INTERACTION KERNELS FOR $A = 4n$ BINARY CLUSTER SYSTEMS [†]

Y. SUZUKI [††], E. J. RESKE and K. T. HECHT

Physics Department, University of Michigan, Ann Arbor, MI 48109, USA

Abstract: Bargmann transform techniques used to calculate norm kernels for nuclear cluster systems have been generalized to evaluate interaction kernels for central interactions of gaussian form for binary cluster systems made up of SU(4)-scalar ($A = 4n$) cluster fragments with internal functions of good SU(3) symmetry and equal oscillator width parameters. The technique involves a reduction from A-particle orbital states of space symmetry characterized by 4-columned Young tableaux to $\frac{1}{4}A$-particle states of single-column symmetry. The interaction kernels are built partly through a convolution of the single-column Bargmann transforms of the Fourier components of basic one-body operators. Bargmann transforms of single-column type have been evaluated in algebraic form for a two-body gaussian interaction and for the one-body Fourier kernel, $\sum_j \exp(i\mathbf{q} \cdot \mathbf{r}_j)$, for the following A-particle systems and cluster decompositions: $A = 12$, $\alpha + {}^8\text{Be}$; $A = 16$, $\alpha + {}^{12}\text{C}$, ${}^8\text{Be} + {}^8\text{Be}$; $A = 20$, $\alpha + {}^{16}\text{O}$, ${}^8\text{Be} + {}^{12}\text{C}$; and $A = 24$, ${}^{12}\text{C} + {}^{12}\text{C}$, ${}^8\text{Be} + {}^{16}\text{O}$, $\alpha + {}^{20}\text{Ne}$. The construction of the Bargmann transform for the full A-particle system is illustrated with a simple example. The example also shows how the coordinate space matrix elements needed for the evaluation of RGM and GCM kernels can be extracted from appropriate expansions of this Bargmann transform by purely algebraic techniques.

1. Introduction

The microscopic cluster model [1]) provides a basis for a number of interesting nuclear problems, including not only the study of possible cluster structure within nuclei, but also the determination of effective potentials for the scattering of heavy ions and a treatment of nuclear reactions in which the Pauli principle is properly incorporated into the theory. Practical calculations in the framework of a microscopic cluster model, however, are dependent on the availability of techniques for the evaluation of challenging norm and interaction kernels. In the early calculations of the Wildermuth school [2]) resonating group (RGM) kernels were calculated by cluster coordinate techniques which involved the direct evaluation of the multi-dimensional integrals. A refinement of the RGM technique has led to the Hackenbroich computer chain [3]) which has proved practical for the calculation of light systems [4]). The calculations of Tang and coworkers [5]) in which RGM kernels are calculated by the complex generator coordinate

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[††] Nishina Memorial Foundation Fellow, on leave of absence from the Department of Physics, Niigata University, Niigata 950–21, Japan.

technique (based on a single Fourier transformation), the generator coordinate (GCM) calculations of Weiguny and coworkers [6], the computer codes of Tohsaki-Suzuki [7] for both RGM and GCM kernels based on a double Fourier transformation, and the complex generator coordinate technique of Horiuchi [8] essentially all involve at some stage a variant of the Margenau-Bloch-Brink technique [9] for the evaluation of the many-particle matrix elements. The computational difficulties associated with these techniques have restricted the applications. As a result detailed, practical calculations in the framework of the microscopic cluster model have been limited to very light nuclei or to simple systems involving closed shell nuclei such as $\alpha + {}^{16}O$, ${}^{16}O + {}^{16}O$, or $\alpha + {}^{40}Ca$.

Powerful techniques have recently been developed [10, 11] for the evaluation of norm and overlap kernels for a fully antisymmetrized cluster basis, but the calculation of interaction kernels still presents a challenging problem. Much of the recent progress in the calculation of norm and overlap kernels has been aided by the introduction of the Bargmann-Segal transform. It is the purpose of the present investigation to show that the Bargmann transform technique can also be used to advantage in the calculation of interaction kernels. The essential point of the present method centers around the fact that the antisymmetrization operation and the transformation to cluster-relative motion coordinates can be carried out *algebraically* by computer in the Bargmann space.

The calculation of norm and overlap kernels is greatly simplified by the use of SU(3)-coupled cluster model functions. The SU(3) symmetry furnishes a useful tool because of (i) the SU(3)-scalar character of the antisymmetrization operator, (ii) the inherent SU(3) symmetry of the ground-state wave functions of nuclear cluster fragments in the $A = 4$–20 mass range, and (iii) the possibility of exploiting SU(3)-recoupling techniques if the cluster functions are expanded in an SU(3)-coupled basis. The simplifications introduced by the SU(3)-scalar property of the operator no longer apply for the interaction kernel. Even in this case, however, an expansion in SU(3) irreducible tensor components may facilitate the calculations in an oscillator basis [12] and may be particularly useful for light $4n$ self-conjugate nuclei [13]. In such nuclei the wave functions of the ground states and of rotational band members based on the ground states are represented reasonably well by states of simple SU(3) symmetry $(\lambda 0)$ or (0μ). States of such SU(3) symmetry can be obtained from a single, Slater determinant intrinsic wave function or SU(3) coherent state. The calculations are simplest for $4n$ self-conjugate nuclei, yet these include some of the most interesting problems which remain to be investigated by the microscopic cluster model. The ${}^{12}C + {}^{12}C$ system with its rich spectrum of so-called quasi molecular resonances forms a prime example. The important exit channels include the $\alpha + {}^{20}Ne$ and ${}^{8}Be + {}^{16}O$ fragmentation which also fall into this category.

The present investigation is limited to binary cluster systems. The A-particle nuclear system is decomposed into fragments with mass numbers f and $A - f$, alternately f' and $A - f'$, where each fragment $(f, A - f, f'$ and $A - f')$ is a $4n$ self-conjugate nucleus and is assumed to have an orbital function of highest possible space symmetry, of

[44...4] character. The calculation of interaction kernels is simplest for such systems. The fragment internal functions are built from spin-isospin saturated products. As a result, the calculation of matrix elements of the hamiltonian can be reduced to the calculation of matrix elements of orbital functions only. This reduces the calculation from one involving A-particle states of space symmetry characterized by 4-columned Young tableaux to one involving $\frac{1}{4}A$-particle totally antisymmetric orbital states characterized by a single-column tableau. This reduction process is explained in sect. 2. For the 2-body interaction it involves the introduction of the Fourier transform $\tilde{u}(q)$ of the radial part of the interaction $u(r)$. The full A-particle hamiltonian matrix element can be obtained from three basic matrix elements involving totally antisymmetric $\frac{1}{4}A$-particle orbital functions, (i) the norm or overlap matrix element, i.e. the matrix element of the antisymmetrizer, (ii) the matrix elements of basic 1-body operators including the Fourier kernel $\sum_j \exp\{i(q \cdot r_j)\}$, and (iii) a 2-body matrix element of purely orbital type.

The Bargmann transforms of the three types of basic operators are given in sect. 3. The single-column Bargmann transforms are first expressed in single-particle Bargmann space variables. The transformation to Bargmann space cluster-internal and cluster-relative motion coordinates is made in sect. 4. For cluster fragments in the $A = 4$–20 mass range only a few cluster-internal degrees of freedom carry oscillator excitations. The dependence of the single-column Bargmann transforms on these internal Bargmann space variables is extracted in this section in terms of SU(3) coherent states or in some cases a simple superposition of SU(3) coherent states. Detailed calculations are carried out for the single-column Bargmann transforms needed for the following systems: (i) the 12-particle system with cluster decomposition $\alpha + {}^8$Be, (ii) the $A = 16$ system with cluster decompositions $\alpha + {}^{12}$C, and ^{8}Be + ^{8}Be, (iii) the $A = 20$ system with cluster decompositions $\alpha + {}^{16}$O, and ^{8}Be + ^{12}C, and (iv) the $A = 24$ system with cluster decompositions ^{12}C + ^{12}C, $\alpha + {}^{20}$Ne and ^{8}Be + ^{16}O. The combination of single-column Bargmann transforms to construct the Bargmann transform of the hamiltonian for the A-particle system of full 4-columned Young symmetry is carried out in sect. 5. It involves the convolution of 1-body operators in the Fourier decomposition of the 2-body interaction.

The complicated $3A$-dimensional coordinate space matrix elements needed for the calculation of both RGM and GCM interaction kernels can be extracted from the full Bargmann transform by purely algebraic techniques through the expansion of the Bargmann transform in appropriately coupled Bargmann space functions. The relationship between the needed coordinate space matrix elements and the coefficients in these expansions is established in sect. 6. Two cases are considered. In the first an expansion of the cluster basis in SU(3)-coupled harmonic oscillator functions (with equal width parameters) is used. In the second an angular momentum coupled cluster basis is combined with locally peaked radial relative motion functions. In both expansions cluster-internal functions are SU(3) coupled to functions with definite internal SU(3) quantum numbers. The Bargmann space expansions are illustrated in some detail with the simplest of our examples, the $\alpha + {}^8$Be cluster system, in sect. 7.

Finally, sect. 8 gives a summary and discusses the difficulties which may be encountered in generalizing our technique to more complicated cluster systems.

2. The hamiltonian. Reduction to matrix elements of $\frac{1}{4}A$-particle orbital functions

Both in the Bargmann technique with intrinsic states of SU(3) symmetry and the various generator coordinate methods of calculating interaction kernels, the actual calculations involve matrix elements between wave functions of antisymmetrized products of single-particle functions. For $4n$ self-conjugate nuclei with spin-isospin saturated cluster fragments the reduction to the calculation of matrix elements involving $\frac{1}{4}A$-particle orbital functions only is best carried out by the methods of Brink [9]), and the notation of ref. [9]) is therefore followed in this section.

The A-particle wave functions are built from a set of independent single-particle functions $\phi_i \xi_{\alpha_i}$, (alternately $\psi_i \xi_{\alpha_i}$), where i stand for the orbital quantum numbers and α_i for the spin-isospin quantum numbers, (with $\alpha_i \equiv m_s m_t = +\frac{1}{2}+\frac{1}{2}, +\frac{1}{2}-\frac{1}{2}, -\frac{1}{2}+\frac{1}{2}, -\frac{1}{2}-\frac{1}{2}$, for $i = 1,\ldots,4$),

$$\Phi = (A!)^{-\frac{1}{2}} \mathscr{A} \left\{ \prod_{i=1}^{4} (\phi_1(r_i)\xi_{\alpha_i}(\sigma_i, \tau_i)\phi_2(r_{i+4})\xi_{\alpha_i}(\sigma_{i+4}, \tau_{i+4})\ldots) \right\},$$

$$\Psi = (A!)^{-\frac{1}{2}} \mathscr{A} \left\{ \prod_{i=1}^{4} (\psi_1(r_i)\xi_{\alpha_i}(\sigma_i, \tau_i)\psi_2(r_{i+4})\xi_{\alpha_i}(\sigma_{i+4}, \tau_{i+4})\ldots) \right\}. \quad (1)$$

Here $\mathscr{A}$ is the A-particle antisymmetrizer, normalized according to $\mathscr{A} \equiv \sum (-1)^{\sigma(P)} P$, where the sum runs over the $A!$ permutations, P. For the hamiltonian

$$H = \sum_{i=1}^{A} t_i + \sum_{i<j}^{A} V_{ij} \quad (2)$$

with a central two-body interaction

$$V(r) = u(r)(W + BP_\sigma - HP_\tau - MP_\sigma P_\tau), \quad (3)$$

the sums over spin-isospin quantum numbers $\alpha_i, \alpha_j, \ldots$ can be carried out explicitly, and the A-particle matrix element is reduced to

$$\langle \Phi, H\Psi \rangle = 4\langle \Phi, \Psi \rangle \sum_{ij} \langle \phi_i, t\psi_j \rangle (B^{-1})_{ji}$$

$$+ \langle \Phi, \Psi \rangle \sum_{ijkl} \langle \phi_i \phi_j | u | \psi_k \psi_l \rangle [X_d (B^{-1})_{ki}(B^{-1})_{lj} + X_e (B^{-1})_{kj}(B^{-1})_{li}], \quad (4)$$

where the sums are over orbital quantum numbers $i, j, \ldots$ only, and the direct and

exchange terms of the potential energy are given in terms of the exchange parameters by

$$X_d = \tfrac{1}{2}(16W + 8B - 8H - 4M), \qquad X_e = \tfrac{1}{2}(16M + 8H - 8B - 4W). \tag{5}$$

The matrix B and its inverse B^{-1} are given by the single-particle overlap integrals

$$B_{ij} = \langle \phi_i, \psi_j \rangle, \tag{6}$$

with

$$\langle \Phi, \Psi \rangle = (\det B)^4. \tag{7}$$

Except for a norm factor of $(\det B)^3$, and a spin-isospin coefficient of 4, the kinetic energy part of the hamiltonian is reduced to a matrix element involving totally anti-symmetric $\tfrac{1}{4}A$-particle orbital functions, by the relation

$$\det B \sum_{ij} \langle \phi_i, t\psi_j \rangle (B^{-1})_{ji} = \langle \phi_1 \phi_2 \dots \phi_{A/4} \left| \sum_{i=1}^{A/4} t_i \mathscr{A} \right| \psi_1 \psi_2 \dots \psi_{A/4} \rangle, \tag{8}$$

where $\mathscr{A}$ is now a $\tfrac{1}{4}A$-particle antisymmetrizer. A trivial correction for c.m. motion is still to be made (see sect. 4).

