int_{-L}^{+L} (dw''v + qwv - \nu wv + v''v)\, dx \\ &= \int_{-L}^{+L} w\{dv'' + qv - \nu v\}\, dx - \int_{-L}^{+L} (v')^2 dx \\ &= -\int_{-L}^{+L} (v')^2 dx < 0. \end{aligned}$$

Thus, there is only one value of d, $\bar{d}$, for which $\nu(d, L) = 0$ and from the implicit function theorem we obtain that $\bar{d}(L)$ is a smooth function of L.

However, an argument similar to the one above yields

$$\frac{\partial \nu}{\partial L} = \int_{-L}^{+L} \frac{\partial u_0}{\partial L} (v)^2 dx + 2(v'(L))^2,$$

where we have used the fact that u_0 and thus v, is symmetric about $x = 0$. Now, an elementary analysis shows that $\partial u_0/\partial L$ is positive so that $\partial \nu/\partial L$, and thus $\bar{d}'$, is positive since $\bar{d}'(\partial \nu/\partial d) + (\partial \nu/\partial L) = 0$. This completes the proof of the lemma.

With the aid of this lemma we are able to prove the following theorem.

THEOREM 3.4. *If $L \geq L_\gamma$ and $d > \bar{d}(L)$ then $(u_0, 0)$ is a global attractor for all non-negative solutions of* $(0.1)_{\text{d}}$, (0.2). *If $L \geq L_\gamma$ and $0 < d < \bar{d}(L)$ then $(u_0, 0)$ is unstable as a steady-state solution of* $(0.1)_{\text{d}}$, (0.2).

Proof. If $0 < d < \bar{d}(L)$ it follows from Lemma 3.3 and Theorem 2.7(A) that the spectrum of the linearization $A(d, L)$ has positive elements. The instability of $(u_0, 0)$ follows from Theorem 2.5.

If $d > \bar{d}(L)$ then from Corollary 2.4 we see that for $\epsilon > 0$ there is a $T > 0$ such that $u(x, t) \leq u_0(x) + \epsilon$ for $|x| \leq L$ and $t \geq T$. Now, if in the second equation of $(0.1)_{\text{d}}$ we multiply by v, integrate over $(-L, +L)$, and use (2.8), we get

$$\begin{aligned}\frac{d}{dt}\int_{-L}^{+L} v^2 dx &= -d\int_{-L}^{+L} (v')^2 dx + \int_{-L}^{+L} [-v^3 + mv^2(u - \gamma)]\, dx \\ &\leq -d\int_{-L}^{+L} (v')^2 + m\int_{-L}^{+L} v^2(u_0 - \gamma) dx + m\epsilon\int_{-L}^{+L} v^2 dx \\ &\leq (\nu_1 + m\epsilon)\int_{-L}^{+L} v^2 dx\end{aligned}$$

for $t \geq T$. Because $d > \bar{d}(L)$ we see that $\nu_1 + m\epsilon < 0$ for all sufficiently small ϵ. Moreover, from Gronwall's inequality we obtain

$$\int_{-L}^{+L} [v(t, x)]^2 dx \leq \text{const.}\, e^{(\nu_1 + m\epsilon)t},$$

from which it follows that $v(t, \cdot) \to 0$ in L^2 as $t \to \infty$. From this, as was shown in [6] or [17], it follows that $v(t, x)$ converges to zero uniformly in $[-L, +L]$ as $t \to \infty$. Thus, the argument used in the proof of Theorem 2.3 can be applied to show that $u(t, x)$ converges uniformly to u_0 in $[-L, +L]$ as $t \to \infty$. This completes the proof of Theorem 3.4.

The next theorem shows that the stability lost by $(u_0, 0)$ as d decreases across $\bar{d}$ is picked up by a new branch of nonnegative solutions bifurcating from $(u_0, 0)$.

THEOREM 3.5. *There is a* $\delta > 0$ *such that for all* $d \in (\bar{d}(L), \bar{d}(L) - \delta)$, *there is a steady-state solution* (u_1, v_1) *of* $(0.1)_d$, (0.2) *having the following properties*:

(A) $0 < u_1 < u_0$ *and* $0 < v_1$ *on* $(-L, +L)$,

(B) (u_1, v_1) *converges to* $(u_0, 0)$ *as* $d \uparrow \bar{d} \equiv \bar{d}(L)$,

(C) (u_1, v_1) *is asymptotically stable*,

(D) (u_1, v_1) *is the only other steady-state solution near* $(u_0, 0)$.

Proof. Our proof is based upon Lemma 1.1 and Theorem 1.7 of [8]. We fix L and define the operator F: $C_0^2 \times C_0^2 \times (\bar{d} - \epsilon, \bar{d} + \epsilon) \to C$ by

$$F(u, v; d) = \begin{bmatrix} u'' + f(u) - uv \\ dv'' - v^2 + mv(u - \gamma) \end{bmatrix}.$$

We then have $F(u_0, 0; d) = 0$ for all $d > 0$. To show bifurcation at $d = \bar{d}$ we consider the linearization of F about $(u_0, 0)$:

$$F_U(u_0, 0; d)\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} w'' + f'(u_0)w - u_0 z \\ dz'' + m(u_0 - \gamma) \end{bmatrix}. \tag{3.15}$$

U denotes the generic point (u, v). As we saw in Theorem 2.7 (where F_U was referred to as A) the spectrum of F_U is purely discrete and is $\{\mu_1, \mu_2, \dots\} \cup \{\nu_1, \nu_2, \dots\}$. For all positive values of d the μ_j's are negative and $\nu_j < \nu_1$. As shown in the preceding discussion (Lemma 3.3), the principal eigenvalue, ν_1, is a strictly monotone decreasing function of d and $\nu_1(\bar{d}) = 0$.

According to Lemma 1.1 of [8], to show bifurcation at $d = \bar{d}$ we must demonstrate the following three claims:

(α) the null space of $F_U(u_0, 0; \bar{d})$ has dimension one.

(β) the range of $F_U(u_0, 0; \bar{d})$ has codimension one.

(γ) if $[\bar{w}, \bar{z}]^t$ is the eigenfunction of $F_U(u_0, 0; \bar{d})$ that corresponds to $\nu_1(\bar{d}) = 0$ then

$$F_{Ud}(u_0, 0; \bar{d})\begin{bmatrix} \bar{w} \\ \bar{z} \end{bmatrix} \notin \text{Range}\,\{F_U(u_0, 0; \bar{d})\},$$

where F_{Ud} is the mixed second Fréchet derivative with respect to U and d.

To prove these claims note first that $\bar{w}$ and $\bar{z}$ satisfy

$$\begin{aligned} \bar{w}'' + f'(u_0)\bar{w} &= u_0 \bar{z} \\ \bar{d}\bar{z}'' + m(u_0 - \gamma)\bar{z} &= 0 \end{aligned}, \qquad \bar{w}(\pm L) = \bar{z}(\pm L) = 0. \tag{3.16}$$

We thus see that $\bar{z}$ is a principal eigenfunction of the scalar operator

$\bar{d}D^2 + m(u_0 - \gamma)$. It is a classical result that this eigenspace is one dimensional and that we may choose $\bar{z} > 0$ on $(-L, +L)$. Again using the result of [19] that zero is not in the spectrum of $D^2 + f'(u_0)$ we see that for each $\bar{z}$ there is a uniquely determined $\bar{w}$. This proves claim (α).

If (g, h) is in the range of $F_U(u_0, 0; \bar{d})$ then there must be a solution (w, z) of the following system:

$$\begin{matrix} w'' + f'(u_0)w = u_0 z + g \\ \bar{d}z'' + m(u_0 - \gamma)z = h \end{matrix}, \qquad w(\pm L) = z(\pm L) = 0.$$

From classical results concerning one equation we see that there is a solution of the second equation if and only if $\langle \bar{z}, h \rangle = 0$, where $\langle\ \rangle$ denotes the inner product in $L^2(-L, +L)$. For each such solution, z, and for any g there is a uniquely determined solution w, where again we use the results of [19]. Thus the range consists of all pairs (g, h) which are orthogonal in L^2 to $(0, \bar{z})$, and therefore claim (β) is true.

Finally, note that

$$F_{Ud}(u_0, 0; \bar{d})\begin{bmatrix} \bar{w} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & D^2 \end{bmatrix}\begin{bmatrix} \bar{w} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{z}'' \end{bmatrix}.$$

But,

$$\left\langle \begin{bmatrix} 0 \\ \bar{z}'' \end{bmatrix}, \begin{bmatrix} 0 \\ \bar{z} \end{bmatrix} \right\rangle = \int_{-L}^{+L} \bar{z}''\bar{z}\, dx = -\int_{-L}^{+L} (\bar{z}')^2\, dx \neq 0.$$

In view of the description of the range given in the preceding paragraph, this demonstrates the validity of (γ).

We now can apply Lemma 1.1 of [8] to obtain functions $\phi(s)$, $\psi(s)$ and $d(s)$, defined for all real s such that $|s|$ is sufficiently small, and these functions have the following properties:

(i) $d(0) = \bar{d}$.

(ii) $\phi(s)$, $\psi(s)$ are in C_0^2 and $\phi(0) \equiv \psi(0) \equiv 0$ on $(-L, +L)$.

(iii) $F(u_1(s), v_1(s); d(s)) \equiv 0$, where

$$u_1(s) = u_0 + s(\bar{w} + \phi(s)), \qquad v_1(s) = s\bar{z} + s\psi(s). \tag{3.17}$$

(iv) (u_1, v_1) is the only other solution of $F(u, v; d(s)) = 0$ in a neighborhood of $(u_0, 0)$.

Since we have chosen $\bar{z} > 0$ we see that $v_1(s) > 0$ for $s > 0$. It is also clear that $u_1 \geq 0$ for all sufficiently small s. But from Corollary 2.4 we know that every nonnegative solution (u, v) of $(0.1)_d$, (0.2) must satisfy

$$\limsup_{t\to\infty} u(x, t) \leq u_0(x).$$

We thus see that for $s > 0$ we have $u_1(x) \leq u_0(x)$. Using (3.17), this in turn implies that $\bar{w}(x) \leq 0$. In fact, the following stronger statement is true; namely,

$$\bar{w}(x) < 0 \qquad \text{for } |x| < L. \tag{3.18}$$

This holds because if $\bar{w}(\bar{x}) = 0$, $|x| \neq L$, then from (3.16) we have

$$\bar{w}''(\bar{x}) = u_0(\bar{x})\bar{z}(\bar{x}) > 0,$$

which contradicts the fact that zero is the maximum value of $\bar{w}$ on $[-L, +L]$. The fact that $\bar{w}$ is negative will allow us to show that $d(s) < \bar{d}$ for $s > 0$ and therefore that (u_1, v_1) is stable.

To this end let K denote the injection of C^2 into C. Then

$$\left\langle K\begin{bmatrix} \bar{w} \\ \bar{z} \end{bmatrix}, \begin{bmatrix} 0 \\ \bar{z} \end{bmatrix} \right\rangle = \int_{-L}^{+L} \bar{z}^2 dx > 0,$$

so that $K(\bar{w}, \bar{z})^t \notin \text{Range}\{F_U(u_0, 0; \bar{d})\}$. Thus $\nu_1(\bar{d}) = 0$ is a "K-simple" eigenvalue of $F_U(u_0, 0; \bar{d})$ (cf [8]). Thus, if $\pi(s)$ denotes the principal eigenvalue of the linearization of F about $(u_1(s), v_1(s))$ with $d = d(s)$, then $\pi(0) = \nu_1(\bar{d}) = 0$ and (Theorem 1.7 of [8]) $\pi(s)$ has the same sign as

$$-sd'(s)\frac{\partial \nu_1}{\partial d}(\bar{d}).$$

But as we saw in the proof of Lemma 3.3, $\partial \nu_1/\partial d$ is negative so that $\pi(s)$ has the same sign as $sd'(s)$. Thus, if $d'(0) < 0$ then $(u_1(s), v_1(s))$ is asymptotically stable for $s > 0$, and we may choose d as the bifurcation parameter rather than s. We see then that the proof of Theorem 3.5 will be complete once we show that $d'(0) < 0$. We do this by directly computing $d'(0)$.

To this end let us consider the second equation satisfied by (u_1, v_1):

$$d(s)v_1'' - v_1^2 + mv_1(u_1 - \gamma) = 0.$$

Substitute here the expressions for u_1 and v_1 given in (3.17), divide by s, differentiate with respect to s, and finally set $s = 0$. The result is

$$d'(0)\bar{z}'' + \bar{d}\psi_s'' - \bar{z}^2 + m\psi_s(u_0 - \gamma) + m\bar{z}\bar{w} = 0,$$

where ψ_s is the derivative of ψ with respect to s and all quantities are evaluated at $s = 0$. If we now multiply by $\bar{z}$ and then integrate we obtain,

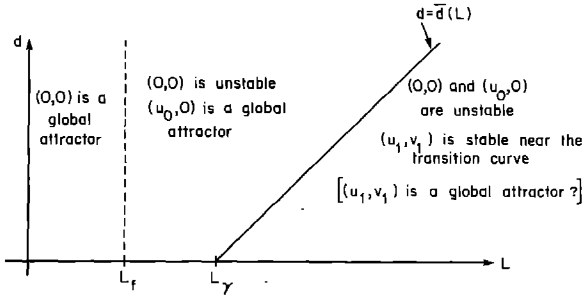

FIGURE 5

after several "integration-by-parts" and using (3.16),

$$d'(0)\int_{-L}^{+L}(\bar{z}')^2\,dx = \int\{\bar{d}\psi_s''\bar{z} + m\psi_s\bar{z}(u_0-\gamma)\} - \int\bar{z}^3 + \int m\bar{z}^2\bar{w}$$

$$= \int\psi_s\{\bar{d}\bar{z}'' + m\bar{z}(u_0-\gamma)\} - \int\bar{z}^3 + m\int\bar{z}^2\bar{w}$$

$$= -\int\bar{z}^3 + m\int\bar{z}^3\bar{w},$$

where all integrals are over the interval $|x| \le L$. Thus,

$$d'(0) = \frac{-\int z^3 + m\int\bar{z}^2\bar{w}}{\int(\bar{z}')^2}. \tag{3.19}$$

Since $\bar{w} < 0$, and $\bar{z} > 0$, this shows that $d'(0) < 0$ and completes the proof of Theorem 3.5.

We can now summarize our discussion of nonnegative steady-state solutions by the bifurcation diagram in Fig. 5.

CONJECTURE. On the basic of extensive computer simulation we believe that (u_1, v_1) is defined throughout $0 < d < \bar{d}(L)$; that it is a global attractor for nonnegative solutions of $(0.1)_d$, (0.2); that it is the only nonnegative steady-state solution other than $(0,0)$ and $(u_0,0)$; and that as $d \downarrow 0$, it converges to the compactly supported steady-state solution constructed in Section 1 (cf. Theorem 1.3 and Fig. 3).

II. Cubic Nonlinearity

In Sections 4 and 5 we shall consider our Eqs. $(0.1)_d$, (0.2) under the assumption that f is a cubic polynomial. In 4 we assume that $f(u) =$

$au(u-b)(1-u)$ so that the prey species is an "asocial" population (cf. [2]). In Section 5 we briefly discuss other cubic polynomials in order to indicate the richness of the structure of the bifurcations. The solution set in the cubic case differs from that in the quadratic case primarily in the presence of more than one stable equilibrium solution. We shall choose the parameters to satisfy certain general conditions, so as to contrast the differences between the cubic and quadratic cases. Moreover, in order that the paper does not become too lengthy, we shall go into detail only in the places where we consider the differences between the two types of nonlinearities to be sharp. The reader who has come this far along should have no difficulty, using our outlines, in completing the arguments.