After rearrangement of the interaction part of the matrix element (int),

$$\text{int} = (X_d + X_e)(\det B)^4 \sum_{ijkl} \langle \phi_i \phi_j | u | \psi_k \psi_l \rangle (B^{-1})_{ki} (B^{-1})_{lj}$$

$$- X_e (\det B)^4 \sum_{ijkl} \langle \phi_i \phi_j | u | \psi_k \psi_l \rangle [(B^{-1})_{ki}(B^{-1})_{lj} - (B^{-1})_{kj}(B^{-1})_{li}], \tag{9}$$

the second term has been put into a form which can be converted directly to a matrix element involving totally antisymmetric $\tfrac{1}{4}A$-particle orbital functions, by the relation

$$\tfrac{1}{2}\det B \sum_{ijkl} \langle \phi_i \phi_j | u | \psi_k \psi_l \rangle [(B^{-1})_{ki}(B^{-1})_{lj} - (B^{-1})_{kj}(B^{-1})_{li}]$$

$$= \langle \phi_1 \phi_2 \dots \left| \sum_{i<j}^{A/4} u(r_{ij}) \mathscr{A} \right| \psi_1 \psi_2 \dots \rangle. \tag{10}$$

To convert the first term of eq. (9) into the appropriate form it is convenient to introduce the Fourier transform of $u(r)$

$$u(r) = \int dq \, \hat{u}(q) e^{\sqrt{2}iq \cdot r} \tag{11}$$

and use the analogue of relation (8) for the basic one-body operator, $\sum_j \exp\{\sqrt{2}i$

$\times (\boldsymbol{q} \cdot \boldsymbol{r}_j)\}$. Thus

$$(\det B)^2 \sum_{ijkl} \langle \phi_i \phi_j | u | \psi_k \psi_l \rangle (B^{-1})_{ki}(B^{-1})_{lj}$$

$$= \int d\boldsymbol{q}\, \hat{u}(\boldsymbol{q})(\det B) \sum_{ik} \langle \phi_i | e^{\sqrt{2}i\boldsymbol{q}\cdot\boldsymbol{\eta}} | \psi_k \rangle (B^{-1})_{ki} (\det B) \sum_{jl} \langle \phi_j | e^{-\sqrt{2}i\boldsymbol{q}\cdot\boldsymbol{\eta}} | \psi_l \rangle (B^{-1})_{lj}$$

$$= \int d\boldsymbol{q}\, \hat{u}(\boldsymbol{q}) \langle \phi_1 \phi_2 \ldots \Big| \sum_{j=1}^{A/4} e^{\sqrt{2}i\boldsymbol{q}\cdot\boldsymbol{r}_j} \mathscr{A} \Big| \psi_1 \psi_2 \ldots \rangle \langle \phi_1 \phi_2 \ldots \Big| \sum_{j=1}^{A/4} e^{-\sqrt{2}i\boldsymbol{q}\cdot\boldsymbol{r}_j} \mathscr{A} \Big| \psi_1 \psi_2 \ldots \rangle. \quad (12)$$

With the use of eqs. (8)–(12) the full A-particle hamiltonian matrix is reduced to

$$\langle \Phi, H\Psi \rangle = 4 \mathscr{H}_N^3 \mathscr{H}_T$$

$$+ (X_d + X_e) \mathscr{H}_N^2 \int d\boldsymbol{q}\, \hat{u}(\boldsymbol{q}) \mathscr{H}_E(\boldsymbol{q}) \mathscr{H}_E(-\boldsymbol{q}) - 2 X_e \mathscr{H}_N^3 \mathscr{H}_U, \quad (13)$$

where the matrix elements

$$\mathscr{H}_N = \langle \phi_1 \phi_2 \ldots | \mathscr{A} | \psi_1 \psi_2 \ldots \rangle, \quad (14)$$

$$\mathscr{H}_T = \langle \phi_1 \phi_2 \ldots \Big| \sum_{j=1}^{A/4} t_j \mathscr{A} \Big| \psi_1 \psi_2 \ldots \rangle, \quad (15)$$

$$\mathscr{H}_E(\boldsymbol{q}) = \langle \phi_1 \phi_2 \ldots \Big| \sum_{j=1}^{A/4} e^{i\sqrt{2}\boldsymbol{q}\cdot\boldsymbol{r}_j} \mathscr{A} \Big| \psi_1 \psi_2 \ldots \rangle, \quad (16)$$

$$\mathscr{H}_U = \langle \phi_1 \phi_2 \ldots \Big| \sum_{i<j}^{A/4} u(r_{ij}) \mathscr{A} \Big| \psi_1 \psi_2 \ldots \rangle, \quad (17)$$

are matrix elements involving $\frac{1}{4}A$-particle totally antisymmetric orbital functions only. $\mathscr{H}_N$ is the norm or overlap matrix; that is, the matrix element of the $\frac{1}{4}A$-particle antisymmetrizer. The hamiltonian matrix is now expressed in terms of $\mathscr{H}_N$ and the basic one-body matrix elements $\mathscr{H}_T$ and $\mathscr{H}_E$, as well as the orbital 2-body matrix element $\mathscr{H}_U$, where all of these are quantities pertaining to the $\frac{1}{4}A$-particle system of orbital symmetry characterized by a single-column Young tableau.

If a Fourier analysis is made of the general one-body operator (e.g. the nuclear charge or current density operator), the full A-particle matrix element of such an operator is reduced to the calculation of $\mathscr{H}_N$ and $\mathscr{H}_E$ by the relation

$$\langle \Phi \Big| \sum_{j=1}^{A} e^{i\sqrt{2}\boldsymbol{q}\cdot\boldsymbol{r}_j} \Big| \Psi \rangle = 4 \mathscr{H}_N^3 \mathscr{H}_E(\boldsymbol{q}). \quad (18)$$

3. The Bargmann transform

Although a *numerical* evaluation of the basic matrix elements, eqs. (14)–(17), can be achieved most directly by the Brink technique, i.e. by relations such as those of eqs. (8) and (10), the evaluation of such matrix elements in the *analytic* form vital for RGM calculations is complicated for all but the lightest systems because of the difficulties associated with the inversion of the overlap matrix B. The evaluation is greatly simplified by the introduction of the Bargmann transform of the operators $\mathcal{C}\cdot\mathcal{A}$, with $\mathcal{C} = 1, \sum_j t_j, \sum_j \exp\{i\sqrt{2}q \cdot r_j\}$, and $\sum_{ij} \mu(r_{ij})$. The Bargmann transforms of the operators $\mathcal{C}\cdot\mathcal{A}$ for the many-particle system are obtained if the single-particle functions ϕ_i, ψ_i in the multi-dimensional spatial integrals are replaced by

$$\phi_i^* \rightarrow A(\bar{k}_i, r_i), \qquad \psi_i \rightarrow A(k_i^*, r_i), \tag{19}$$

where

$$A(k, r) = \pi^{-\frac{3}{4}} e^{-\frac{1}{2}k \cdot k + \sqrt{2}k \cdot r - \frac{1}{2}r \cdot r} \tag{20}$$

is the kernel function which generates the Bargmann transform in the 3-dimensional single-particle space. The properties of the Bargmann transform which are most relevant for the microscopic cluster model have been discussed in ref. [11]. The notation will follow that of ref. [11]. Thus the r_i are dimensionless single-particle coordinates; that is, r_i is equal to the physical single-particle coordinate vector divided by the oscillator width parameter, $[\hbar/m\omega]^{\frac{1}{2}}$. The same notation was implied in sect. 2 where q is also a dimensionless vector, that is the physical q (measured in fm^{-1}) divided by $[m\omega/\hbar]^{\frac{1}{2}}$. This investigation will be restricted to cluster fragments with equal width parameters. The generalization of the Bargmann transform to the case of fragments with unequal width parameters is straightforward. However, the elimination of spurious center-of-mass motion excitations would then require special techniques.

The $\bar{k}_i$ and k_i are the Bargmann space single-particle vectors. The Bargmann transforms of $\mathcal{C}\cdot\mathcal{A}$ are easy to evaluate in these single-particle variables. The extraction of matrix elements for a particular cluster decomposition of the A-particle system then involves a simple transformation from single-particle to cluster-internal and cluster-relative motion variables, where this is given by the same orthogonal transformation matrix in real space and in Bargmann space variables. The essential point of the present method centers around the fact that the antisymmetrization operation and the transformation to cluster-relative motion variables can be carried out algebraically by computer for the $\frac{1}{4}A$-particle Bargmann transforms characterized by a single-column Young tableau. The extraction of cluster-internal excitations corresponding to cluster fragments in minimum Pauli-allowed states of simple SU(3) symmetry is facilitated by the harmonic oscillator generating function property of $A(k, r)$. The transformation to Bargmann space cluster-internal and cluster-relative motion variables is carried out in sect. 4. In this section we first give the Bargmann transforms of the basic operators in single-particle Bargmann space variables.

The Bargmann transforms of the operators $\mathcal{C}\cdot\mathcal{A}$ are to be denoted by $H_{\mathcal{C}}(\bar{k}, k^*)$ where $\bar{k}$ stands for $\bar{k}_1, \bar{k}_2, \ldots$,

$$H_{\mathcal{C}}(\bar{k}, k^*) = \langle A(\bar{k}, r_1)^* A(\bar{k}_2, r_2)^* \ldots | \mathcal{C}\cdot\mathcal{A} | A(k_1^*, r_1) A(k_2^*, r_2) \ldots \rangle. \tag{21}$$

[As in ref. [11] the Bargmann variables in the bra part of a matrix element will always carry a bar.]

(a) *The single-column norm.* With $\mathcal{C} = 1$, the Bargmann transform of the ($\frac{1}{4}A$-particle) antisymmetrizer, built with permutation operators P, is given by

$$H_N(\bar{k}, k^*) = \sum_P (-1)^{\sigma(P)} \exp\left\{\sum_j \bar{k}_j \cdot Pk_j^*\right\}. \tag{22}$$

(b) *The kinetic energy operator.* With $\mathcal{C} = \sum_i t_i$,

$$H_T(\bar{k}, k^*) = \tfrac{1}{2}\hbar\omega\left\{\sum_P (-1)^{\sigma(P)} \sum_i [\tfrac{3}{2} - \tfrac{1}{2}(\bar{k}_i - Pk_i^*)^2] \exp\left[\sum_j (\bar{k}_j \cdot Pk_j^*)\right]\right\}. \tag{23}$$

(c) *The basic 1-body operator.* With $\mathcal{C} = \sum_j e^{i\sqrt{2}q \cdot r_j}$,

$$H_E(\bar{k}, k^*) = e^{-\frac{1}{2}q \cdot q} \sum_P (-1)^{\sigma(P)} \sum_j [\exp\{iq \cdot (\bar{k}_j + Pk_j^*)\}] \exp\left\{\sum_i (\bar{k}_i \cdot Pk_i^*)\right\}. \tag{24}$$

(d) *A gaussian 2-body interaction.* With $\mathcal{C} = \sum_{i<j} u(r_{ij})$ and $u(r_{ij}) = e^{-\frac{1}{2}\beta r_{ij}^2}$ where β, like r_{ij}, is dimensionless,

$$H_U(\bar{k}, k^*) = (1+\beta)^{-\frac{3}{2}} \sum_P (-1)^{\sigma(P)} \sum_{i<j} \exp[-\tfrac{1}{2}g(\bar{k}_i - \bar{k}_j + Pk_i^* - Pk_j^*)^2]$$

$$\times \exp\left\{\sum_l (\bar{k}_l \cdot Pk_l^*)\right\}, \tag{25a}$$

where

$$g = \beta/2(1+\beta). \tag{25b}$$

In eqs. (22)–(25) the sums over particle indices i, j, l run from 1 to $\frac{1}{4}A$, the sums over permutations contain $(\frac{1}{4}A)!$ terms.

4. The Bargmann transform in cluster coordinates

The transformation from the single-particle k_i to Bargmann space cluster-internal and cluster-relative motion variables is achieved by the same $A \times A$ orthogonal transformation matrix which effects this transformation in real space. For cluster fragments in the $A = 4$–20 mass range, however, only a few cluster internal degrees of freedom carry oscillator excitations. As in ref. [11] the cluster Bargmann space vectors are denoted by K_i, K, $K_{\text{c.m.}}$ where a subscript i denotes a cluster-internal variable, K (without a subscript) denotes the relative motion vector of the binary cluster system,
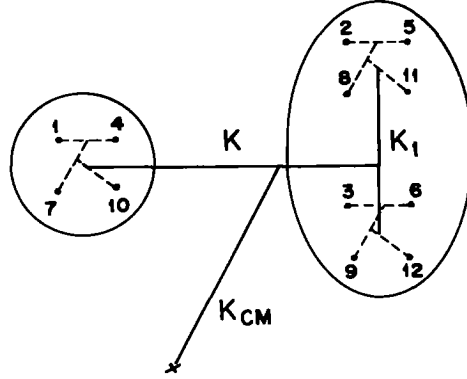
Fig. 1. Illustration of the Bargmann space cluster variables for the $x + {}^8$Be system. Cluster-internal variables such as $\sqrt{\frac{1}{2}}(k_1 - k_4)$ or $\sqrt{\frac{1}{6}}(k_2 + k_5 - 2k_8)$ (dashed lines), can carry no oscillator excitation. They are given no special designation since they are set equal to zero in the final expansions in K-space variables.

and $K_{\text{c.m.}}$ is the Bargmann space vector for the motion of the center of mass of the A-particle system. (See fig. 1.) For $4n$ self-conjugate nuclei, the four single-particle variables k_i, $k_{A/4+i}$, $k_{2A/4+i}$, $k_{3A/4+i}$, (with $i = 1,\ldots,\frac{1}{4}A$), carry the *same* dependence on K, $K_{\text{c.m.}}$, K_1, K_2, ..., whereas cluster-internal variables built from linear combinations of such a set of four (with the same i), can carry no oscillator excitations if the cluster fragments are in minimum Pauli-allowed states of highest possible SU(3) symmetry. For such a cluster decomposition into fragments with mass numbers f and $A - f$, the single-particle Bargmann space variables are transformed according to

$$k_{nA/4+i} = \frac{1}{[f]^{\frac{1}{2}}} \left(-\left[\frac{A-f}{A}\right]^{\frac{1}{2}} K + \left[\frac{f}{A}\right]^{\frac{1}{2}} K_{\text{c.m.}} \right) + \ldots \qquad \text{for } i \leq \tfrac{1}{4}f,$$

$$k_{nA/4+i} = \frac{1}{[A-f]^{\frac{1}{2}}} \left(\left[\frac{f}{A}\right]^{\frac{1}{2}} K + \left[\frac{A-f}{A}\right]^{\frac{1}{2}} K_{\text{c.m.}} \right) + \ldots \qquad \text{for } \tfrac{1}{4}f < i \leq \tfrac{1}{4}A, \quad (26)$$

where $n = 0, 1, 2, 3$, and where the terms abbreviated by $+\ldots$ stand for the cluster-internal degrees of freedom K_i, specific to each particular cluster system.

In principle, the extraction of a many-particle matrix element of $\mathcal{C}\cdot\mathcal{A}$ in the appropriate cluster basis requires three basic steps: (i) a transformation from Bargmann single-particle variables k_i to Bargmann space cluster variables K_i, K, (ii) the combination of single-column Bargmann transforms by the anlogue of eq. (13) to construct the Bargmann transform of $\mathcal{C}\cdot\mathcal{A}$ for the full A-particle system, and (iii) an expansion of this transform in the cluster variables K_i, K retaining only the needed low powers of the internal K_i which correspond to cluster fragments with internal wave functions in minimum Pauli-allowed states of highest possible orbital symmetry. In step (i) the key

relation follows from the orthogonality of the transformation from the k_j to K, $K_{c.m.}$, $K_i, \ldots$:

$$\prod_{j=1}^{A} A(k_j, r_j) = A(K, R) A(K_{c.m.}, R_{c.m.}) \prod_i A(K_i, R_i). \tag{27}$$

In step (iii) the generating function property of $A(K_i, R_i)$ is vital

$$A(K_i, R_i) = \sum_Q \sum_\alpha P(K_i)_\alpha^{(Q0)} \phi(R_i)_\alpha^{(Q0)*}. \tag{28}$$

Here $\phi_\alpha^{(Q0)}$ is a 3-dimensional harmonic oscillator function in the variable R_i, $P(K_i)_\alpha^{(Q0)}$ is its Bargmann transform, and α is any set of convenient subgroup labels of the SU(3) representation $(Q0)$. For the $\alpha + {}^8\text{Be}$ cluster system of fig. 1, e.g., only the internal variable K_1 carries oscillator excitations. To construct a ${}^8\text{Be}$ internal function with $(\lambda\mu) = (40)$ each single-column transform must contribute one oscillator excitation. In the K-space expansions of such a transform therefore the only terms which can contribute to the $\alpha + {}^8\text{Be}$ matrix elements are those which are of first power in $\bar{K}_1$ (and of first power in K_1^*).