4. Equilibrium Solutions in the Case of Asocial Prey Population

We consider Eqs. $(0.1)_d$, (0.2), where

$$f(u) = au(u-b)(1-u), \qquad 0 < b < 1/2, \tag{4.1}$$

and $b < \gamma < 1$. It is easy to check that the equations admit arbitrarily large invariant rectangles and that all steady-state solutions again lie in the region Σ defined by (1.1). If we set $v \equiv 0$ in $(0.1)_d$, and seek steady-state solutions, we see that u must be a solution of the problem

$$u'' + f(u) = 0, \qquad -L < x < L, \qquad u(\pm L) = 0. \tag{4.2}$$

From the results in [20], we know that there is a positive number L_f such that if $L < L_f$, $u \equiv 0$ is the only solution of (4.2). If $L = L_f$, (4.2) admits a unique nonconstant solution, and if $L > L_f$, (4.2) has precisely two nonconstant solutions u_1 and u_2. These are depicted in Fig. 6, the phase plane portrait of (4.2), where we use the notation $p_i = u_i'(-L)$, $i = 1, 2$. Observe that every solution of (4.2) must lie in the "fish" region of Fig. 6 and each

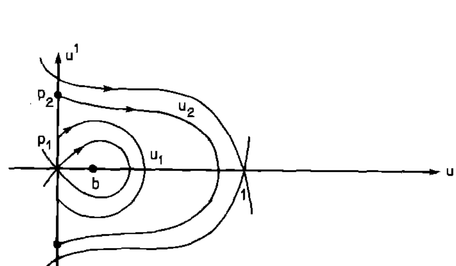

Figure 6

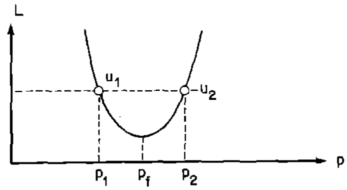

FIGURE 7

solution is characterized by its initial velocity $p = u'(-L)$. We further recall from [20], that the bifurcation diagram for nonconstant solutions of (4.2) takes the form of Fig. 7; of course $u \equiv 0$ is a solution of (4.2) for all $L > 0$.

We can also consider solutions of (4.2) as steady-state solutions of the time dependent problem

$$U_t = U_{xx} + f(U), \qquad -L < x < L, t > 0, U(\pm L, t) = 0, t > 0. \tag{4.3}$$

Concerning their stability, we have the following theorem, whose proof was given in [4, 5].

THEOREM 4.1. *If $L < L_f$, then $u \equiv 0$ is a global attractor for solutions of* (4.3). *If $L > L_f$, then $u_0 \equiv 0$ and u_2 are stable solutions of* (4.3), *and u_1 has a one-dimensional unstable manifold. In fact there are (heteroclinic) solutions $v_0(x, t)$ and $v_2(x, t)$ of* (4.3) *which connect u_1 to the two attractors u_0 and u_1, in the sense that* $\lim_{t\to-\infty} v_i(x, t) = u_1(x)$, *and* $\lim_{t\to\infty} v_i(x, t) = u_i(x)$, *uniformly on $|x| \geq L$, for $i = 0, 2$. (The fact that the convergence is uniform follows, for example, from the results in* [15].)

We turn now to the steady-state solutions of $(1.2)_0$, (1.3); i.e., to the solutions of (1.7), (1.8). From the second equation in (1.7), we see that we must always have $v = 0$ or $v = m(u - \gamma)$, (and of course, $v \geq 0$!). In order to study the structure of these solutions, we define the number τ by

$$\int_0^\tau f(u)\,du = 0. \tag{4.4}$$

Note that $b < \tau < 1$ since $0 < b < 1/2$.

LEMMA 4.2. *Let τ be defined by* (4.4) *and assume that*

$$\tau < \gamma < 1. \tag{4.5}$$

Define h by

$$\begin{aligned} h(u) &= f(u), && \text{if } u < \gamma \\ &= f(u) - mu(u - \gamma), && \text{if } u \geq \gamma. \end{aligned} \tag{4.6}$$

Then h has exactly two positive roots, r_1 and r_2, where $r_1 = b < \gamma < r_2 < 1$. Moreover, $h(0) = 0$ and $\int_0^{r_2} h(u)du > 0$.

Proof. Let

$$\phi(u) = f(u) - mu(u - \gamma). \tag{4.7}$$

Obviously $h(0) = 0$ and $h(b) = f(b) = 0$. Since $\phi(\gamma) = f(\gamma) > 0 > \phi(1)$, ϕ, and thus, h, has a root r_2, $\gamma < r_2 < 1$. We show that h has no other root in $u > \gamma$. It obviously suffices to assume $\phi'(0) \neq (0)$. Since $\phi(0) = 0$, $\phi'(0) < 0$ implies ϕ has a root theorem between 0 and γ, while if $\phi'(0) > 0$, has a negative root. Finally, since $\gamma > \tau$,

$$\int_0^{r_2} h(u)du = \int_0^{\gamma} h(u)du + \int_{\gamma}^{r_2} h(u)du = \int_{\tau}^{\gamma} f(u)du + \int_{\gamma}^{r_2} h(u)du > 0.$$

The proof of the lemma is complete.

It follows from this lemma that the phase plane for the problem

$$u'' + h(u) = 0, \qquad -L < x < L, u(\pm L) = 0, \tag{4.8}$$

has the same qualitative features as that of (4.2). The point here is that h' being discontinuous at $u = \gamma$ plays no role whatsoever in the orbit structure of the phase plane, and classical (i.e., C^2) solutions of (4.8), as well as of the problem

$$U_t = U_{xx} + h(U), \qquad |x| < L, t > 0;\ U(+L, t) \equiv 0 \text{ for } t > 0, \tag{4.9}$$

exist as usual. In particular we will show in the Appendix that the bifurcation diagram for the solutions of (4.8) is qualitatively similar to Fig. 7. Therefore, there is a number $L_h > 0$ such that if $L < L_h$ then the only solution of (4.8) is identically zero, while if $L > L_h$ there are precisely two nonconstant solutions. For $L = L_h$ we define p_h in the same way that we earlier defined p_f.

In preparation for the construction of solutions of (1.7), (1.8) we define the number $L_\gamma > 0$ by

$$L_\gamma = \int_0^{\gamma} \frac{du}{(2F(\gamma) - 2F(u))^{1/2}}, \tag{4.10}$$

where $F'(u) \equiv f(u)$ and $F(0) = 0$. Thus, referring to Fig. 8, if u_γ is the unique solution of (4.2) satisfying $u_\gamma(0) = \gamma$, then $2L_\gamma$ is length of the interval on which it resides i.e. L_γ is the "time" the orbit takes to travel from the line $u = 0$ to $u = \gamma$. Note that it is because we assume $\tau > \gamma$ that L_γ and u_γ are well defined. Note also that $L_\gamma \leq L_f$. Finally, if we denote by

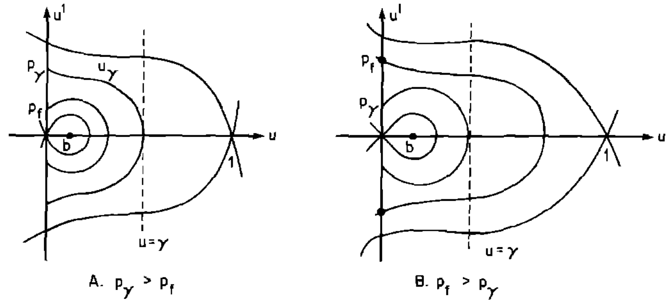

FIGURE 8

u_f the unique nonconstant solution of (4.2) when $L = L_f$ and if we set

$$p_f = u_f'(-L_f) \qquad \text{and} \qquad p_\gamma = u_\gamma'(-L_\gamma),$$

then we have $L_\gamma = L_f$ iff $p_f = p_\gamma$ while $L_\gamma > L_f$ otherwise.