The direct method for the evaluation of the matrix elements sketched here is complicated by the presence of the antisymmetrizer, and the process is actually carried out by somewhat more indirect means. This more indirect method uses the fact that the Bargmann kernel function is related to a 0s-oscillator function centered at $\sqrt{2}k$,

$$A(k, r) = \phi_{0s}(r, \sqrt{2}k) e^{\frac{1}{2}k \cdot k}. \tag{29}$$

Differentiations with respect to $k_\alpha (\alpha = x, y, z)$ convert 0s-oscillator functions to 0p-oscillator functions:

$$\frac{\partial}{\partial k_\alpha} A(k, r) = \{\phi_{0p_\alpha}(r, \sqrt{2}k) e^{\frac{1}{2}k \cdot k} + \ldots\}. \tag{30}$$

The term abbreviated by $+\ldots$ is not written explicitly. The term indicates that there is an additional 0s term. It is not written explicitly because such a 0s term is generally annihilated by the action of the antisymmetrizer if a 0p-oscillator function is needed in the construction of the many-particle matrix elements. Similarly, sd-shell functions are created by appropriate double k-variable differentiations. A transformation from the single-particle Bargmann space variables, k, to cluster variables, via eq. (26), in which $K_{c.m.}$ and K_i are set equal to zero, leads to a product of single-particle oscillator functions centered at position vectors appropriate for fragments f and $A-f$. The action of the antisymmetrizer is carried out in a straightforward fashion in the space of single-particle variables. The transformation to cluster-relative motion K-space variables is also straightforward. Finally, the dependence on the internal Bargmann space variables can be regained through the construction of SU(3)-coherent states of the simple SU(3) symmetries appropriate for the cluster-internal K_i in the K-space expansions of the single-column Bargmann transforms.

The method is best illustrated with a few examples of the type to be treated in this investigation.

4.1. THE 1+2 PARTICLE SYSTEM

For the 12-particle system with an $\alpha + {}^8$Be cluster decomposition, the single-column Bargmann functions are built from 3-particle systems where one of the particles comes from cluster fragment of mass number $f = 4$, and the other two from cluster fragment $A - f = 8$. For this system

$$\left[\frac{\partial}{\partial \bar{k}_{3\alpha}} \left\{ \prod_{i=1}^{3} A(\bar{k}_i, r_i) \right\} \right]_{\bar{K}_i = 0, K_{c.m.} = 0}$$

$$= e^{\bar{K}^2/8} \phi_{0s}(r_1, -[2(A-f)/fA]^{\frac{1}{2}}\bar{K}) \phi_{0s}(r_2, +[2f/A(A-f)]^{\frac{1}{2}}\bar{K})$$

$$\times \{ \phi_{0p_z}(r_3, +[2f/A(A-f)]^{\frac{1}{2}}\bar{K}) + \ldots \}, \tag{31}$$

with $\alpha = x, y$ or z. The operation implied by the notation for the left-hand side of eq. (31) involves

(i) differentiation with respect to the single-particle Bargmann space variable,

(ii) subsequent transformation from the $\bar{k}_i$ to $\bar{K}, K_i, K_{c.m.}$, and

(iii) the setting to zero of all but the cluster-relative motion $\bar{K}$-space variable.

(The term $+\ldots$ is included for completeness; it again indicates the presence of a 0s component for particle 3, also centered at $\sqrt{2f/(A-f)A}\,\bar{K}$. This term is annihilated by the subsequent action of the antisymmetrizer and can thus be omitted henceforth.) When applied to the bra side of the Bargmann transform the above operation yields spatial functions of oscillator excitations appropriate for the internal functions of the cluster fragments. To regain the proper dependence on the corresponding internal Bargmann space variables K_i (the single Bargmann space variable $\bar{K}_1$ in this example), it is sufficient to take the scalar product

$$\bar{K}_1 \cdot \left[\frac{\partial}{\partial \bar{k}_3} \left\{ \prod_{i=1}^{3} A(\bar{k}_i, r_i) \right\} \right]_{\substack{\bar{K}_i = 0 \\ K_{c.m.} = 0}} = e^{\bar{K}^2/8} \sum_{\alpha} P(\bar{K}_1)_{\alpha}^{(10)} \phi(1)^{(00)} \phi(2)^{(00)} \phi(3)_{\alpha}^{(10)}, \tag{32}$$

where (1), (2) and (3) are shorthand notations for the arguments of eq. (31), and the oscillator labels 0s, 0p are replaced by the SU(3) labels (00), (10). The state in eq. (32) is an SU(3)-coherent state of irreducible character $(\lambda\mu) = (10)$. Using the properties of the K-space functions the product of four such coherent states will yield an internal state of SU(3) character $(\lambda_{A-f}\mu_{A-f}) = (40)$ where the desired subgroup label can be extracted by the appropriate SU(3) coupling.

The combination of the operations implied by eqs. (31) and (32) when applied to both the bra ($\bar{K}$) variables and the ket (K^*) variables of the Bargmann transform of $\mathcal{O}\mathcal{A}$ leads to the single-column Bargmann space functions of SU(3) symmetry appropriate for the cluster-internal wave functions. By defining functions F which are the generalizations

of the functions introduced in ref. [11]) for the norm kernel ($\mathcal{C} = 1$) [see eq. (57) of ref. [11]]
the necessary single-column Bargmann transforms are obtained at once. For the $1+2$
particle system, we define

$$F_\mathcal{C}\begin{pmatrix} 0;0\alpha \\ 0;0\beta \end{pmatrix} \equiv \left[\frac{\partial}{\partial \bar{k}_{3\alpha}} \frac{\partial}{\partial k_{3\beta}^*} H_\mathcal{C}(\bar{k}, k^*)\right]_{K_i = 0, K_j^* = 0, K_{c.m.} = 0} \tag{33}$$

The single-column Bargmann transform $H_\mathcal{C}(\bar{k}, k^*)$ is defined by eq. (21). The operations
implied by the right-hand side are identical to those introduced in connection with eq.
(31). With this F,

$$\sum_{\alpha\beta} P(\bar{K}_1)_\alpha^{(10)} P(K_1^*)_\beta^{(01)} F_\mathcal{C}\begin{pmatrix} 0;0\alpha \\ 0;0\beta \end{pmatrix} \equiv H_\mathcal{C}\begin{pmatrix} 1(00) & 1(00) \\ 2(10) & 2(10) \end{pmatrix}, \tag{34}$$

where the $H_\mathcal{C}(\begin{smallmatrix} 1(00) & 1(00) \\ 2(10) & 2(10) \end{smallmatrix})$ are the generalizations to the arbitrary operator $\mathcal{C}$ of the single-
column Bargamann space functions introduced in ref. [11]) for the norm ($\mathcal{C} = 1$). [See, in
particular appendix C of ref. [11]), and note the slight generalization of notation. The
factor $\exp\{\eta(\bar{K} \cdot K^*)\}$ of ref. [11]) has also been absorbed into the definition of $H_\mathcal{C}$.]

4.2. THE 1+3 PARTICLE SYSTEM

For the 16-particle system with an $\alpha + {}^{12}C$ cluster decomposition, the single-column
Bargmann functions are built from $1+3$ particle systems where the 3 particles from
cluster fragment $A - f = 12$ must carry SU(3) character $(\lambda\mu) = (01)$ built from the ${}^{12}C$
Bargmann space internal functions $\bar{K}_1$ and $\bar{K}_2$. With $\sqrt{\frac{1}{2}}[\bar{K}_1 \times \bar{K}_2] \equiv K_{12}$, the SU(3)
$(\lambda\mu) = (01)$ coherent states can be constructed by

$$\sqrt{\tfrac{1}{2}}[\bar{K}_1 \times \bar{K}_2] \cdot \left[\left[\frac{\partial}{\partial \bar{k}_3} \times \frac{\partial}{\partial \bar{k}_4}\right]\left\{\prod_{i=1}^4 A(\bar{k}_i, r_i)\right\}\right]_{K_i = 0, K_{c.m.} = 0}$$

$$= e^{\bar{K}^2/8} \sum_{\alpha\beta\gamma} \varepsilon_{\alpha\beta\gamma} P(\bar{K}_{12})_\alpha^{(01)} \phi(1)^{(00)} \phi(2)^{(00)} \phi(3)_\beta^{(10)} \phi(4)_\gamma^{(10)}, \tag{35}$$

where the $\phi(i)$ with $i = 2, 3, 4$ are oscillator functions centered at $+\sqrt{2f/(A-f)A}\,\bar{K}$,
while $\phi_{0s}(1)$ is centered at $-\sqrt{2(A-f)/Af}\,\bar{K}$. The appropriate single-column
Bargmann space function is now

$$\sum_{\alpha\beta\gamma}\sum_{\alpha'\beta'\gamma'} \varepsilon_{\alpha\beta\gamma} P(\bar{K}_{12})_\alpha^{(01)} \varepsilon_{\alpha'\beta'\gamma'} P(K_{12}^*)_{\alpha'}^{(10)} F_\mathcal{C}\begin{pmatrix} 0;0\beta\gamma \\ 0;0\beta'\gamma' \end{pmatrix}$$

$$\equiv H_\mathcal{C}\begin{pmatrix} 1(00) & 1(00) \\ 3(01) & 3(01) \end{pmatrix}. \tag{36}$$

4.3. THE 2+3 PARTICLE SYSTEM

As a further example, we consider the 20-particle system with a ^{8}Be$+$ ^{12}C cluster decomposition. The single-column Bargmann space functions for this system are built from $2+3$ particle systems where the 2 particles from fragment $f = 8$ must carry SU(3) character $(\lambda\mu) = (10)$ while the 3 particles from fragment $A - f = 12$ must carry SU(3) character $(\lambda\mu) = (01)$. The appropriate SU(3) coherent states are constructed from

$$\sum_{\alpha\beta} \bar{K}_{1\alpha} \sqrt{\tfrac{1}{2}} [\bar{K}_2 \times \bar{K}_3]_\beta \left[\frac{\partial}{\partial \bar{k}_{2\alpha}} \left[\frac{\partial}{\partial \bar{k}_4} \times \frac{\partial}{\partial \bar{k}_5} \right]_\beta \left\{ \prod_{i=1}^{5} A(\bar{k}_i, r_i) \right\} \right]_{\substack{\bar{K}_i = 0 \\ \bar{K}_{\text{c.m.}} = 0}}$$

$$= e^{\bar{K}^2/8} \sum_{\alpha\beta\gamma\nu} P(\bar{K}_1)_\alpha^{(10)} P(\bar{K}_{23})_\beta^{(01)} \varepsilon_{\beta\gamma\nu} \phi(1)^{(00)} \phi(2)_\alpha^{(10)} \phi(3)^{(00)} \phi(4)_\gamma^{(10)} \phi(5)_\nu^{(10)}. \quad (37)$$

Now

$$\sum_{\alpha\beta\gamma\nu} \sum_{\alpha'\beta'\gamma'\nu'} P(\bar{K}_1)_\alpha^{(10)} P(\bar{K}_{23})_\beta^{(01)} \varepsilon_{\beta\gamma\nu} P(K_1^*)_{\alpha'}^{(01)} P(K_{23}^*)_{\beta'}^{(10)} \varepsilon_{\beta'\gamma'\nu'}$$

$$\times F_\epsilon \begin{pmatrix} 0\alpha ; 0\gamma\nu \\ 0\alpha' ; 0\gamma'\nu' \end{pmatrix} \equiv H_\epsilon \begin{pmatrix} 2(10) \\ 3(01) \end{pmatrix} \begin{pmatrix} 2(10) \\ 3(01) \end{pmatrix}. \quad (38)$$

The corresponding single-column Bargmann space function needed for a matrix element connecting a ^{8}Be$+$ ^{12}C cluster function to an $\alpha + {}^{16}$O cluster function is given by

$$\sum_{\alpha\beta\gamma\nu} P(\bar{K}_1)_\alpha^{(10)} P(\bar{K}_{23})_\beta^{(01)} \varepsilon_{\beta\gamma\nu} F_\epsilon \begin{pmatrix} 0\alpha ; 0\gamma\nu \\ 0 ; 0xyz \end{pmatrix} \equiv H_\epsilon \begin{pmatrix} 2(10) \\ 3(01) \end{pmatrix} \begin{pmatrix} 1(00) \\ 4(00) \end{pmatrix}, \quad (39)$$

where the SU(3)-scalar function $\sqrt{\tfrac{1}{6}}(K_1^* \cdot [K_2^* \times K_3^*])$ for the ^{16}O fragment need not be written explicitly.

4.4. THE 1+5 PARTICLE SYSTEM

Cluster systems with sd shell fragments can also be included. For the 24-particle system with an $\alpha + {}^{20}$Ne cluster decomposition, e.g., the $1+5$ particle single-column functions are given by

$$H_\epsilon \begin{pmatrix} 1(00) \\ 5(20) \end{pmatrix} \begin{pmatrix} 1(00) \\ 5(20) \end{pmatrix} \equiv \sum_{\alpha\beta\gamma\delta} P(\bar{K}_4)_{(\alpha\beta)}^{(20)} P(K_4^*)_{(\gamma\delta)}^{(02)} F_\epsilon \begin{pmatrix} 0 ; 0xyz(\alpha\beta) \\ 0 ; 0xyz(\gamma\delta) \end{pmatrix}, \quad (40)$$

where $(\alpha\beta)$ in F_σ denotes $[1 + \delta_{\alpha\beta}]^{-\frac{1}{2}} (\partial^2 / \partial \bar{k}_{6\alpha} \partial \bar{k}_{6\beta})$ and

$$P(\bar{K}_4)_{(\alpha\beta)}^{(20)} \equiv [1 + \delta_{\alpha\beta}]^{-\frac{1}{2}} \bar{K}_{4\alpha} \bar{K}_{4\beta}.$$

Single-column Bargmann space functions of the type

$$H_\sigma \begin{pmatrix} l(\lambda_l \mu_l) \\ n(\lambda_n \mu_n) \end{pmatrix} \begin{pmatrix} l'(\lambda_{l'} \mu_{l'}) \\ n'(\lambda_{n'} \mu_{n'}) \end{pmatrix},$$

with $l = \frac{1}{4}f$, $n = \frac{1}{4}(A - f)$, needed for the calculation of cluster norm and overlap matrix elements ($\mathcal{O} = 1$) have been evaluated in ref. [11] for many simple cluster systems with $l + n \leq 6$. [See in particular, the tabulation in appendix C of ref. [11].] In ref. [11] the $(\lambda_l \mu_l)$ and $(\lambda_n \mu_n)$ were coupled to a specific resultant single column $(\lambda_c \mu_c)$. At present, we shall reserve this coupling to a later stage in the calculation. This simplifies the calculation of the $H_{\mathcal{O}}(\vdots | \vdots)$. The $H_{\mathcal{E}}$ of ref. [11] with $(\lambda_l \mu_l)$ and $(\lambda_n \mu_n)$ coupled to $(\lambda_c \mu_c)$ were calculated by means of auxiliary coefficients $y_{m(\lambda'\mu')}$ which had to be evaluated for each $(\lambda_c \mu_c)$ via specific SU(3) coupling coefficients and a knowledge of the $F_{\mathcal{E}}$. Since the $H_{\mathcal{E}}$, as defined in this investigation, automatically have the desired $(\lambda_l \mu_l)$, $(\lambda_n \mu_n)$, $(\lambda_{l'} \mu_{l'})$ and $(\lambda_{n'} \mu_{n'})$, their evaluation via relations such as those of eqs. (34), (36) and (38)–(40) is somewhat simpler. In the present method the single column $(\lambda_l \mu_l)$ are first combined to the resultant $(\lambda_f \mu_f)$ when the single-column Bargmann transforms are combined to A-particle Bargmann transforms of full 4-columned Young symmetry. Similarly, the single column $(\lambda_n \mu_n)$ are first combined to resultant $(\lambda_{A-f} \mu_{A-f})$. The $(\lambda_f \mu_f)$ and $(\lambda_{A-f} \mu_{A-f})$ of the f- and $(A-f)$-particle cluster fragments are then combined to resultant A-particle internal SU(3) symmetry $(\lambda_c \mu_c)$. This process therefore is now part of the SU(3)-recoupling transformations of the Bargmann space internal K_i of the full A-particle Bargmann transforms and is thus discussed in sects. 5–7.