In order to simplify the analysis, we make the additional assumption that

$$\gamma > \sigma = (1 + b)/2, \tag{4.11}$$

where σ is the solution of the equation $f(u) = uf'(u)$, (cf. [19, 20]). Concerning solutions of (4.8) and (4.9), we have the following proposition. (Recall u_i, $i = 1, 2$, was defined earlier, cf. Fig. 6.)

PROPOSITION 4.4. (A) *If* $p_\gamma \geq p_f$ *then* $L_h = L_f \leq L_\gamma$. *In this case* (4.8) *has precisely two nonconstant solutions,* $\tilde{u}_1$ *and* $\tilde{u}_2$, *with* $\tilde{u}_2'(-L) > \tilde{u}_1'(-1) > 0$. *Moreover,* $\tilde{u}_1 = u_1$ *while* $\tilde{u}_2 = u_2$ *only if* $L_f < L < L_\gamma$.

(B) *If* $p_\gamma < p_f$, *then* $L_f < L_h \leq L_\gamma$, *and* $\tilde{u}_1 = u_1$ *if* $L \geq L_\gamma$.

(C) *The solutions* $\tilde{u}_0 \equiv 0$ *and* $\tilde{u}_2$ *are stable solutions of* (4.9), *while* $\tilde{u}_1$ *has a one-dimensional unstable manifold. In fact, there are heteroclinic orbits of* (4.9) *which connect* $\tilde{u}_1$ *to the two attractors* $\tilde{u}_0$ *and* $\tilde{u}_2$.

Proof. Part (C) follows exactly as in Theorem 4.1. Also, a moment's reflection will show that part (A) is true. We thus confine our attention to part (B).

First note that the orbit for $u'' + f(u) = 0$, such that $(u(0), u'(0)) = (\alpha, 0)$, where α is any number satisfying $\tau < \alpha < 1$, yields a solution of (4.2) on an interval $|x| < L(\alpha)$, where

$$L(\alpha) = \int_0^\alpha (2F(\alpha) - 2F(u))^{-1/2}\,du, \qquad F' = f.$$

In the same way we see that there is a solution of (4.8) on the interval

$|x| < \tilde{L}(\alpha)$, where

$$\tilde{L}(\alpha) = \int_0^{\alpha} (2H(\alpha) - 2H(u))^{-1/2}\, du$$

and $H'(u) \equiv h(u)$ and $H(0) = 0$. Since $h(u) \leq f(u)$ and $h(u) < f(u)$ for $u > \gamma$ we see that $\tilde{L}(\alpha) \geq L(\alpha)$ and $\tilde{L}(\alpha) > L(\alpha)$ for $\alpha > \gamma$. When $p_\gamma < p_f$ then it is clear from Fig. 8 that $u_f(0) > \gamma$ so that

$$\begin{aligned} L_h &= \min\{\tilde{L}(\alpha)\colon \tau < \alpha < 1\} = \tilde{L}(\tilde{\alpha}) > L(\tilde{\alpha}) \\ &\geq \min\{L(\alpha)\colon \tau < \alpha < 1\} = L_f. \end{aligned}$$

To see that $L_h \leq L_\gamma$ note that $u_\gamma(x) \leq \gamma$ for all x, $|x| \leq L$, so that u_γ is a solution of (4.8), as well as (4.2). This shows that $L_\gamma \geq L_h$. We also notice that $\tilde{u}_1(x) \leq \gamma$ so that $\tilde{u}_1$ is a solution of (4.2) as well as (4.8), and thus must be the same as u_1. This completes the proof of Proposition 4.4.

Now, just as in Part I, we construct solutions of $(1.2)_d$, (1.3), for $d = 0$, using the solutions of (4.2) and (4.8). If u is a solution of (4.2) then $(u, 0)$ is a solution of $(1.2)_0$, (1.3); if u is a solution of (4.8) then (u, v) is a solution of $(1.2)_0$, (1.3) if $v = \hat{v}(u) \equiv mP(u - \gamma)$, where $P(z) = \max\{0, z\}$. The following theorem summarizes our results for the system.

THEOREM 4.5. *Let f be defined by* (4.1) *and* (4.5) *and let* (4.11) *be satisfied. Then the following statements concerning* $(1.2)_d$ $(d = 0)$, (1.3) *are valid.*

(A) If $L < L_f$ then $U_0 = (0, 0)$ is the only solution.

(B) Suppose $p_\gamma \geq p_f$. Then if $L_f < L \leq L_\gamma$ there are precisely two nonzero solutions, $U_1 = (u_1, 0)$ and $U_2 = (u_2, 0)$, where u_1 and u_2 are the solutions of (4.2) discussed in Theorem 4.1. If $L_\gamma < L$ then in addition to U_1 and U_2 there is a third nonzero solution: $\tilde{U}_2 = (\tilde{u}_2, \tilde{v}_2)$, where $\tilde{u}_2$ is as in Proposition 4.4 and $\tilde{v}_2 = \hat{v}(\tilde{u}_2)$. $\tilde{U}_2$ bifurcates from U_2 in the sense that $\tilde{U}_2$ converges uniformly to U_2 as $L \downarrow L_\gamma$.

(C) Suppose $p_\gamma < p_f$. If $L_f < L < L_h$ then U_1 and U_2 are the only nonzero solutions. If $L_h < L \leq L_\gamma$ there are two additional nonzero solutions, $\tilde{U}_1 = (\tilde{u}_1, \hat{v}(\tilde{u}_1))$ and $\tilde{U}_2 = (\tilde{u}_2, \hat{v}(\tilde{u}_2))$. For $L > L_\gamma$ we have $U_1 \equiv \tilde{U}_1$ so that there are three nonzero solutions: U_1, U_2 and $\tilde{U}_2$.

The situation described in this theorem is nicely illustrated in the two bifurcation diagrams of Fig. 9.

As in the case of quadratic case there is a degeneracy when L crosses L_γ. There is again a continuum of *generalized* solutions but because the situation is so similar to that described in Part I, we shall not discuss it further.

We turn now to a discussion of the stability of the equilibrium solutions which we have constructed. The first step is taken in the following proposi-

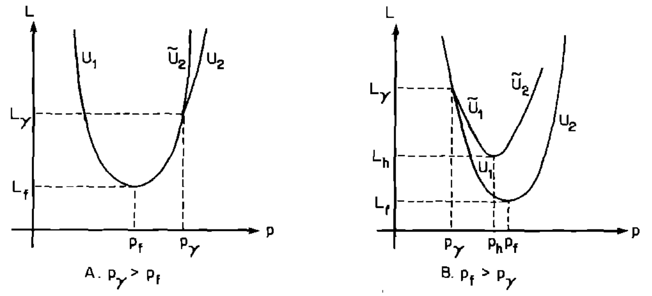

FIGURE 9

tion which yields information concerning the "domains of attraction" of the stable steady-state solutions of the single equations (4.3) and (4.9). A novelty of our discussion is the use of heteroclinic orbits as comparison functions.

PROPOSITION 4.6. *Let $U(x, t)$ be the solution of* (4.3) *with $U(x,0) \geq 0$ for $|x| \leq L$. If $L < L_f$ then $U(x, t) \to 0$ uniformly in x as $t \to \infty$. If $L > L_f$ and $U(x,0) \geq u_1(x)$ (resp. $U(x,0) \leq u_1(x)$) but $U(x,0) \not\equiv u_1(x)$ then $U(x, t) \to u_2(x)$ (resp. $U(x, t) \to 0$) uniformly in x as $t \to \infty$. An analogous statement is valid for solutions of* (4.9).