The computer code used in ref. [11] to calculate the functions $F_{\mathcal{E}}$ with $\mathcal{O} = 1$ has now been generalized to include the basic one- and two-body operators enumerated in sect. 3. For the kinetic energy operator, however, no new Bargmann space functions are needed since the Bargmann transform of T can be related to that for the norm. Before proceeding to the Bargmann transform of the interaction this relationship will be established.

The kinetic energy operator. The Bargmann transform of T for the full A-particle system, characterized by a 4-columned tableau, can be obtained from the single-column transforms $H_N(\bar{k}, k^*)$ and $H_T(\bar{k}, k^*)$ of eqs. (22) and (23) by the Bargmann space analogue of eq. (13) (T-term). Since we want the kinetic energy relative to the c.m., a subtraction for the c.m. motion must be made. The Bargmann transform of the translationally invariant T for the A-particle system (full 4-columned symmetry) is therefore

$$4H_T(\bar{k}, k^*)H_N^3(\bar{k}, k^*) - H_T^{c.m.}$$

$$= 4\left\{\tfrac{1}{2}\hbar\omega\left(\sum_P^{(A/4)!}(-1)^{\sigma(P)}\sum_{i=1}^{A/4}[\tfrac{3}{2} - \tfrac{1}{2}(\bar{k}_i - Pk_i^*)^2]\exp\left\{\sum_{j=1}^{A/4}\bar{k}_j \cdot Pk_j^*\right\}\right)\right\}$$

$$\times H_N^3(\bar{k}, k^*) - \tfrac{1}{2}\hbar\omega[\tfrac{3}{2} - \tfrac{1}{2}(K_{c.m.} - K_{c.m.}^*)^2]H_N^4(\bar{k}, k^*)$$

$$= \tfrac{1}{2}\hbar\omega[\tfrac{3}{2}(A-1) - \tfrac{1}{2}\sum_{i=1}^A(\bar{k}_i^2 + k_i^{*2}) + \tfrac{1}{2}(K_{c.m.} - K_{c.m.}^*)^2]H_N^4(\bar{k}, k^*)$$

$$+ 4H_N^3\tfrac{1}{2}\hbar\omega\left[\frac{d}{dZ}\sum_P^{(A/4)!}(-1)^{\sigma(P)}\exp\left\{\sum_{j=1}^{A/4}\sqrt{Z}\bar{k}_j \cdot P\sqrt{Z}k_j^*\right\}\right]_{Z=1}, \qquad (41)$$

where the simple identity

$$\sum_P (-1)^{\sigma(P)} \sum_i (\bar{k}_i \cdot Pk_i^*) \exp\left\{\sum_j \bar{k}_j \cdot Pk_j^*\right\}$$

$$= \left[\frac{\mathrm{d}}{\mathrm{d}Z} \sum_P (-1)^{\sigma(P)} \exp\left\{\sum_j \sqrt{Z}\bar{k}_j \cdot P\sqrt{Z}k_j^*\right\}\right]_{Z=1} \qquad (42a)$$

has been used to convert the second term to a useful form. In this form the action of the operation

$$\left[\frac{\partial}{\partial \bar{k}_{i_\alpha}} \cdots \frac{\partial}{\partial k_{j_\beta}^*}\right]_{\bar{K}_i = 0, K_j^* = 0, K_{c.m.} = 0}$$

defined by the functions F_ℓ, of eq. (33), effectively converts the second term of eq. (41) to

$$4H_N^3 \tfrac{1}{2}\hbar\omega \left[\frac{\mathrm{d}}{\mathrm{d}Z}\left\{Z^{\frac{1}{2}q_{int}} H_N\binom{l(\lambda_l\mu_l)}{n(\lambda_n\mu_n)}\bigg| \begin{matrix} l'(\lambda_{l'}\mu_{l'}) \\ n'(\lambda_{n'}\mu_{n'}) \end{matrix}\bigg)_{\substack{K \to \sqrt{Z}K \\ K^* \to \sqrt{Z}K^*}}\right\}\right]_{Z=1}, \qquad (42b)$$

where q_{int} is the total number of k-space derivatives in F_N, that is the total number of internal oscillator quanta carried by the cluster fragments f, $A-f$, f' and $A-f'$ in the single-column transform, summing over cluster fragments in both bra and ket. (For the example of eq. (38), $q_{int} = 1+2+1+2 = 6$.)

In the first term of eq. (41) the sum

$$\sum_i^A (\bar{K}_i^2 + k_i^{*2}) = \sum_i (\bar{K}_i^2 + K_i^{*2}) + \bar{K}^2 + K^{*2} + K_{c.m.}^2 + K_{c.m.}^{*2} \qquad (43)$$

can effectively be replaced by $\bar{K}^2 + K^{*2}$ if all cluster fragments have internal functions of minimum Pauli-allowed oscillator excitation. If a $\bar{K}_i^2$ for a particular internal cluster degree of freedom were retained in the sum (43) within the first term of eq. (41), the expansion of H_N^4 in internal K-space variables would perforce contribute two fewer powers of this $\bar{K}_i$. This would correspond to an A-particle norm transform with Q_{int} two less than the minimum Pauli-allowed oscillator excitation, and such a norm transform would be identically equal to zero.

Thus eq. (41) for the full A-particle Bargmann transform of T can be converted to

$$H_T(K, K^*) = \tfrac{1}{2}\hbar\omega\left\{[\tfrac{3}{2}(A-1)+\tfrac{1}{2}Q_{int}-\tfrac{1}{2}(K^2+K^{*2})]H_N(K, K^*)\right.$$

$$\left. + \left[\frac{\mathrm{d}}{\mathrm{d}Z} H_N(\sqrt{Z}K, \sqrt{Z}K^*)\right]_{Z=1}\right\}, \qquad (44)$$

where $Q_{int} = 4q_{int}$ and where $H_T(\bar{K}, K^*)$ and $H_N(\bar{K}, K^*)$, with variables $\bar{K}, K^*$ (rather than $\bar{k}, k^*$), stand for the (properly normalized) Bargmann transform of the full A-particle system of 4-columned symmetry. The Bargmann transform of the kinetic

energy operator can thus be calculated from a knowledge of $H_N(\bar{K}, K^*)$. The combination of four single-column functions of type $H(_{(A-f)/4\colon\colon|_{(A-f')/4\colon\colon}})$ to build the (properly normalized) Bargmann transforms $H_\ell(\bar{K}, K^*)$ is discussed in detail in sects. 5–7. Before proceeding to the general cases the single-column functions $H_\ell(_{(A-f)/4\colon\colon|_{(A-f')/4\colon\colon}})$ for operators, $\mathcal{O} = \sum_j \exp\sqrt{2}\mathrm{i}(q \cdot r_j)$ and $\mathcal{O} = \sum_{i<j} u(r_{ij})$ have to be calculated.

Single-column functions of this type can in principle be evaluated by straightforward application of equations such as eqs. (34), (36), or (38)–(40). In actual practice, the sums over $\alpha, \beta, \ldots$ in these equations could contain a huge number of terms. Straightforward application of eq. (34) would involve a sum over 3^2 or 9 F_ℓ functions; for eq. (38) it would involve a sum over 3^6 or 729 F_ℓ functions. In actual practice the two F_ℓ functions with $\alpha\beta = zz$ and zx are sufficient for the evaluation of the $H_\ell(\colon\colon|\colon\colon)$ of eq. (34); while a choice of 9 F_ℓ functions are sufficient for the $H_\ell(\colon\colon|\colon\colon)$ of eq. (38). This is so because the structure of the $H_\ell(k, k^*)$, eqs. (22)–(25), and the nature of the $F_\ell(\bar{-}\ \bar{-}\ \bar{-}\ |\ \bar{-}\ \bar{-}\ \bar{-})$ permit us to enumerate all possible scalar products which can be constructed from the Bargmann space vectors $\bar{K}_1, \ldots \bar{K}_{23}, \ldots K_1^*, \ldots K, K^*$, and q (in the case of H_E). The coefficients of the various possible scalar products can be determined by assigning specific orientations to $\bar{K}_1, \ldots K_1^*, \ldots$. The process can be understood most easily by examining the structure of some of the results.

For the tabulation of the results it is useful to define a few short-hand symbols. Assuming that the $\bar{K}$ variables or the bra side of a matrix element always correspond to cluster decomposition into fragments of mass numbers f and $A-f$, while the K^* variables or the ket side of a matrix element correspond to a decomposition into fragments of mass numbers $f', A-f'$ (including, of course, the possibility $f' = f$), then it is useful to define

$$\mathscr{K} = \left[\frac{A}{f(A-f)}\right]^{\frac{1}{2}} \bar{K}, \qquad \mathscr{K}^* = \left[\frac{A}{f'(A-f')}\right]^{\frac{1}{2}} K^*. \tag{45}$$

In terms of these variables the quantities e^p, defined in ref. [11], have the value

$$\mathrm{e}^p \equiv \exp\left(-p_x \mathscr{K} \cdot \mathscr{K}^*\right), \qquad (\mathrm{e}^p \mathrm{e}^q = \mathrm{e}^{p+q}). \tag{46}$$

The H_E can be expressed in terms of these e^p and exponentials in the functions $\theta(a, b)$ with $a, b = 1$ or 2, where

$$\theta(1, 1) \equiv i\boldsymbol{q} \cdot \left[-\left(\frac{A-f}{A}\right)\mathscr{K} - \left(\frac{A-f'}{A}\right)\mathscr{K}^*\right],$$

$$\theta(2, 1) \equiv i\boldsymbol{q} \cdot \left[\frac{f}{A}\mathscr{K} - \left(\frac{A-f'}{A}\right)\mathscr{K}^*\right],$$

$$\theta(1, 2) \equiv i\boldsymbol{q} \cdot \left[-\left(\frac{A-f}{A}\right)\mathscr{K} + \frac{f'}{A}\mathscr{K}^*\right], \tag{47}$$

$$\theta(2, 2) \equiv i\boldsymbol{q} \cdot \left[\frac{f}{A}\mathscr{K} + \frac{f'}{A}\mathscr{K}^*\right].$$

These satisfy the simple relation

$$\theta(a, b) - \theta(a', b') = iq \cdot [(a-a')\mathscr{K} + (b-b').\mathscr{K}^*],$$ (48)

which is useful in the convolution of the Fourier transform.

Finally, for a gaussian interaction the H_U can be expressed in terms of the e^p and exponentials in functions ϕ, with

$$
\begin{aligned}
e^{\phi(1, 0)} &\equiv \exp\left[-\tfrac{1}{2}g\mathscr{K} \cdot \mathscr{K}\right], \\
e^{\phi(0, 1)} &\equiv \exp\left[-\tfrac{1}{2}g\mathscr{K}^* \cdot \mathscr{K}^*\right], \\
e^{\phi(1, 1)} &\equiv \exp\left[-\tfrac{1}{2}g(\mathscr{K} + \mathscr{K}^*) \cdot (\mathscr{K} + \mathscr{K}^*)\right], \\
e^{\phi(1, -1)} &\equiv \exp\left[-\tfrac{1}{2}g(\mathscr{K} - \mathscr{K}^*) \cdot (\mathscr{K} - \mathscr{K}^*)\right],
\end{aligned}
$$ (49a)

where g is defined in eq. (25b), and $\phi(i, j)$ can be expressed by

$$\phi(i, j) = -\tfrac{1}{2}g(i\mathscr{K} + j.\mathscr{K}^*)^2.$$ (49b)

The $\alpha + {}^8Be$ single-column transforms. The evaluation of eq. (34) for the single-column Bargmann transform of $\mathscr{C} = \sum_j \exp\{i\sqrt{2}(q \cdot r_j)\}$ for the 12-particle system with an $\alpha + {}^8Be$ cluster decomposition, $(f' = f = 4, A - f' = A - f = 8)$ leads to

$$
\begin{aligned}
H_E\begin{pmatrix} 1(00) \\ 2(10) \end{pmatrix}\begin{pmatrix} 1(00) \\ 2(10) \end{pmatrix} = &\exp\left[\tfrac{1}{4}\frac{f(A-f)}{A}\ \mathscr{K} \cdot \mathscr{K}^*\right]\exp[-\tfrac{1}{2}q \cdot q] \\
&\times [(\bar{K}_1 \cdot K_1^*)\{e^{\theta(1, 1)} - e^1 e^{\theta(1, 2)} - e^1 e^{\theta(2, 1)} + (2 - e^1)e^{\theta(2, 2)}\} \\
&+ (K_1^* \cdot \mathscr{K})(\bar{K}_1 \cdot \mathscr{K}^*)\{-e^1 e^{\theta(1, 2)} - e^1 e^{\theta(2, 1)} - e^1 e^{\theta(2, 2)}\} \\
&+ (iq \cdot \bar{K}_1)(K_1^* \cdot \mathscr{K})\{e^1 e^{\theta(2, 1)} - e^1 e^{\theta(2, 2)}\} \\
&+ (iq \cdot K_1^*)(\bar{K}_1 \cdot \mathscr{K}^*)\{e^1 e^{\theta(1, 2)} - e^1 e^{\theta(2, 2)}\} \\
&+ (iq \cdot \bar{K}_1)(iq \cdot K_1^*)(1 - e^1)e^{\theta(2, 2)}].
\end{aligned}
$$ (50)

From the general structure of the single-column transform, eq. (24), it can be seen that the coefficients of the various exponentials must be built from products of factors of the type $(iq \cdot a)$, $(iq \cdot iq)$, or $(a \cdot b)$ where $a(b)$ are one of the vectors $\mathscr{K}$, $\mathscr{K}^*$, $\bar{K}_1$, K_1^*. However, the q-independent scalar products $(a \cdot b)$ must arise from a factor with the same structure as the norm and must therefore be SU(3)-scalars; they therefore include only the possibilities $(\bar{K}_1 \cdot K_1^*)$, $(K_1^* \cdot \mathscr{K})$, $(\bar{K}_1 \cdot \mathscr{K}^*)$, and $(\mathscr{K} \cdot \mathscr{K}^*)$. From the definition of the single-column function $H_{\mathscr{C}}$ each coefficient must contain both $\bar{K}_1$ and K_1^* to the first power. Thus the q-independent coefficients which are linear in both $\mathscr{K}$ and $\mathscr{K}^*$, for example, must be of the form $[A(\mathscr{K} \cdot K_1^*)(\bar{K}_1 \cdot \mathscr{K}^*) + B(\bar{K}_1 \cdot K_1^*)(\mathscr{K} \cdot \mathscr{K}^*)]$ where A and B are easily evaluated from the q-independent coefficients (linear in $\mathscr{K}$ and $\mathscr{K}^*$) in the various exponential terms of the two functions $F_E(\begin{smallmatrix} 0;\ 0z \\ 0;\ 0z \end{smallmatrix})$ and $F_E(\begin{smallmatrix} 0;\ 0z \\ 0;\ 0x \end{smallmatrix})$. The result [cf. eq. (50)] shows that $B = 0$.