Proof. The case of $L < L_f$ is proved just as in Proposition 2.1. If $L > L_f$ and $U(x,0) \geq u_1(x)$, $|x| \leq L$, then from $U_t - U_{xx} - f(U) = 0 = -u_1'' - f(u_1)$ it follows that $U(x, t) \geq u_1(x)$ for all $t \geq 0$ and $|x| \leq L$. In fact, since $U(x,0) \not\equiv u_1(x)$ we can conclude that $U(x, t) > u_1(x)$ for $t > 0$, $|x| < L$, and that $\mp[U_x(\pm L, t) - u_1'(\pm L)] > 0$. Thus, if $t_0 > 0$ is arbitrary and $v_2(x, t)$ is the (heteroclinic) solution connecting u_1 to u_2 (cf. Proposition 4.1) then we can find $\bar{t} < 0$ such that $v_1(x, \bar{t}) < U(x, t_0)$ for $|x| < L$. If we let $w(x, t) = v_1(x, t - t_0 + \bar{t})$ then $w(x, t_0) < U(x, t_0)$ so that $w(x, t) < U(x, t)$ for all $t \geq t_0$, $|x| < L$. It thus follows that

$$\limsup_{t \to \infty} U(x, t) \geq u_2(x), \; |x| \leq L.$$

Since u_1 and u_2 are the only positive equilibria it follows from the gradient-like nature of the equation, as in the proof of Proposition 2.1, that $U(x, t) \to u_2(x)$ uniformly in x as $t \to \infty$. The proofs of the other statements of Proposition 4.6 proceed in similar fashion.

The following theorem deals with the stability properties of the solutions of the system (for $d = 0$) which were constructed in Theorem 4.5.

THEOREM 4.7. *Under the assumptions of Theorem* 4.5, *the following statements concerning nonnegative solutions of* $(1.2)_0$, (1.3) *are valid.*

(A) For all $L > 0$, U_0 is asymptotically stable (i.e., is an attractor); if $L < L_f$ then U_0 is a global attractor (i.e., is the uniform limit of all positive solutions).

(B) If $L > L_f$ then U_1 is unstable while U_2 is stable when $u_2(0) < \gamma$ (i.e., when $p_f < p_\gamma$ and $L_f < L < L_\gamma$) but unstable when $u_2(0) > \gamma$ (i.e., when $p_f > p_\gamma$ or $L > L_\gamma$).

Proof. Let (u, v) be a nonnegative solution of $(1.2)_0$, (1.3). If U is the solution of (4.3) such that $U(x,0) = u(x,0)$ then from (4.12)

$$u_t - u_{xx} - f(u) = -uv \leq 0 = U_t - U_{xx} - f(U)$$

it follows from the comparison theorem ([0]) that $u(x, t) \leq U(x, t)$. Thus, from Proposition 4.6 it follows that $u(x, t)$ converges uniformly to zero as $t \to \infty$ if $L < L_f$ or if $u(x,0) < u_1(x)$. From this we conclude, as in the proof of Theorem 2.2, that v also converges uniformly to zero and thus part (A) is seen to be true. This also shows that U_1 is unstable for all $L > L_f$.

Now let us consider the case $L > L_f$, where $u_2(0) < \gamma$. This occurs when $p_f < p_\gamma$ and $L_f < L < L_\gamma$. Note that $u_2(x) < \gamma$ for all x, $|x| \leq L$. If we assume that $U(x,0) > u_1(x)$ for $-L < x < +L$ then we know from Proposition 4.6 that $U(x, t) \to u_2(x)$. It therefore follows from (4.12) that

$$\limsup_{t \to \infty} u(x, t) \leq u_2(x) \tag{4.13}$$

uniformly on $|x| \leq L$. Since $u_2(x) < \gamma$ it follows as in the proof of Theorem 2.3 that $v(t, x) \to 0$ and that $u(x, t) \to u_2(x)$ as $t \to \infty$. This shows that U_2 is an attractor if $u_2(0) < \gamma$. That U_2 is unstable when $u_2(0) > \gamma$ is proved just as in the proof of Theorem 3.2. This completes the proof of Theorem 4.7.

Remark. It is easy to see that when $L > L_f$ and $u_2(0) > \gamma$ there is a continuum of generalized equilibria just as in the quadratic case (Theorem 3.3). For the sake of brevity we shall omit a discussion of this situation.

Let us now turn to the case where $d > 0$. Note that U_0, U_1 and U_2 are equilibrium solutions of $(0.1)_d$, (0.2) in this case as well. The following theorem summarizes their stability properties.

THEOREM 4.8. *For all $L > 0$, $d \geq 0$, $U_0 = (0, 0)$ is an attractor for nonnegative solutions of $(0.1)_d$, (0.2). If $L < L_f$ then U_0 is a global attractor. If $L > L_f$, U_1 is an unstable equilibrium solution while U_2 is an attractor for $d > d(L)$ and is unstable for $d < d(L)$. $d(L) \equiv 0$ for $L_f < L < L_\gamma$ while $d(L) > 0$, $d'(L) > 0$ for $L > L_\gamma$.*

The situation described in this theorem is illustrated in Fig. 10. The proof of Theorem 4.8 is similar to those of Theorems 2.2, 2.3, and 3.4 and, therefore,

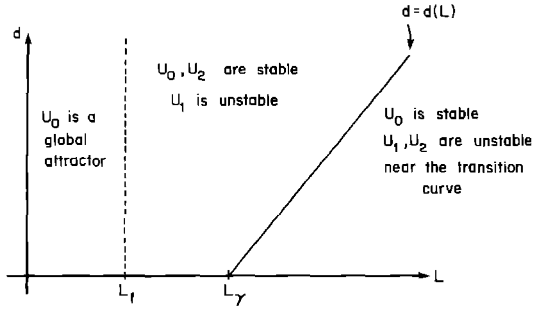

FIGURE 10

we omit the details. Just as in the case of quadratic f it would seem to us that there is precisely one stable nonzero solution when $d < d(L)$ and this solution converges to U_2 as $d \uparrow d(L)$ while it converges to U_2 as $d \downarrow 0$. This conjecture is supported by extensive computer simulations. For d near $d(L)$ it is further supported by a bifurcation analysis similar to that in Section 3.

5. Some Interesting Bifurcation Diagrams

In this section, we shall discuss, rather briefly, the Eqs. $(0.1)_d$, (0.2) where $f(u)$ is of the form

$$f(u) = (u-a)(u-b)(1-u), \qquad a<b<0<1, \tag{5.1}$$

where a and b are close to zero, and $0<\gamma<1$. If we consider the problem (4.2), for this f, we know, from the results in [20], that the global bifurcation diagram for the number of nonconstant solutions is given by Fig. 11. That is, if $L_f < L < \bar{L}_f$, there are three solutions u_0, u_1, u_2 while if $L > \bar{L}_f$ or $L < L_f$, there is only one; u_0 or u_2, respectively. We shall see below that this yields some quite different and interesting bifurcation diagrams for solutions of the steady-state equations $(1.2)_0$, (1.3).

Before we consider the steady-state solutions, it is first necessary to show that the problem $(0.1)_d$, (0.2) admits globally defined solutions whenever the

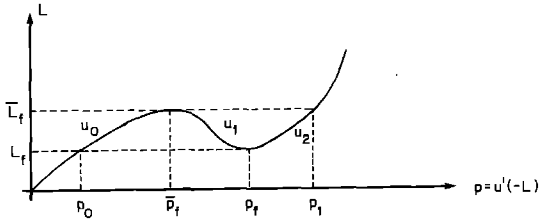

FIGURE 11

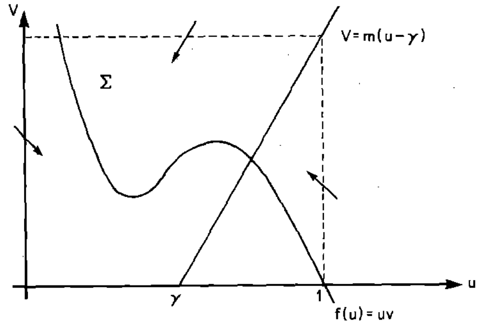

FIGURE 12

initial data functions $u(x,0)$, and $v(x,0)$ are bounded smooth, nonnegative functions. To this end, we consider the isoclines of the vector field $V = (f(u) - uv,\ v[-v + m(u-\gamma)])$, in the quarter plane $u \geq 0$, $v \geq 0$. Along the curve $uv = f(u)$, if $u \neq 0$, we have $v = f(u)/u$; hence $v' = 0$ whenever $uf'(u) = f(u)$. But since a and b are close to zero, we have, from the results in [20], that $v' = 0$ exactly twice in $u > 0$. It follows that the flow of the orbits of V can be described by Fig. 12. We see at once from this picture that Eqs. $(0.1)_d$ admit arbitrarily large bounded invariant rectangles in $u \geq 0$, $v \geq 0$ so from the results in [3], solutions exist globally in time. Furthermore, the region Σ given by (1.1) contains all the steady-state solutions.

Next, we define the function h by (4.6), where f is given by (5.1). If $\phi(u) = f(u) - mu(u-\gamma)$, then since $\phi(0) > 0$, $\phi(b) > 0$ and $\phi > 0$ if $u \ll -1$, ϕ has two roots in $u < 0$. Since $\phi(\gamma) > 0$ and $\phi(1) < 0$, ϕ, and hence h, has precisely one root r in the interval $\gamma < u < 1$; in fact this is the only positive root of h. It follows that h has the same qualitative form as f. We can assume that a and b are chosen so close to zero that whenever $r \geq \gamma$, Eq. (4.7), for this h, has the qualitative form of Fig. 13, if m is sufficiently small (see the Appendix).

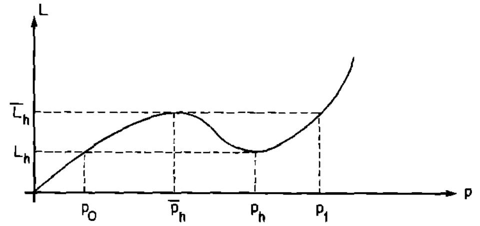

FIGURE 13

We shall discuss only the steady-state solutions of $(0.1)_0$, (0.2), and leave the case $d > 0$ to the reader. Thus, we seek solutions of $(1.2)_0$, (1.3), where f is defined by (5.1). There are several cases to consider, each one being qualitatively distinct from the others. Recalling the definition of L_γ in (4.10), and the definitions of L_f and $\bar{L}_f$ in Fig. 11, we see that there are three different cases to consider; namely

$$\text{(i) } L_\gamma < L_f, \quad \text{(ii) } L_f < L_\gamma < \bar{L}_f, \quad \text{(iii) } L_\gamma > \bar{L}_f. \tag{5.2}$$

Now rather than give a complete and exhaustive study of all these cases, we shall content ourselves to first describe the solutions, then to give the bifurcation diagrams, and finally to state the stability results; the reader should have little difficulty in verifying our statements. In what follows we shall denote by u_f and $\bar{u}_f$ the solutions of (4.2) which correspond to interval lengths L_f and $\bar{L}_f$, respectively, and which satisfy $u_f'(-L_f) = p_f$ and $\bar{u}_f'(-\bar{L}_f) = \bar{p}_f$.

Observe first that h has the same basic qualitative properties as f; namely that $h(a) = h(b) = 0$, and $h(r) = 0$ for some r, $0 < r < 1$. Thus it is easy to see, using elementary phase-plane considerations, that problem (4.8), admits nonconstant solutions for all $L > 0$. Thus, if p_γ is defined as in Section 4 (cf. Fig. 10), we have that whenever $p > p_\gamma$, there are solutions of $(1.2)_0$ of the form (u, v), where u satisfies (4.8), $u'(-L) = p$, and $v(x) = m(u(x) - \gamma)$ if $u(x) \geq \gamma$, while $v(x) = 0$ if $u(x) < \gamma$.

We now find all the solutions of $(1.2)_0$, (1.3) by considering successively the cases (5.2) above. In the interest of brevity, we shall not give a complete classification; rather, we shall merely show some of the interesting new phenomenon. We begin first with some general remarks. Referring to Fig. 14, the phase plane for (4.2), (with f given by (5.1)!), we have marked off the points p_0, $\bar{p}_f$ and p_f on the line $u = 0$, and the corresponding L intervals between, them which we obtain from Fig. 13. In addition we have marked off points P, Q, R, S and T on the line $u' = 0$; these points will be needed to distinguish the different cases in (5.2) and will represent the various possible positions of the line $u = \gamma$. Finally, we recall once again that p_γ denotes the unique point on the line $u = 0$ with the property that the orbit of the flow (4.2) which passes through p_0, is tangent to the line $u = \gamma$, (p_γ will vary of course, depending on whether the line $u = \gamma$ is in one of the four positions P, Q, R, S or T.) These correspond, respectively, to the cases

$$p_\gamma < p_0, \qquad p_0 < p_\gamma < \bar{p}_f, \qquad \bar{p}_f < p_\gamma < p_f, \qquad p_f < p_\gamma < p_1, \qquad p_1 < p_\gamma, \tag{5.3}$$

where we recall p_1 is defined in Fig. 11. In what follows, u denotes a solution of (4.2) and $\tilde{u}$ denotes a solution of (4.8).

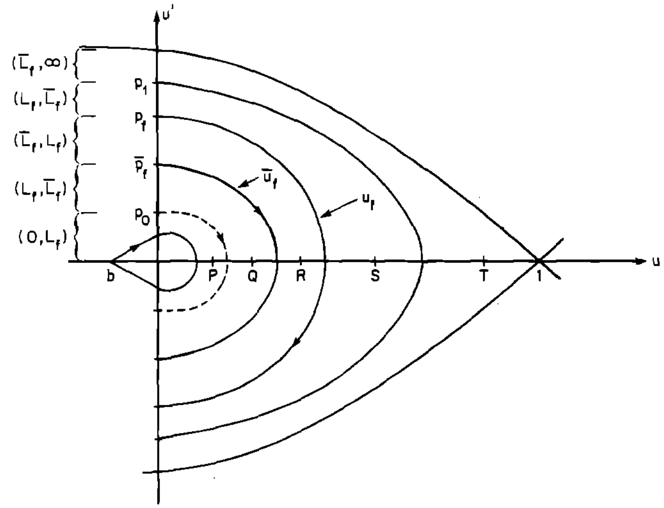

FIGURE 14

Observe that solutions of (4.8) can differ from solutions of (4.2) only when $u'(-L) < p_\gamma$; for smaller initial derivatives, the two equations yield identical solutions (cf. Fig. 14). Moreover, by the argument in the proof of Proposition 4.4 B, if $p_\gamma < p$, $T(p) < \tilde{T}(p)$, where T and $\tilde{T}$ denote, respectively, the "time maps" corresponding to (4.2) and (4.8), respectively. With these two observations, we can construct the bifurcation diagram for small $m > 0$.

We begin with the case $p_\gamma < p_0$ (i.e., $\gamma = P$ in Fig. 14). If $L < L_\gamma$, then the only solution of $(1.2)_0$, (1.3) is $U_0 = (u_0, v)$. If $L_\gamma < L < L_f$, then we have the two solutions U_0 and $\tilde{U}_0 = (\tilde{u}_0, \tilde{v}_0)$, $\tilde{U}_0$ bifurcating out of U_0. If $L_f < L < \bar{L}_f$, we first get the two additional solutions $U_1 = (u_1, 0)$ and $U_2 = (u_2, 0)$, $(L_f < L_h)$, and then when L crosses L_h, we obtain two additional solutions $\tilde{U}_1 = (\tilde{u}_1, \tilde{v}_1)$ and $\tilde{U}_2 = (\tilde{u}_2, \tilde{v}_2)$. Now since m is small, we have $L_f < L_h < \bar{L}_f < \bar{L}_h$ so that when L exceeds $\bar{L}_f$, the solutions U_1 and U_0 cancel each other, and then when $L > \bar{L}_h$, the solutions $\tilde{U}_0$ and $\tilde{U}_1$ cancel. Hence, for $L > \bar{L}_h$, we are left only with the two solutions U_2 and $\tilde{U}_2$. We depict this case in Fig. 15, where we have also denoted the number of solutions in the various regions of the L axis. Moreover, a superscript s or u on a term U_i indicates whether the function is a stable or unstable solution of $(1.2)_0$, (1.3), respectively. We shall briefly discuss this at the end of this section.