The evaluation of eq. (34) for $\mathscr{C} = \sum_{i<j}\mu(r_{ij})$ with u of gaussian form [eq. (25)], leads

to the single-column transform ($\alpha + {}^8$Be cluster decomposition)

$$H_U \begin{pmatrix} 1(00) & | & 1(00) \\ 2(10) & | & 2(10) \end{pmatrix} = (1-2g)^{\frac{3}{2}} \exp\left[\frac{1}{4}\frac{f(A-f)}{A}\, \mathscr{K}^- \cdot \mathscr{K}^*\right]$$

$$\times [P(10000)\{(1-2g)+(-1+2g)e^1e^{\phi(1,0)}+(-1+2g)e^1e^{\phi(0,1)}$$

$$+(2-g)e^{\phi(1,1)}+(-1-g)e^1e^{\phi(1,-1)}\}$$

$$+P(01100)\{(-1+2g)e^1e^{\phi(1,0)}+(-1+2g)e^1e^{\phi(0,1)}+g^2e^{\phi(1,1)}-(1-g)^2e^1e^{\phi(1,-1)}\}$$

$$+P(00011)\{g^2e^{\phi(1,1)}-g^2e^1e^{\phi(1,-1)}\}$$

$$+P(00110)\{(g-2g^2)e^1e^{\phi(1,0)}+g^2e^{\phi(1,1)}+(-g+g^2)e^1e^{\phi(1,-1)}\}$$

$$+P(01001)\{(g-2g^2)e^1e^{\phi(0,1)}+g^2e^{\phi(1,1)}+(-g+g^2)e^1e^{\phi(1,-1)}\}], \qquad (51)$$

where

$$P(abcde) \equiv (\bar{K}_1 \cdot K_1^*)^a (\bar{K}_1 \cdot \mathscr{K}^*)^b (K_1^* \cdot \mathscr{K}^-)^c (\bar{K}_1 \cdot \mathscr{K}^-)^d (K_1^* \cdot \mathscr{K}^*)^e. \qquad (52)$$

Note that H_U (unlike H_E) contains scalar products of the form $(\bar{K}_1 \cdot \mathscr{K}^-)$, SU(3) (20) tensors, and $(K_1^* \cdot \mathscr{K}^*)$, SU(3) (02) tensors. Note also that both H_U and H_E are invariant under the transformation $\bar{K}_1 \leftrightarrow K_1^*$, $\mathscr{K}^- \leftrightarrow \mathscr{K}^*$ (interchange of bra and ket).

Finally, note that the single-column norm transforms, H_N, can be read from $H_E(q = 0)$ or from $H_U(g = 0)$ through the relations $H_N = (\frac{1}{4}A)^{-1}H_E(q = 0)$ or $H_N = [\frac{1}{2} \cdot \frac{1}{4}A(\frac{1}{4}A-1)]^{-1}H_U(g = 0)$. For the $\alpha + {}^8$Be system this leads to [cf. appendix C of ref. [11])]

$$H_N \begin{pmatrix} 1(00) & | & 1(00) \\ 2(10) & | & 2(10) \end{pmatrix} = \exp\left(\tfrac{2}{3}\mathscr{K}^- \cdot \mathscr{K}^*\right)\{(\bar{K}_1 \cdot K_1^*)(e^0 - e^1) - e^1(\bar{K}_1 \cdot \mathscr{K}^*)(K_1^* \cdot \mathscr{K}^-)\}. \quad (53)$$

Single-column functions H_E and H_U have been evaluated for the following cluster systems: $A = 16$, $\alpha + {}^{12}$C and ^{8}Be + ^{8}Be cluster decompositions; $A = 20$, $\alpha + {}^{16}$O and ^{8}Be + ^{12}C cluster decompositions; and $A = 24$, $\alpha + {}^{20}$Ne, ^{8}Be + ^{16}O and ^{12}C + ^{12}C cluster decompositions. Since the expressions for some of the heavier systems are somewhat lengthy, the results are given in tabular form. The appendix illustrates the results with the ^{12}C + ^{12}C and ^{8}Be + ^{16}O cluster decompositions of the $A = 24$ system. Complete results are tabulated in ref. [15]).

5. A-particle Bargmann transforms

The combination of single-column Bargmann transforms H_N, H_E, H_U to construct the Bargmann transform of the interaction for the A-particle system of full 4-columned Young symmetry is carried out in this section by the Bargmann space analogue of eq. (13). For a gaussian interaction the Fourier transform [normalized according to eq.

(11)] is given by

$$\hat{u}(q) = \left(\frac{\gamma}{\pi}\right)^{\frac{3}{2}} e^{-\gamma q^2}, \quad \text{for} \quad u(r) = e^{-\frac{1}{2}\beta r^2}, \tag{54}$$

with

$$\gamma = \frac{1}{2g} - 1, \qquad g = \frac{\beta}{2(1+\beta)}.$$

For the interaction V, with arbitrary exchange mixture, the full A-particle Bargmann transform $H_V(\bar{K}, K^*)$ is thus given by

$$H_V(\bar{K}, K^*) = C\left\{(X_d + X_e)H_N^2 \left(\frac{\gamma}{\pi}\right)^{\frac{3}{2}} \int dq\, e^{-\gamma q \cdot q} H_E(q) H_E(-q) - 2X_e H_N^3 H_U\right\}, \tag{55}$$

where the exchange coefficients X_d, X_e are given by eq. (5), and H_N, H_E and H_U are single-column transforms of the type given in eqs. (50) and (51) for the $\alpha + ^8$Be system. Since these single-column transforms are constructed for cluster-internal functions with single-column SU(3) symmetries appropriate for fragments of mass numbers $\frac{1}{4}f$ and $\frac{1}{4}(A-f)$, a transformation must be made to internal functions of full 4-columned symmetry. For systems with a ^{8}Be fragment in bra each of the four single-column factors H_e of eq. (55) carries an internal function $P(\bar{K}_1)^{(10)}$ coupled to other K-space functions. With a ^{12}C fragment in bra there is a similar $P(\bar{K}_{23})^{(01)}$. A recoupling procedure is used to combine the four $\bar{K}_1$ factors to $P(\bar{K}_1)^{(40)}$ and the four K_{23} factors to $P(\bar{K}_{23})^{(04)}$. Finally the ^{8}Be symmetry $(\lambda_f \mu_f) = (40)$ is coupled with the ^{12}C symmetry $(\lambda_{A-f}\mu_{A-f}) = (04)$ to resultant SU(3) symmetry $(\lambda_c \mu_c)$. The details of this recoupling process are best illustrated by specific examples (see sect. 7). Since the single-column functions H_N, H_E, H_U are constructed with normalized single-column internal functions, such as $P(\bar{K}_1)^{(10)}$, a renormalization factor, C, is needed in eq. (55) to build the $H_N(\bar{K}, K^*)$ for full 4-columned symmetry with normalized internal functions such as $P(\bar{K}_1)_z^{(40)}$ for ^{8}Be fragments, or with $(\lambda_f \mu_f) = (04)$ for ^{12}C fragments, or $(\lambda_{A-f}\mu_{A-f}) = (80)$ for a ^{20}Ne fragment. Thus, the coefficient C contains a factor $[(1!)^4/4!]^{\frac{1}{2}}$ for each ^{8}Be and each ^{12}C fragment in bra and ket and a factor $[(2!)^4/8!]^{\frac{1}{2}}$ for each ^{20}Ne fragment. In addition, a factor $\sqrt{\frac{1}{2}}$ is needed if the two cluster fragments in the bra are identical, similarly for the case of identical fragments in the ket. Thus $C = \frac{1}{2}[1/4!]^2$ for ^{12}C + ^{12}C fragment decomposition in both bra and ket, whereas $C = \sqrt{\frac{1}{2}}[1/4!]^1[2^4/8!]^{\frac{1}{2}}$ for ^{12}C + ^{12}C fragment decomposition in bra and $\alpha + ^{20}$Ne fragments in ket, for example.

Except for the details of the recoupling process the evaluation of the Bargmann transform $H_V(\bar{K}, K^*)$ is thus reduced to the evaluation of the convolution integral $\int dq\, e^{-\gamma q^2} H_E(q) H_E(-q)$. From the form of the single-column transforms $H_E(q)$ [cf. eq. (50)] and the relations (47) and (48) it can be seen that this integral consists of a sum of

terms which, except for a q-independent factor, have the general form

$$\left(\frac{\gamma}{\pi}\right)^{\frac{3}{2}} \int dq\, e^{-(\gamma+1)q^2} \prod_{j=1}^{n} (iq \cdot a_j)(iq \cdot iq)^m \exp iq \cdot [(a-a')\mathscr{K} + (b-b').\mathscr{K}^*]$$

$$= (2g)^{\frac{1}{2}n+m}(1-2g)^{\frac{3}{2}} \left\{ \frac{1}{\pi^{\frac{3}{2}}} \int dq \exp[-q'^2 - 2iq' \cdot \kappa] \prod_{j=1}^{n} (iq' \cdot a_j)(iq' \cdot iq')^m \right\}, \quad (56a)$$

where the substitution $q = (\gamma+1)^{-\frac{1}{2}}q' = (2g)^{\frac{1}{2}}q'$ has been made, and where we have defined

$$\kappa = -(\tfrac{1}{2}g)^{\frac{1}{2}}[(a-a')\mathscr{K} + (b-b')\mathscr{K}^*], \quad (56b)$$

with

$$-\kappa^2 = \phi(a-a', b-b'), \quad (56c)$$

where $\phi(i,j)$ is defined in eq. (49).

The basic integral can be evaluated in general form:

$$\frac{1}{\pi^{\frac{3}{2}}} \int dq'\, \exp[-q'^2 - 2iq' \cdot \kappa] \prod_{j=1}^{n} (iq' \cdot a_j)(iq' \cdot iq')^m$$

$$= e^{-\kappa^2} \sum_{p=0}^{[n/2]} (-\tfrac{1}{2})^p [a_1 a_2 \cdots a_n | \kappa]^{(p)} c_m(p), \quad (57)$$

where

$$c_0(p) = 1,$$

$$c_m(p) = (-1)^m \sum_{r=0}^{m} C_r^{(m)}(n+\tfrac{3}{2}-p)(\kappa \cdot \kappa)^r, \qquad \text{for } m > 0$$

with

$$C_r^{(m)}(x) = (-1)^r \sum_{s=r}^{m} \binom{m}{s}\binom{s}{s-r} \frac{\Gamma(s)}{\Gamma(r)} \frac{\Gamma(x+m-s)}{\Gamma(x)}, \qquad \text{for } r > 0,$$

$$C_0^{(m)}(x) = \frac{\Gamma(x+m)}{\Gamma(x)} + \sum_{s=1}^{m} \frac{m!}{(m-s)!} \frac{1}{s} \frac{\Gamma(x+m-s)}{\Gamma(x)}, \quad (58)$$

and where the symbol $[a_1 a_2 \ldots a_n | \kappa]^{(p)}$ defines a function involving scalar products of the vectors a_j with κ in which p contractions have been made. With no contractions ($p = 0$),

$$[a_1 a_2 \ldots a_n | \kappa]^{(0)} \equiv (a_1 \cdot \kappa)(a_2 \cdot \kappa) \ldots (a_n \cdot \kappa). \quad (59a)$$

With one contraction ($p = 1$), the symbol defines the sum of $\tfrac{1}{2}n(n-1)$ terms:

$$[a_1 a_2 \ldots a_n | \kappa]^{(1)} \equiv (a_1 \cdot a_2)(a_3 \cdot \kappa) \ldots (a_n \cdot \kappa) + (a_1 \cdot a_3)(a_2 \cdot \kappa) \ldots (a_n \cdot \kappa)$$

$$+ \ldots + (a_{n-1} \cdot a_n)(a_1 \cdot \kappa) \ldots (a_{n-2} \cdot \kappa). \quad (59b)$$

With two contractions ($p = 2$), the symbol defines the sum of $\frac{1}{8}n(n-1)(n-2)(n-3)$ terms:

$$[a_1 a_2 \ldots a_n | \kappa]^{(2)} \equiv (a_1 \cdot a_2)(a_3 \cdot a_4)(a_5 \cdot \kappa)\ldots(a_n \cdot \kappa)$$

$$+ (a_1 \cdot a_3)(a_2 \cdot a_4)(a_5 \cdot \kappa)\ldots(a_n \cdot \kappa)+\ldots \quad (59c)$$

etc., where $[a_1 a_2 \ldots a_n | \kappa]^{(p)}$ is homogeneous of degree $n - 2p$ in κ:

$$[a_1 a_2 \cdots a_n | \sigma \kappa]^{(p)} = \sigma^{n-2p}[a_1 a_2 \ldots a_n | \kappa]^{(p)}. \quad (60)$$

From the general form of eq. (57), and the fact that $(a - a')$ [similarly $(b - b')$] $= \pm 1, 0$ [see eq. (56b)], we note that the convolution integral leads to terms of the same form as those appearing in H_U; that is exponentials of the type $e^{\phi(i, j)}$, defined by eqs. (49) (or e^0), multiplied by scalar product factors of the type $(\bar{K}_1 \cdot K_1^*)$, $(\bar{K}_1 \cdot \mathscr{K})$,... and the additional $\mathscr{K}$-dependent factors e^p of eq. (46).

As a specific example we give the results for the $\alpha + {}^8\text{Be}$ decomposition of the $A = 12$ system. In terms of the $P(abcde)$ defined by eq. (52), and H_N given by eq. (53), we get

$$\left(\frac{\gamma}{\pi}\right)^{\frac{3}{2}} \int d\mathbf{q} e^{-\gamma \mathbf{q} \cdot \mathbf{q}} H_E(\mathbf{q}) H_E(-\mathbf{q}) = (1 - 2g)^3 [H_N^2 \{3 + 2e^{\phi(1, 0)} + 2e^{\phi(0, 1)} + 2e^{\phi(1, -1)}\}$$

$$+ \exp\left[\tfrac{2}{3}\mathscr{K} \cdot \mathscr{K}^*\right] H_N(P(10000)\{(-2 - 2g + 2ge^1) + (2 - 2g + 2ge^1)[e^{\phi(1, 0)}$$

$$+ e^{\phi(0, 1)}] + 2e^{\phi(1, 1)} - 4e^{\phi(1, -1)}\}$$

$$+ P(00110)(2g)\{(g + (1 - g)e^1)e^{\phi(1, 0)} - e^1 e^{\phi(1, -1)}\}$$

$$+ P(01001)(2g)\{(g + (1 - g)e^1)e^{\phi(0, 1)} - e^1 e^{\phi(1, -1)}\}$$

$$+ P(01100)(2g)\{2e^1 e^{\phi(1, 0)} + 2e^1 e^{\phi(0, 1)} + 2e^1 e^{\phi(1, -1)}\})$$

$$+ \exp\left[\tfrac{4}{3}\mathscr{K} \cdot \mathscr{K}^*\right](P(20000)\{(4 - 2g + 2g^2) + 2g(1 - 2g)e^1 + 2g^2 e^2$$

$$+ (-4 + 2g - 2ge^1)[e^{\phi(1, 0)} + e^{\phi(0, 1)}] + (2 - 2g + 2ge^1)e^{\phi(1, 1)} + 2e^{\phi(1, -1)}\}$$

$$+ P(10110)(2g)\{(-g + (-2 + g)e^1)e^{\phi(1, 0)} + (g + (1 - g)e^1)e^{\phi(1, 1)} + e^1 e^{\phi(1, -1)}\}$$

$$+ P(11001)(2g)\{(-g + (-2 + g)e^1)e^{\phi(0, 1)} + (g + (1 - g)e^1)e^{\phi(1, 1)} + e^1 e^{\phi(1, -1)}\}$$

$$+ P(11100)(2g)\{e^2 + (-2ge^1 + (-1 + 2g)e^2)[e^{\phi(1, 0)} + e^{\phi(0, 1)}]$$

$$+ (g + (2 - g)e^1)e^{\phi(1, 1)} + (-2e^1 + e^2)e^{\phi(1, -1)}\}$$

$$+ P(00200)(\bar{K}_1 \cdot \bar{K}_1)(2g)\{e^2 - e^2 e^{\phi(0, 1)}\}$$

$$+ P(02000)(K_1^* \cdot K_1^*)(2g)\{e^2 - e^2 e^{\phi(1, 0)}\}$$

$$+ P(02200)(2g^2)\{e^2 e^{\phi(1, 0)} + e^2 e^{\phi(0, 1)} + e^2 e^{\phi(1, -1)}\}$$

$$+ P(01210)(2g^2)\{(ge^1 + (1 - g)e^2)e^{\phi(1, 0)} - e^2 e^{\phi(1, -1)}\}$$

$$+ P(02101)(2g^2)\{(ge^1 + (1 - g)e^2)e^{\phi(0, 1)} - e^2 e^{\phi(1, -1)}\}$$

$$+ P(01111)(2g^2)e^2e^{\phi(1, -1)}$$

$$+ P(10011)(2g^2)(1 - e^1)e^{\phi(1, 1)}$$

$$+ P(00101)(\bar{K}_1 \cdot \bar{K}_1)(2g^2)(-e^1 + e^2)e^{\phi(0, 1)}$$

$$+ P(01010)(K_1^* \cdot K_1^*)(2g^2)(-e^1 + e^2)e^{\phi(1, 0)}$$

$$+ (\bar{K}_1 \cdot \bar{K}_1)(K_1^* \cdot K_1^*)(g^2)(1 - e^1)^2)]. \tag{61}$$

For the heavier systems the convolution integral can lead to a large number of terms. However, the systematics of eqs. (57)–(59) are such that the process can be computerized easily.

6. Relationship between coordinate space matrix elements and expansions of the Bargmann transform

The Bargmann transform $H_V(\bar{K}, K^*)$ of the interaction can be used to extract both RGM and GCM interaction kernels by purely algebraic techniques. For this purpose, however, it is necessary to convert eq. (55) to a more useable form. If a harmonic oscillator expansion can be used for the calculation of an RGM kernel, an expansion in an SU(3)-coupled oscillator basis is particularly useful [cf., e.g., eq. (109) of ref. [11]]. If harmonic oscillator expansions lead to convergence problems a basis in terms of more conventional angular momentum coupled-channel functions with cluster-internal functions of good SU(3) symmetry may be more appropriate. In the partial-wave decomposition of a GCM kernel such a basis with locally peaked radial functions is particularly useful. For both cases the needed coordinate space matrix elements (complicated 3A-dimensional integrals), can be read from an expansion of the Bargmann transform $H_V(\bar{K}, K^*)$ in terms of the analogous appropriately coupled K-space functions. In this section we shall first establish in a general way the relationship between the coordinate space matrix elements and the coefficients of the K-space expansions. The specific example of the $\alpha + {}^8\text{Be}$ cluster system is then used to illustrate the process in some detail.

6.1. EXPANSION IN AN SU(3)-COUPLED OSCILLATOR BASIS

To establish the desired relationship in the harmonic oscillator basis the Bargmann transform of the operator $\mathcal{O} \cdot \mathscr{A}$ is first transformed to cluster variables with the use of eq. (27)

$$H_\ell = \left\langle \prod_{j=1}^A A(\bar{k}_j, r_j)^* \middle| \mathcal{O} \cdot \mathscr{A} \middle| \prod_{j=1}^A A(k_j^*, r_j) \right\rangle$$

$$= \left\langle \prod_i A(\bar{K}_i, \bar{R}_i)^* A(\bar{K}, \bar{R})^* A(K_{\text{c.m.}}, R_{\text{c.m.}})^* \middle| \mathcal{O} \cdot \mathscr{A} \middle| \prod_j A(K_j^*, R_j) A(K^*, R) A(K_{\text{c.m.}}^*, R_{\text{c.m.}}) \right\rangle. \tag{62}$$

The expansion of each $A(K_i, R_i)$ in 3-dimensional harmonic oscillator functions by eq. (28), followed by successive SU(3)-coupling of both coordinate-space and K-space functions, leads to [12])

$$
H_\ell = \sum_{(\lambda_c \mu_c)Q(\lambda\mu)\alpha} \sum_{(\lambda'_c \mu'_c)Q'(\lambda'\mu')\alpha'}
$$

$$
\times \left[[P(K_1,\ldots)^{(\lambda_f \mu_f)} \times P(\bar{K}_i,\ldots)^{(\lambda_A - f \mu_A - f)}]^{(\lambda_c \mu_c)} \times P(\bar{K})^{(Q0)}\right]^{(\lambda\mu)}_\alpha
$$

$$
\times \left[[P(K_1^*,\ldots)^{(\mu_{f'} \lambda_{f'})} \times P(K_j^*,\ldots)^{(\mu_A - f' \lambda_A - f')}]^{(\mu'_c \lambda'_c)} \times P(K^*)^{(0Q')}\right]^{(\mu'\lambda')}_{\tilde{\alpha}'}
$$

$$
\times \left\langle [\phi^{(\lambda_f \mu_f)} \times \phi^{(\lambda_A - f \mu_A - f)}]^{(\lambda_c \mu_c)} \times \phi(\bar{R})^{(Q0)}\right]^{(\lambda\mu)}_\alpha \phi(R_{\text{c.m.}})^{(00)} | \mathcal{C}\cdot\mathcal{A}|
$$

$$
\times \left[[\phi^{(\lambda_{f'} \mu_{f'})} \times \phi^{(\lambda_A - f' \mu_A - f')}]^{(\lambda'_c \mu'_c)} \times \phi(R)^{(Q'0)}\right]^{(\lambda'\mu')}_{\alpha'} \phi(R_{\text{c.m.}})^{(00)} \right\rangle . \tag{63}
$$

Here $K_{\text{c.m.}}$ and the cluster-internal K-space variables which carry no oscillator excitations have been set equal to zero, and only the terms corresponding to minimum Pauli-allowed excitations in the variables K_i have been retained. The square brackets denote SU(3)-coupling. If we now imagine that the operator $\mathcal{C}$ is expanded in SU(3)-irreducible tensor components

$$
\mathcal{C} = \sum_{(\lambda_0 \mu_0)\alpha_0} \mathcal{C}^{(\lambda_0 \mu_0)}_{\alpha_0}, \tag{64}
$$

a generalized Wigner-Eckart theorem can be expressed by

$$
\left\langle [\ldots]^{(\lambda\mu)}_\alpha | \mathcal{C}\cdot\mathcal{A} | [\ldots]^{(\lambda'\mu')}_{\tilde{\alpha}'} \right\rangle
$$

$$
= \sum_{(\lambda_0 \mu_0)\alpha_0} \sum_{\rho_0} \left\langle [\ldots]^{(\lambda\mu)}_\alpha \| \mathcal{C}^{(\lambda_0 \mu_0)}\cdot\mathcal{A} \| [\ldots]^{(\lambda'\mu')}_{\tilde{\alpha}'} \right\rangle_{\rho_0}
$$

$$
\times \left\langle (\lambda\mu)\alpha; (\mu'\lambda')\tilde{\alpha}' | (\lambda_0 \mu_0)\alpha_0 \right\rangle_{\rho_0}. \tag{65}
$$

In both eqs. (63) and (65) the phase factor associated with complex conjugation [11,14]) has been included in the definition of $\tilde{\alpha}'$. (Complex conjugation converts $\phi^{(\lambda'\mu')}_{\alpha'}$ into $\phi^{(\mu'\lambda')}_{\tilde{\alpha}'}$.) By combining eqs. (63) and (65), the desired K-space expansion of $H_\mathcal{C}$ is attained,

$$
H_\ell = \sum_{(\lambda_c \mu_c)Q(\lambda\mu)} \sum_{(\lambda'_c \mu'_c)Q'(\lambda'\mu')} \sum_{(\lambda_0 \mu_0)\alpha_0 \rho_0} c^{(\lambda_0 \mu_0)\rho_0}_{aa'}
$$

$$
\times \left[[[P(K_1,\ldots)^{(\lambda_f \mu_f)} \times P(\bar{K}_i,\ldots)^{(\lambda_A - f \mu_A - f)}]^{(\lambda_c \mu_c)} \times P(\bar{K})^{(Q0)}]^{(\lambda\mu)}\right.
$$

$$
\times [[P(K_1^*,\ldots)^{(\mu_{f'} \lambda_{f'})} \times P(K_j^*,\ldots)^{(\mu_A - f' \lambda_A - f')}]^{(\mu'_c \lambda'_c)} \times P(K^*)^{(0Q')}]^{(\mu'\lambda')}]^{(\lambda_0 \mu_0)\rho_0}_{\alpha_0}, \tag{66a}
$$

where the subscripts $a(a')$ in the expansion coefficient are shorthand symbols for

$$
a \equiv [[(\lambda_f \mu_f)(\lambda_A - f \mu_A - f)](\lambda_c \mu_c)(Q0)](\lambda\mu),
$$

$$
a' \equiv [[(\lambda_{f'} \mu_{f'})(\lambda_A - f' \mu_A - f')](\lambda'_c \mu'_c)(Q'0)](\lambda'\mu'),
$$

and the expansion coefficient $c_{aa'}^{(\lambda_0\mu_0)\rho_0}$ is the SU(3)-reduced matrix element of the operator $\mathcal{O}_{x_0}^{(\lambda_0\mu_0)}.\mathcal{A}$,

$$c_{aa'}^{(\lambda_0\mu_0)\rho_0} = \langle [[\phi^{(\lambda_f\mu_f)} \times \phi^{(\lambda_A - f\mu_A - f)}]^{(\lambda_c\mu_c)} \times \phi(\bar{R})^{(Q0)}]^{(\lambda\mu)}\phi(R_{\mathrm{c.m.}})^{(00)}\|$$

$$\times \mathcal{O}^{(\lambda_0\mu_0)}.\mathcal{A}\|[[\phi^{(\lambda_f'\mu_f')} \times \phi^{(\lambda_A - f'\mu_A - f')}]^{(\lambda_c'\mu_c')} \times \phi(R)^{(Q'0)}]^{(\lambda'\mu')}\phi(R_{\mathrm{c.m.}})^{(00)}\rangle_{\rho_0}. \quad (66b)$$

By expanding the Bargmann transform $H_{\mathcal{E}}(\bar{K}, K^*)$ in the SU(3)-coupled K-space functions $[\ldots]_{x_0}^{(\lambda_0\mu_0)\rho_0}$ of eq. (66a) the needed coordinate space matrix elements can thus be read from the coefficients in this expansion.

6.2. EXPANSION IN AN ANGULAR-MOMENTUM COUPLED CHANNEL SPIN BASIS

In the evaluation of a GCM kernel a slightly different K-space expansion is useful. With cluster fragment internal wave functions of good SU(3) symmetry the A-particle basis function of the GCM can be expressed by

$$\mathcal{A}\{[\phi^{(\lambda_f\mu_f)} \times \phi^{(\lambda_A - f\mu_A - f)}]_{\kappa_c I_c M_c}^{(\lambda_c\mu_c)}\Gamma(R_{\mathrm{ph}}, X)\phi(R_{\mathrm{c.m.}})\}, \quad (67)$$

where $\Gamma(R_{\mathrm{ph}}, X)$ is a gaussian wave packet which describes the fluctuation of the (physical) relative motion coordinate R_{ph} around its mean value X

$$\Gamma(R_{\mathrm{ph}}, X) = (\nu/\pi)^{\frac{3}{4}}\exp[-\tfrac{1}{2}\nu(R_{\mathrm{ph}} - X)^2]. \quad (68)$$

The Bargmann kernel function $A(K, R)$, with real K, has essentially this form

$$A(K, R) = \pi^{-\frac{3}{4}}e^{\frac{1}{2}K^2}e^{-\frac{1}{2}(R - \sqrt{2}K)^2} = \nu^{-\frac{3}{4}}e^{\frac{1}{2}K^2}\Gamma(R_{\mathrm{ph}}, X),$$

with

$$\nu = m\omega f(A - f)/\hbar A \quad \text{and} \quad X = [2\hbar A/m\omega f(A - f)]^{\frac{1}{2}}K; \quad (69)$$

and it is now useful to restrict the oscillator expansions and successive SU(3) couplings in $H_{\mathcal{E}}$ to the cluster-internal factors $A(K_i, R_i)$ [cf. eq. (62)]. To extract the *radial* coordinate R for the cluster-relative motion in a partial-wave expansion it is useful to expand $\Gamma(R_{\mathrm{ph}}, X)$ and hence $\exp[-\tfrac{1}{2}K^2]A(K, R)$ according to

$$e^{-\frac{1}{2}K^2}A(K, R) = \pi^{-\frac{3}{4}}\sum_l i_l(\sqrt{2}KR)e^{-(R^2 + 2K^2)/2} \times 4\pi\sum_m Y_{lm}(\hat{R})^*Y_{lm}(\hat{K}), \quad (70)$$

where $i_l(x) = \sqrt{\pi/2x}\,I_{l+\frac{1}{2}}(x)$ and $I_{l+\frac{1}{2}}(x)$ is a modified Bessel function, and $\hat{R}$, $\hat{K}$ are unit vectors. An angular momentum coupling of the spherical harmonics $Y_{lm}(\hat{R})$ [and $Y_{lm}(\hat{K})$], with cluster-internal functions of angular momentum $I_c M_c$, yields the angular momentum coupled-channel functions

$$([\phi^{(\lambda_f\mu_f)} \times \phi^{(\lambda_A - f\mu_A - f)}]_{\kappa_c I_c}^{(\lambda_c\mu_c)} \times Y_l(\hat{R}))_{M_J}^J \quad (71)$$

(and the corresponding K-space functions). In eq. (71) the square bracket denotes SU(3)-coupling, whereas the round bracket denotes ordinary angular momentum coupling. In terms of such functions the analogue of eq. (63) gives the expansion of the Bargmann transform of $\mathcal{O}\cdot\mathscr{A}$ in the form

$$
\begin{aligned}
H_{\mathcal{O}} = \sum_{(\lambda_c\mu_c)\kappa_c I_c} \sum_{(\lambda'_c\mu'_c)\kappa'_c I'_c} \sum_{ll'} \sum_{JM_JJ'M'_J} \\[4pt]
\times\, 4\pi^{\frac{1}{4}}e^{\frac{1}{2}\bar{K}^2}([P(\bar{K}_1,\ldots)^{(\lambda_f\mu_f)} \times P(\bar{K}_i\ldots)^{(\lambda_{A-f}\mu_{A-f})}]^{(\lambda_c\mu_c)}_{\kappa_c I_c} \times Y_l(\hat{\bar{K}}))^J_{M_J} \\[4pt]
\times\, 4\pi^{\frac{1}{4}}e^{\frac{1}{2}K^2}([P(K_1^*,\ldots)^{(\mu_f\lambda_f)} \times P(K_j^*,\ldots)^{(\mu_{A-f}\lambda_{A-f})}]^{(\mu'_c\lambda'_c)}_{\kappa'_c I'_c} \times Y_{l'}(\hat{K}))^{J'}_{-M'_J}(-1)^{J'-M'_J} \\[4pt]
\times\, \langle([\phi^{(\lambda_f\mu_f)} \times \phi^{(\lambda_{A-f}\mu_{A-f})}]^{(\lambda_c\mu_c)}_{\kappa_c I_c} \times Y_l(\hat{\bar{R}}))^J_{M_J} e^{-\frac{1}{2}(R^2+2\bar{R}^2)} i_l(\sqrt{2}\bar{K}\bar{R})\phi(R_{\mathrm{c.m.}})^{(00)}| \\[4pt]
\times\, \mathcal{O}\cdot\mathscr{A}|([\phi^{(\lambda_{f'}\mu_{f'})} \times \phi^{(\lambda_{A-f'}\mu_{A-f'})}]^{(\lambda'_c\mu'_c)}_{\kappa'_c I'_c} \times Y_{l'}(\hat{R}))^{J'}_{M'_J} e^{-\frac{1}{2}(R^2+2K^2)} i_{l'}(\sqrt{2}KR)\phi(R_{\mathrm{c.m.}})^{(00)}\rangle.
\end{aligned}
$$
(72)