Next, consider successively the cases $p_0 < p_\gamma < \bar{p}_f$ (i.e., $\gamma = Q$ in Fig. 14), $\bar{p}_f < p_\gamma < p_f$ (cf. Fig. 14 with $\gamma = R$), $p_f < p_\gamma < p_1$, and $p_1 < p_\gamma$ (c.f. Fig. 14 with $\gamma = S$). The bifurcation pictures are in Figs. 16–19, respectively. (Note

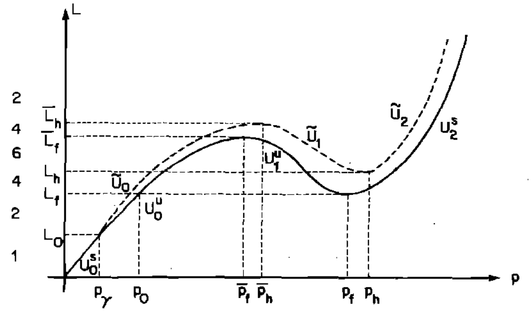

FIG. 15. $p_\gamma < p_0$.

that since m is small, we have, by continuity that $L_\gamma > L_h$ in Fig. 16; similar things hold in the other pictures.)

Finally, we say a few words on the stability of the solutions $U_i = (u_i, 0)$ in the various cases (5.2); the reader should have no difficulty in verifying our statements. In fact, the stability statements follow from the corresponding stability of the u_i considered as a solution of the scalar equation (4.2), while the instability statements could in addition, be due to a bifurcation, with a resulting loss of stability of U_i; we have already observed this phenomenon in Section 4.

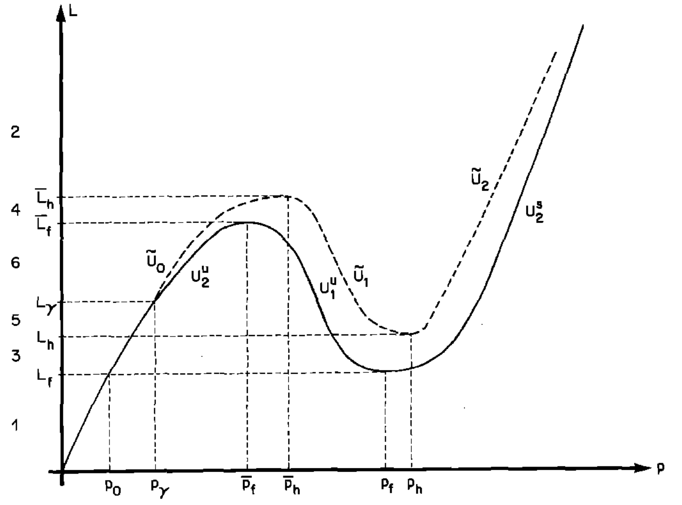

FIG. 16. $p_0 < p_\gamma < \bar{p}_f$.

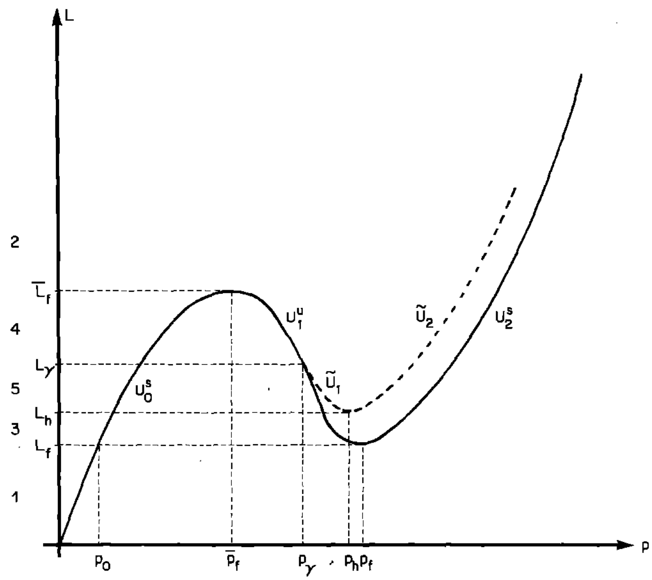

FIG. 17. $p_f < p_\gamma < p_f$.

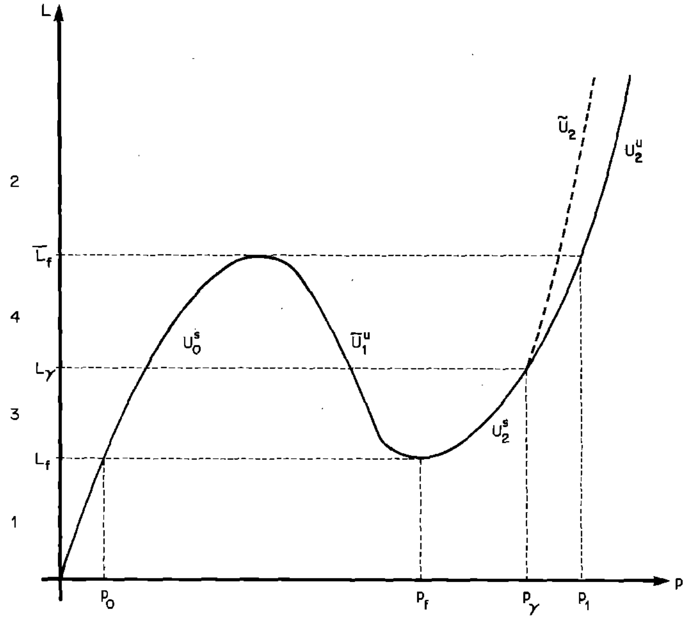

FIG. 18. $p_f < p_\gamma < p_1$.

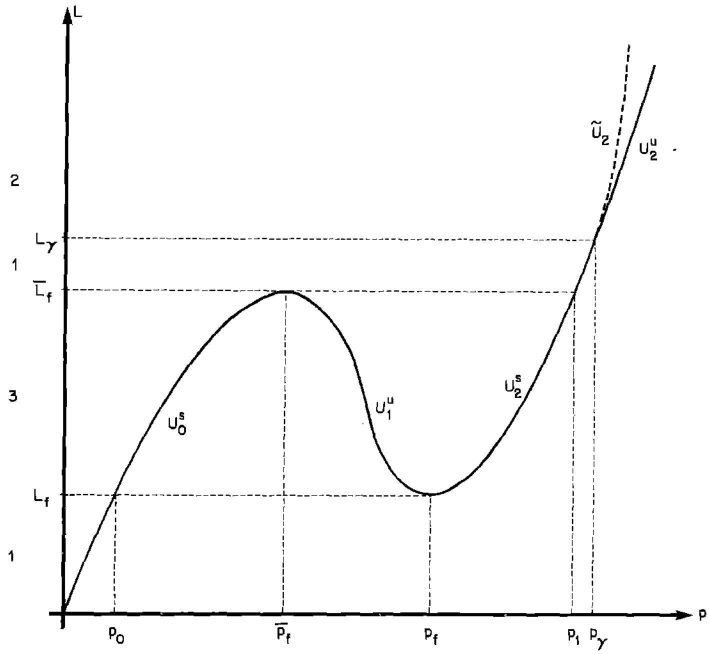

FIG. 19. $p_1 < p_\gamma$.