If the operator $\mathcal{O}$ can be expanded in terms of spherical tensors

$$
\mathcal{O} = \sum_{k_0 m_0} \mathcal{O}^{k_0}_{m_0},
$$
(73)

it is useful to use the Wigner-Eckart theorem in the form

$$
\begin{aligned}
\langle[\ldots]^J_{M_J}|\mathcal{O}^{k_0}_{m_0}\cdot\mathscr{A}|[\ldots]^{J'}_{M'_J}\rangle \\[4pt]
= \langle JM_JJ'-M'_J|k_0m_0\rangle(-1)^{J'-M'_J}\langle[\ldots]^J\|\mathcal{O}^{k_0}\cdot\mathscr{A}\|[\ldots]^{J'}\rangle.
\end{aligned}
$$
(74)

(Note the unconventional order of J, J' and k_0 in the angular momentum Wigner coefficient which is convenient for the generalization to SU(3) [cf. eq. (65)]. The angular momentum-reduced matrix element of eq. (74) thus differs from the conventional angular momentum-reduced matrix element by a factor of $[2k_0+1]^{\frac{1}{2}}$). For the central interaction of eq. (3), $\mathcal{O}$ is a pure $k_0 = 0$ operator. In the more general case the expansion of the Bargmann transform of $\mathcal{O}\cdot\mathscr{A}$ can be put in the form

$$
\begin{aligned}
H_{\mathcal{O}} = \sum_{(\lambda_c\mu_c)\kappa_c I_c} \sum_{(\lambda'_c\mu'_c)\kappa'_c I'_c} \sum_{ll'} \sum_{JJ'} \sum_{k_0m_0} 16\sqrt{\pi}\,e^{\frac{1}{2}(\bar{R}^2+K^2)} \\[4pt]
\times\, (([P(\bar{K}_1,\ldots)^{(\lambda_f\mu_f)} \times P(\bar{K}_i,\ldots)^{(\lambda_{A-f}\mu_{A-f})}]^{(\lambda_c\mu_c)}_{\kappa_c I_c} \times Y_l(\hat{\bar{K}}))^J \\[4pt]
\times\, ([P(K_1^*,\ldots)^{(\mu_f\lambda_f)} \times P(K_j^*,\ldots)^{(\mu_{A-f}\lambda_{A-f})}]^{(\mu'_c\lambda'_c)}_{\kappa'_c I'_c} \times Y_{l'}(\hat{K}))^{J'})^{k_0}_{m_0}\mathscr{K}(\bar{K},K)^{k_0}_{aa'},
\end{aligned}
$$
(75a)

where a and a stand for the channel quantum numbers

$$
a \equiv ([(\lambda_f\mu_f)(\lambda_{A-f}\mu_{A-f})](\lambda_c\mu_c)\kappa_c I_c l)J,
$$

$$
a' \equiv ([(\lambda_{f'}\mu_{f'})(\lambda_{A-f'}\mu_{A-f'})](\lambda'_c\mu'_c)\kappa'_c I'_c l')J',
$$

$$\mathcal{K}(\bar{K}, K)_{aa'}^{k0}$$

$$= \langle ([\phi^{(\lambda_f \mu_f)} \times \phi^{(\lambda_A - f \mu_{A-f})}]_{\kappa_c l_c}^{(\lambda_c \mu_c)} \times Y_l(\hat{\bar{R}}))^{J} i_l(\sqrt{2}\bar{K}\bar{R}) e^{-\frac{1}{4}(R^2 + 2\bar{K}^2)} \phi(\mathbf{R}_{\text{c.m.}})^{(00)}\|$$

$$\times \mathcal{C}^{k0} \mathcal{A}\|([\phi^{(\lambda_f \mu_f)} \times \phi^{(\lambda_A - f \mu_{A-f})}]_{\kappa_c l_c}^{(\lambda_c \mu_c)} \times Y_{l'}(\hat{R}))^{J} i_{l'}(\sqrt{2}KR) e^{-\frac{1}{4}(R^2 + 2K^2)} \phi(\mathbf{R}_{\text{c.m.}})^{(00)}\rangle.$$

$$(75b)$$

This coordinate-space matrix element with the radial function $i_l(\sqrt{2}KR)\exp\{-\frac{1}{4}(R^2 + 2K^2)\}$ is precisely the spatial integral which is needed for a partial-wave decomposition of a GCM kernel with gaussian, locally peaked radial functions and with cluster channel functions specified by angular momentum quantum numbers I_c, l, J. To extract the needed spatial integrals from the Bargmann transform of $\mathcal{C}\mathcal{A}$ it is only necessary to restrict the Bargmann space relative motion variables, $\bar{K}$ and K to be real and expand $H_\ell(\bar{K}, K^*)$ in the angular-momentum coupled K-space functions $\{16\sqrt{\pi}\exp\frac{1}{2}(\bar{K}^2 + K^2)(\ldots)_{m_0}^{k0}\}$ of eq. (75a). The needed $3A$-dimensional spatial integrals, functions of $\bar{K}$ and K, are then the coefficients in this expansion.

7. A specific example. The $\alpha + {}^8$Be system

It will be useful to illustrate the expansion process with a specific example, the simplest of our cluster systems. For the $\alpha + {}^8$Be system the single-column transform H_U of eq. (51) can be written in the form

$$H_U = \exp\left[\frac{1}{4}\frac{f(A-f)}{A}\mathcal{K}\cdot\mathcal{K}^*\right](1 - 2g)^{\frac{1}{2}}\sum P(abcde)G_{abcde}, \qquad (76)$$

where the G_{abcde} are functions of $\mathcal{K}\cdot\mathcal{K}^*$, $\mathcal{K}\cdot\mathcal{K}$ and $\mathcal{K}^*\cdot\mathcal{K}^*$ through the exponentials $e^{\phi(i,j)}$ and e^p.

In terms of these functions the $H_N^3 H_U$ term of the full-A-particle transform $H_V(\bar{K}, K^*)$ of eq. (55) can be written as

$$H_N^3 H_U = (1 - 2g)^{\frac{1}{2}} \exp\left[\frac{f(A-f)}{A}\mathcal{K}\cdot\mathcal{K}^*\right]\sum_l \binom{3}{l}(-1)^l e^l(1 - e^1)^{3-l}$$

$$\times \{P(4 - l\, ll\, 00)G_{10000}$$

$$+ P(3 - l\, l+1\, l+1\, 00)G_{01100} + P(3 - l\, l\, l+1\, 10)G_{00110} + P(3 - l\, l+1\, l\, 01)G_{01001}$$

$$+ P(3 - l\, ll\, 11)G_{00011}\}. \qquad (77)$$

A similar expansion of $H_N^2 \int dq e^{-\gamma q^2} H_E(q) H_E(-q)$ leads to $H_V(\bar{K}, K^*)$ in the form

$$H_V(\bar{K}, K^*) = \sum_{s=1}^{9} \sum_l (\bar{K}_1 \cdot K_1^*)^a (\bar{K}_1 \cdot K^*)^b (K_1^* \cdot \bar{K})^c f_s G_{s,l}(\bar{K}, K^*), \qquad (78)$$

TABLE 1

The functions f_s for eq. (78)

s	a	b	c	f_s
1	$4-l$	l	l	1
2	$3-l$	l	$l+1$	$(\bar{K}_1 \cdot \bar{K}) = \sqrt{3}[P(\bar{K}_1)^{(10)} \times P(\bar{K})^{(10)}]_{L=0}^{(20)}$
3	$3-l$	$l+1$	l	$(K_1^* \cdot K^*) = \sqrt{3}[P(K_1^*)^{(01)} \times P(K^*)^{(01)}]_0^{(02)}$
4	$3-l$	l	l	$(\bar{K}_1 \cdot \bar{K})(K_1^* \cdot K^*) = 3\{\sqrt{\tfrac{1}{6}}[\ldots]_0^{(00)} - \sqrt{\tfrac{5}{6}}[\ldots]_0^{(22)}\}$
				$[\ldots]_0^{(\lambda\mu)} \equiv [[P(\bar{K}_1)^{(10)} \times P(\bar{K})^{(10)}]^{(20)}$
				$\times [P(K_1^*)^{(01)} \times P(K^*)^{(01)}]^{(02)}]_0^{(\lambda\mu)}$
5	$2-l$	l	$l+2$	$(\bar{K}_1 \cdot \bar{K}_1) = \sqrt{6}P(\bar{K}_1)_0^{(20)}$
6	$2-l$	$l+2$	l	$(K_1^* \cdot K_1^*) = \sqrt{6}P(K_1^*)_0^{(02)}$
7	$2-l$	$l+1$	l	$(\bar{K}_1 \cdot \bar{K})(K_1^* \cdot K_1^*)$
				$= \sqrt{3}\sqrt{6}\{\sqrt{\tfrac{1}{6}}[\ldots]_0^{(00)} - \sqrt{\tfrac{5}{6}}[\ldots]_0^{(22)}\}$
				$[\ldots]_0^{(\lambda\mu)}$
				$= [[P(\bar{K}_1)^{(10)} \times P(\bar{K})^{(10)}]^{(20)} \times P(K_1^*)^{(02)}]_0^{(\lambda\mu)}$
8	$2-l$	l	$l+1$	$(K_1^* \cdot K^*)(\bar{K}_1 \cdot \bar{K}_1)$
				$= \sqrt{3}\sqrt{6}\{\sqrt{\tfrac{1}{6}}[\ldots]_0^{(00)} - \sqrt{\tfrac{5}{6}}[\ldots]_0^{(22)}\}$
				$[\ldots]_0^{(\lambda\mu)}$
				$= [P(\bar{K}_1)^{(20)} \times [P(K_1^*)^{(01)} \times P(K^*)^{(01)}]^{(02)}]_0^{(\lambda\mu)}$
9	$2-l$	l	l	$(\bar{K}_1 \cdot \bar{K}_1)(K_1^* \cdot K_1^*) = 6\{\sqrt{\tfrac{1}{6}}[\ldots]_0^{(00)} - \sqrt{\tfrac{5}{6}}[\ldots]_0^{(22)}\}$
				$[\ldots]_0^{(\lambda\mu)} = [P(\bar{K}_1)^{(20)} \times P(K_1^*)^{(02)}]_0^{(\lambda\mu)}$

where the nine functions f_s are defined in table 1, and where the powers a, b and c have a different l-dependence for each s (also given in table 1). The nine $G_{s,l}$ are functions of $(\bar{K} \cdot K^*)$, $(\bar{K} \cdot \bar{K})$, $(K^* \cdot K^*)$ through the exponentials $e^{\phi(i,j)}$, e^p, and $e^{K \cdot K^*}$ and are made to to include all g- and mass-dependent factors. [Note that we have converted the mass-dependent $\mathscr{K}$, $\mathscr{K}^*$ (which were a convenient shorthand notation in sects. 4 and 5) back to $\bar{K}$, K^* (which have a simpler normalization in their SU(3)-tensor character). Note also that the $G_{s,l}$ are independent of the cluster-internal variables $\bar{K}_1, K_1^*$.]

To gain the expansion of $H_V(\bar{K}, K^*)$ in terms of the appropriately coupled K-space functions, of eq. (66) or eq. (75), it is necessary to combine the $\bar{K}_1$ (and K_1^*)-dependent factors by simple SU(3) recoupling techniques of the type used in ref. [11]. [We adhere to the notations of refs. [11, 14].] Since the recoupling procedure is somewhat different for the two cases, they are discussed separately.

7.1. HARMONIC OSCILLATOR EXPANSION

To attain the expansion in the SU(3)-coupled K-space functions of eq. (66) it is convenient to combine all SU(3)-scalar factors first. For this purpose a power series expansion is first made in exponentials of the type $\exp\{(1 - p[A/f(A-f)] \pm g[A/f(A-f)])(\bar{K} \cdot K^*\}$ which contain the SU(3) (00) tensor $(\bar{K} \cdot K^*)$. The SU(3) scalar factors are then expanded by

$$(\bar{K}_1 \cdot K_1^*)^a(\bar{K}_1 \cdot K^*)^b(K_1^* \cdot \bar{K})^c(\bar{K} \cdot K^*)^\alpha = \sum_{(\lambda_1\mu_1)} R_1(abc\alpha; \lambda_1\mu_1)$$

$$\times [[P(\bar{K}_1)^{(a+b, 0)} \times P(\bar{K})^{(c+\alpha, 0)}]^{(\lambda_1\mu_1)} \times [P(K_1^*)^{(0, a+c)} \times P(K^*)^{(0, b+\alpha)}]^{(\mu_1\lambda_1)}]_0^{(00)}, \qquad (79)$$

where

$$R_1(abc\alpha; \lambda_1\mu_1) = [d(\lambda_1\mu_1)(a+b)!(a+c)!(b+\alpha)!(c+\alpha)!]^{\frac{1}{2}}$$

$$\times \begin{bmatrix} (a0) & (c0) & (a+c, 0) \\ (b0) & (\alpha0) & (b+\alpha, 0) \\ (a+b, 0) & (c+\alpha, 0) & (\lambda_1\mu_1) \end{bmatrix} \qquad (80)$$

[see eq. (B.8) of ref.[11]), and note that the SU(3) 9-$(\lambda\mu)$ coefficient, with $\lambda_1 + 2\mu_1 = a+b+c+\alpha$, is equivalent to a simple SU(2) 9-j coefficient].