6. MORE THAN ONE SPACE VARIABLE

In this final short section, we remark that certain of our results can be extended to the case of several space variables. In particular, we shall show how to obtain the bifurcations of the steady-state solutions in this case.

We consider the steady-state equations

$$\nabla^2 u + f(u) - uv = 0, \qquad d\nabla^2 v + v[-v + m(u - \gamma)], \tag{6.1}$$

on a domain $\Omega \subset \mathbb{R}^N$, together with homogeneous Dirichlet boundary conditions

$$(u, v) = (0,0) \text{ on } \partial\Omega. \tag{6.2}$$

We shall discuss only the quadratic case, $f(u) = au(1 - u)$. From the results of [1], for example, we know that the problem

$$\nabla^2 u + f(u) = 0, \qquad u = 0 \text{ on } \partial\Omega \tag{6.3}$$

has a nonzero solution $\bar{u}(x)$, provided that Ω is sufficiently large. (We shall give an alternate simple proof of this fact.) From this, it follows that (6.1), (6.2) admits $(\bar{u}, 0)$ as a solution. We are interested in the bifurcations of this solution, as Ω increases. To this end, we shall only consider the case $d = 0$, since the discussion when $d > 0$ follows exactly as before.

We let $h(u)$ be defined as above in Eq. (4.6) and we consider the problem

$$\nabla^2 u + h(u) = 0, \qquad u = 0 \text{ on } \partial\Omega. \tag{6.4}$$

Our goal is to show that (6.3) has a solution different from $u_0(x)$ provided that Ω is sufficiently large. This will follow from a lemma.

LEMMA. *If Ω is sufficiently large, there is a solution $u_0(x)$ of* (6.3) *with $u_0(x) > \gamma$ for some $x \in \Omega$.*

Proof. Let $w \geq 0$ be a principal eigenfunction of $-\nabla^2$ on Ω, together with homogeneous Dirichlet boundary conditions, and let $\lambda > 0$ be the corresponding eigenvalue. We may assume $w(x) \leq 1$, $x \in \Omega$, and for some $\bar{x} \in \Omega$, $\bar{w}(x) = 1$. If Ω is large and $\epsilon > 0$ is small, $\lambda + a(\gamma + \epsilon - 1) < 0$. If u is the solution of the parabolic equation

$$u_t = \nabla^2 u + f(u), \qquad u = 0 \text{ on } \partial\Omega, \tag{6.5}$$

and $u(x,0) > \gamma w(x)$, $x \in \Omega$, then

$$\begin{aligned} u_t - \nabla^2 u - f(u) = 0 &\geq (\gamma + \epsilon) w[\lambda + a(\gamma + \epsilon - 1)] \\ &= -\nabla^2((\gamma + \epsilon)w) - f((\gamma + \epsilon)w). \end{aligned}$$

Hence by the usual comparison theorem, $u(x, t) \geq (\gamma + \epsilon)w(x)$, $x \in \Omega$, $t > 0$. Since (6.5) is gradient-like with respect to the functional

$$\phi(u) = \int_\Omega \left[\tfrac{1}{2} |\nabla u|^2 - F(u)\right] dx, \qquad F' = f,$$

and (6.5) admits arbitrarily large invariant regions in $u \geq 0$, we obtain that $\lim_{t \to \infty} u(x, t) = u_0(x)$ exists, and u_0 is a solution of (6.3). Since $u_0(x) \geq (\gamma + \epsilon)w(x)$, we see that $u_0(\bar{x}) > \gamma$.

We turn our attention now to (6.3). To obtain a solution, we shall construct upper and lower solutions. Thus, with w as in the lemma, and $\delta > 0$ sufficiently small, δw is an upper solution, and, the function u_0, obtained in the lemma is a lower solution. Hence (6.3) has a solution $u_1(x)$ different from $u_0(x)$. As in the one-dimensional case, we can easily show that the solution $(u_0, 0)$ bifurcates into $(u_0, 0)$ and (u_1, v_1), where again $v_1(x) = 0$ if $u_1(x) \leq \gamma$ and $v_1(x) = m(u_1(x) - \gamma)$ otherwise, so that again $v \equiv 0$ in a neighborhood of the boundary. Although we will not give the details, it is easily seen that we again have a continuum of (generalized) steady-state solutions which reflects the fact that part of the residual spectrum is positive.

APPENDIX

We shall show that for the functions $h(u)$, defined by $h(u) = f(u) - mI(u)u(u-\gamma)$, (where I is the characteristic function of the interval $u \geq \gamma$), their "time" maps (i.e., bifurcation diagrams for solutions of $u'' + h(u) = 0$, $-L < x < L$, $u(\pm L) = 0$), have the same qualitative features as the corresponding "time" maps for solutions of $u'' + f(u) = 0$, $-L < x < L$, $u(\pm L) = 0$. Here f can be any of the three functions we have previously considered in Sections 2, 4, and 5, viz., $f(u) = au(1-u)$, $f(u) = au(u-b)(1-u)$, $f(u) = (u-a)(u-b)(1-u)$, respectively.

In all three cases, if $H' = h$, $H(0) = 0$, then (see [20]), the bifurcation diagrams are given by counting the number of critical points of the "time map"

$$S(\alpha) = \int_0^\alpha \frac{du}{\sqrt{H(\alpha) - H(u)}} = T(\alpha(p)),$$

where $p^2 = 2H(\alpha(p))$. We also have ([20]) the following formula:

$$S'(\alpha) = \int_0^\alpha \frac{\theta(\alpha) - \theta(u)}{(2(H(\alpha) - H(u)))^{3/2}} \frac{du}{\alpha}; \qquad (A_1)$$

where

$$\theta(u) = 2H(u) - uh(u). \qquad (A_2)$$

Furthermore, an easy calculation shows

$$\begin{aligned} \theta'(u) &= f(u) - uf'(u), & u &< \gamma \\ &= f(u) - uf'(u) + mu^2, & u &> \gamma \end{aligned}. \qquad (A_3)$$

Now if $f(u) = au(1-u)$, then

$$\begin{aligned} \theta'(u) &= au^2, & u &\leq \gamma \\ &= (u+m)u^2, & u &> \gamma' \end{aligned}$$

and we see from writing the integral in (A_1) as the sum of two integrals, $S'(\alpha) = \int_0^\gamma + \int_\gamma^\alpha$, that (A_3) implies that $S'(\alpha) > 0$. Hence in this case, there is a unique nonconstant solution of $u'' + h(u) = 0$, $-L < x < L$, $u(\pm L) = 0$, for each $L > L_h$, (see [19]). This proves the last statement in Theorem 1.2.

Suppose next that $f(u) = au(u-b)(1-u)$, $0 < b < 1/2$. Then if $T'(p) = 0$, we have $S'(\alpha) = 0$ and conversely so if $S'(\alpha) = 0$ we have (see [20])

$$2\alpha^2 S''(\alpha) \geq \int_0^\gamma \frac{\alpha\theta'(\alpha) - u\theta'(u)}{(H(\alpha) - H(u))^{3/2}}\,du + \int_\gamma^\alpha \frac{\alpha\theta'(\alpha) - u\theta'(u)}{(H(\alpha) - H(u))^{3/2}}\,du, \tag{A_4}$$

where θ is again defined by (A_2). Now if $\alpha \leq \sigma$, then $S' < 0$ while (A_4) shows that if $\sigma < \alpha$, then $S''(\alpha) > 0$ when $S'(\alpha) = 0$. Hence it follows that $T(p)$ has precisely one critical point, and thus the statement in Proposition 4.4, concerning the number of solutions of (4.7) is valid.

Finally, we consider the case where $f(u) = (u-a)(u-b)(1-u)$, where $a, b < 0$, and a and b are near zero. If m is sufficiently small, one can check that the results of [20] apply. This follows since the bifurcation diagrams depend smoothly on the parameters, see (A_1). Thus Fig. 13 gives the qualitative picture of the solutions of (4.8).

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