The K-space functions of eq. (79) are then combined with the factors f_s of table 1 to yield

$$(\bar{K}_1 \cdot K_1^*)^a(\bar{K}_1 \cdot K^*)^b(K_1^* \cdot \bar{K})^c(\bar{K} \cdot K^*)^\alpha f_s = \sum_{(\lambda_1\mu_1)} R_1(abc\alpha; \lambda_1\mu_1) \sum_{(\lambda_2\mu_2)(\lambda_2'\mu_2')}$$

$$\times \sum_{(\bar{\lambda}\bar{\mu})\bar{\rho}} R_2(\lambda_1\mu_1; \lambda_2\mu_2, \lambda_2'\mu_2'; (\bar{\lambda}\bar{\mu})\bar{\rho}) \times [[P(\bar{K}_1)^{(40)} \times P(\bar{K})^{(Q-2\beta, 0)}]^{(\lambda_2\mu_2)}$$

$$\times [P(K_1^*)^{(04)} \times P(K^*)^{(0, Q'-2\gamma)}]^{(\mu_2'\lambda_2')}]_{L=0}^{(\bar{\lambda}\bar{\mu})\bar{\rho}}, \qquad (81)$$

where:

(i) for $s = 1$: $(\bar{\lambda}\bar{\mu})\bar{\rho} = (00)1$, $R_2 = \delta_{(\lambda_2\mu_2)(\lambda_1\mu_1)}\delta_{(\lambda_2'\mu_2')(\lambda_1\mu_1)}$; $\qquad (82)$

(ii) for $s = 2$: $(\bar{\lambda}\bar{\mu})\bar{\rho} = (20)1$, $R_2 = \delta_{(\lambda_2'\mu_2')(\lambda_1\mu_1)}(-1)^{\lambda_1+\mu_1+\lambda_2+\mu_2}$

$$\times \left[\frac{3d(\lambda_2\mu_2)}{d(20)d(\lambda_1\mu_1)}\right]^{\frac{1}{2}} \begin{bmatrix} (10) & (10) & (20) \\ (a+b, 0) & (c+\alpha, 0) & (\lambda_1\mu_1) \\ (a+b+1, 0) & (c+\alpha+1, 0) & (\lambda_2\mu_2) \end{bmatrix}$$

$$\times \left[\binom{a+b+1}{1}\binom{c+\alpha+1}{1}\right]^{\frac{1}{2}}, \qquad (83)$$

with $a+b+1 = 4, c+\alpha+1 = Q-2\beta; d(\lambda\mu) = \frac{1}{2}(\lambda+1)(\mu+1)(\lambda+\mu+2)$;

(iii) for $s = 3: (\overline{\lambda}\overline{\mu})\overline{\rho} = (02)1, \quad R_2 = \delta_{(\lambda_2\mu_2)(\lambda_1\mu_1)}(-1)^{\lambda_1+\mu_1+\lambda_2+\mu_2}$

$$\times \left[\frac{3d(\lambda_2'\mu_2')}{d(20)d(\lambda_1\mu_1)}\right]^{\frac{1}{2}} \begin{bmatrix} (a+c,0) & (b+\alpha,0) & (\lambda_1\mu_1) \\ (10) & (10) & (20) \\ (a+c+1,0) & (b+\alpha+1,0) & (\lambda_2'\mu_2') \end{bmatrix}$$

$$\times \left[\binom{a+c+1}{1}\binom{b+\alpha+1}{1}\right]^{\frac{1}{2}}, \tag{84}$$

with $a+c+1 = 4; b+\alpha+1 = Q'-2\gamma$;

(iv) for $s = 4: (\overline{\lambda}\overline{\mu})\overline{\rho} = (00)1$ and $(22)\overline{\rho}$,

$$R_2 = 3[\sqrt{\tfrac{1}{6}}\delta_{(\overline{\lambda}\overline{\mu})(00)} - \sqrt{\tfrac{5}{6}}\delta_{(\overline{\lambda}\overline{\mu})(22)}]\left[\frac{d(\lambda_2\mu_2)}{d(20)d(\lambda_1\mu_1)}\right]^{\frac{1}{2}}$$

$$\times U((\lambda_2\mu_2)(\mu_1\lambda_1)(\overline{\lambda}\overline{\mu})(02);(20)_{--}(\mu_2'\lambda_2')_{-}\overline{\rho})$$

$$\times \begin{bmatrix} (a+b,0) & (c+\alpha,0) & (\lambda_1\mu_1) \\ (10) & (10) & (20) \\ (a+b+1,0) & (c+\alpha+1,0) & (\lambda_2\mu_2) \end{bmatrix}\begin{bmatrix} (a+c,0) & (b+\alpha,0) & (\lambda_1\mu_1) \\ (10) & (10) & (20) \\ (a+c+1,0) & (b+\alpha+1,0) & (\lambda_2'\mu_2') \end{bmatrix}$$

$$\times \left[\binom{a+b+1}{1}\binom{c+\alpha+1}{1}\binom{a+c+1}{1}\binom{b+\alpha+1}{1}\right]^{\frac{1}{2}}; \tag{85}$$

(v) for $s = 5: (\overline{\lambda}\overline{\mu})\overline{\rho} = (20)1, \quad R_2 = \delta_{(\lambda_2\mu_2)(\lambda_1\mu_1)}(-1)^{\lambda_1+\mu_1+\lambda_2+\mu_2}$

$$\times \left[\frac{d(\lambda_2\mu_2)}{d(\lambda_1\mu_1)}\right]^{\frac{1}{2}}\left[\binom{a+b+2}{2}\right]^{\frac{1}{2}}$$

$$\times U((20)(a+b,0)(\lambda_2\mu_2)(c+\alpha,0);(a+b+2,0)_{--};(\lambda_1\mu_1)_{--}); \tag{86}$$

(vi) for $s = 6: (\overline{\lambda}\overline{\mu})\overline{\rho} = (02)1, \quad R_2 = \delta_{(\lambda_2\mu_2)(\lambda_1\mu_1)}$

$$\times \left[\frac{d(\lambda_2'\mu_2')}{d(\lambda_1\mu_1)}\right]^{\frac{1}{2}}\left[\binom{a+c+2}{2}\right]^{\frac{1}{2}}$$

$$\times U((02)(0,a+c)(\mu_2'\lambda_2')(0,b+\alpha);(0,a+c+2)_{--};(\mu_1\lambda_1)_{--}); \tag{87}$$

(vii) for $s = 7 : (\bar{\lambda}\bar{\mu})\bar{\rho} = (00)1$ and $(22)\bar{\rho}$,

$$R_2 = \left[\sqrt{\tfrac{1}{6}}\delta_{(\bar{\lambda}\bar{\mu})(00)} - \sqrt{\tfrac{5}{6}}\delta_{(\bar{\lambda}\bar{\mu})(22)}\right]\left[\frac{3d(\lambda_2'\mu_2')}{d(\lambda_1\mu_1)}\right]^{\frac{1}{4}}$$

$$\times \; U((20)(\lambda_1\mu_1)(\bar{\lambda}\bar{\mu})(\mu_2'\lambda_2');(\lambda_2\mu_2)_-\bar{\rho};(02)_{--})$$

$$\times \; U((02)(0,a+c)(\mu_2'\lambda_2')(0,b+\alpha);(0,a+c+2)_{--};(\mu_1\lambda_1)_{--})$$

$$\times \begin{bmatrix} (10) & (10) & (20) \\ (a+b,0) & (c+\alpha,0) & (\lambda_1\mu_1) \\ (a+b+1,0) & (c+\alpha+1,0) & (\lambda_2\mu_2) \end{bmatrix}$$

$$\times \left[\binom{a+b+1}{1}\binom{c+\alpha+1}{1}\binom{a+c+2}{2}\right]^{\frac{1}{4}}; \qquad (88)$$

(viii) for $s = 8 : (\bar{\lambda}\bar{\mu})\bar{\rho} = (00)1$ and $(22)\bar{\rho}$,

$$R_2 = \left[\sqrt{\tfrac{1}{6}}\delta_{(\bar{\lambda}\bar{\mu})(00)} - \sqrt{\tfrac{5}{6}}\delta_{(\bar{\lambda}\bar{\mu})(22)}\right]\left[\frac{3d(\lambda_2'\mu_2')}{d(\lambda_1\mu_1)}\right]^{\frac{1}{4}}$$

$$\times \; U((20)(\lambda_1\mu_1)(\bar{\lambda}\bar{\mu})(\mu_2'\lambda_2');(\lambda_2\mu_2)_-\bar{\rho};(02)_{--})$$

$$\times \; U((20)(a+b,0)(\lambda_2\mu_2)(c+\alpha,0);(a+b+2,0)_{--};(\lambda_1\mu_1)_{--})$$

$$\times \begin{bmatrix} (10) & (10) & (20) \\ (a+c,0) & (b+\alpha,0) & (\lambda_1\mu_1) \\ (a+c+1,0) & (b+\alpha+1,0) & (\lambda_2'\mu_2') \end{bmatrix}$$

$$\times \left[\binom{a+c+1}{1}\binom{b+\alpha+1}{1}\binom{a+b+2}{2}\right]^{\frac{1}{4}}; \qquad (89)$$

(ix) for $s = 9 : (\bar{\lambda}\bar{\mu})\bar{\rho} = (00)1$ and $(22)\bar{\rho}$,

$$R_2 = \left[\sqrt{\tfrac{1}{6}}\delta_{(\bar{\lambda}\bar{\mu})(00)} - \sqrt{\tfrac{5}{6}}\delta_{(\bar{\lambda}\bar{\mu})(22)}\right]\left[\frac{6d(\lambda_2'\mu_2')}{d(\lambda_1\mu_1)}\right]^{\frac{1}{4}}$$

$$\times \; U((20)(\lambda_1\mu_1)(\bar{\lambda}\bar{\mu})(\mu_2'\lambda_2');(\lambda_2\mu_2)_-\bar{\rho};(02)_{--})$$

$$\times \; U((20)(a+b,0)(\lambda_2\mu_2)(c+\alpha,0);(a+b+2,0)_{--};(\lambda_1\mu_1)_{--})$$

$$\times \; U((02)(0,a+c)(\mu_2'\lambda_2')(0,b+\alpha);(0,a+c+2)_{--};(\mu_1\lambda_1)_{--})$$

$$\times \left[\binom{a+b+2}{2}\binom{a+c+2}{2}\right]^{\frac{1}{4}}. \qquad (90)$$

In the final step of the harmonic oscillator expansion, a power series expansion is made in the exponentials containing the (20) tensors $(\boldsymbol{K} \cdot \boldsymbol{K})$ and the (02) tensors $(\boldsymbol{K}^* \cdot \boldsymbol{K}^*)$. The factors $(\bar{\boldsymbol{K}} \cdot \boldsymbol{K})^\beta$, $(\boldsymbol{K}^* \cdot \boldsymbol{K}^*)^\gamma$ are then combined with the K-space functions of eq. (81) to give

$$(\bar{\boldsymbol{K}}_1 \cdot \boldsymbol{K}_1^*)^a (\bar{\boldsymbol{K}}_1 \cdot \boldsymbol{K}^*)^b (\boldsymbol{K}_1^* \cdot \boldsymbol{K})^c f_s (\bar{\boldsymbol{K}} \cdot \boldsymbol{K}^*)^x (\bar{\boldsymbol{K}} \cdot \bar{\boldsymbol{K}})^\beta (\boldsymbol{K}^* \cdot \boldsymbol{K}^*)^\gamma$$

$$= \sum_{(\lambda_1 \mu_1)} R_1(abc\alpha; \lambda_1 \mu_1) \sum_{(\lambda_2 \mu_2)(\lambda_2' \mu_2')} \sum_{(\bar{\lambda}\bar{\mu})\bar{\rho}} R_2(\lambda_1 \mu_1; \lambda_2 \mu_2, \lambda_2' \mu_2'; (\bar{\lambda}\bar{\mu})\bar{\rho})$$

$$\times \sum_{(\lambda_0 \mu_0)\rho_0} R_3(\lambda_2 \mu_2, \lambda_2' \mu_2', (\bar{\lambda}\bar{\mu})\bar{\rho}; (\lambda_0 \mu_0)\rho_0)$$

$$\times [[P(\bar{\boldsymbol{K}}_1)^{(40)} \times P(\bar{\boldsymbol{K}})^{(Q0)}]^{(\lambda\mu)} \times [P(\boldsymbol{K}_1^*)^{(04)} \times P(\boldsymbol{K}_1^*)^{(0Q')}]^{(\mu'\lambda')}]_{L=0}^{(\lambda_0 \mu_0)\rho_0}, \qquad (91)$$

where

$$R_3 = \left\{ \sum_{(\lambda_3 \mu_3)} \langle (2\beta, 0)L = 0; (0, 2\gamma)L = 0 \| (\lambda_3 \mu_3)L = 0 \rangle \right.$$

$$\times \sum_{\rho'} \langle (\bar{\lambda}\bar{\mu})L = 0; (\lambda_3 \mu_3)L = 0 \| (\lambda_0 \mu_0)L = 0 \rangle_{\rho'} \left. \begin{bmatrix} (\lambda_2 \mu_2) & (\mu_2' \lambda_2') & (\bar{\lambda}\bar{\mu})\bar{\rho} \\ (2\beta, 0) & (0, 2\gamma) & (\lambda_3 \mu_3)_- \\ (\lambda\mu) & (\mu'\lambda') & (\lambda_0 \mu_0)\rho_0 \\ - & - & \rho' \end{bmatrix} \right\}$$

$$\times \left[(2\beta + 1)!(2\gamma + 1)! \binom{Q}{2\beta}\binom{Q'}{2\gamma} \right]^{\frac{1}{2}}$$

$$\times U((40)(Q - 2\beta, 0)(\lambda\mu)(2\beta, 0); (\lambda_2 \mu_2)_{--}; (Q0)_{--})$$

$$\times U((04)(0, Q' - 2\gamma)(\mu'\lambda')(0, 2\gamma); (\mu_2' \lambda_2')_{--}; (0Q')_{--}). \qquad (92)$$

The final SU(3)-coupled K-space function is now in the form needed for the harmonic oscillator expansion of eq. (66). The determination of the needed coefficients is achieved by carrying out the sums over all $abc\alpha\beta\gamma$-dependent terms which are contributed by the several terms of eq. (55).

7.2. CHANNEL SPIN EXPANSION

To attain the expansion in the angular-momentum coupled K-space channel functions of eq. (75), the SU(3) recoupling transformations are restricted to the $\bar{\boldsymbol{K}}_1$, $\boldsymbol{K}_1^*$ dependent factors, given by eqs. (79) and (81) with $\alpha = 0$. With $\alpha = 0$, the coefficient R_1 has the simple form

$$R_1(abc0; \lambda_1 \mu_1) = \left[\frac{d(\lambda_1 \mu_1)(a + b - \mu_1)!(a + c - \mu_1)!}{(b - \mu_1)!(c - \mu_1)!} \right]^{\frac{1}{2}} (-1)^{a+b+c+\lambda_1+\mu_1} b! c!. \qquad (93)$$

The coefficients R_2 are those enumerated in eqs. (82)–(90) (with $\alpha = 0$). The SU(3)-coupled K-space functions of eq. (81) can now be converted to angular momentum-coupled K-space functions by means of SU(3) $\supset$ R(3) reduced Wigner coefficients:

$$[[P(\bar{K}_1)^{(40)} \times P(\bar{K})^{(q_10)}]^{(\lambda_2\mu_2)} \times [P(K_1^*)^{(04)} \times P(K^*)^{(0q_2)}]^{(\mu_2'\lambda_2')}]^{(\bar{\lambda}\bar{\mu})\bar{\rho}}_{\bar{L}=0,\,\bar{M}=0}$$

$$= \sum_{L_2\kappa_2'\kappa_2}\sum_{l_cl_1}\sum_{l_c'l_2}\langle(\lambda_2\mu_2)\kappa_2 L_2 ; (\mu_2'\lambda_2')\kappa_2' L_2\|(\bar{\lambda}\bar{\mu})\bar{L}=0\rangle_{\bar{\rho}}$$

$$\times \langle(40)l_c ; (q_10)l_1\|(\lambda_2\mu_2)\kappa_2 L_2\rangle\langle(04)l_c' ; (0q_2)l_2\|(\mu_2'\lambda_2')\kappa_2' L_2\rangle$$

$$\times 4\pi\left[\frac{2^{l_1+l_2}(\frac{1}{2}[q_1+l_1])!(\frac{1}{2}[q_2+l_2])!}{(\frac{1}{2}[q_1-l_1])!(\frac{1}{2}[q_2-l_2])!(q_1+l_1+1)!(q_2+l_2+1)!}\right]^{\frac{1}{2}}\bar{K}^{q_1}K^{q_2}$$

$$\times ((P(\bar{K}_1)^{(40)}_{l_c} \times Y_{l_1}(\hat{\bar{K}}))^{L_2} \times (P(K_1^*)^{(04)}_{l_c'} \times Y_{l_2}(\hat{K}))^{L_2})^{\bar{L}=0}_{\bar{M}=0}, \qquad (94)$$

where we have used

$$P(\bar{K})^{(q0)}_{lm} = \left[\frac{4\pi 2^l(\frac{1}{2}[q+l])!}{(\frac{1}{2}[q-l])!(q+l+1)!}\right]^{\frac{1}{2}}\bar{K}^q Y_{lm}(\hat{\bar{K}}) \qquad (95)$$

[see, e.g. Kramer et al.[1]), although the phase convention is that of refs.[11,14)]. Note that the symbols $Q - 2\beta$, $Q' - 2\gamma$ of eq. (81) have been replaced by the running indices q_1, q_2. Since $q_1 \leq 4$, $q_2 \leq 4$, the SU(3) $\supset$ R(3) (double-barred) Wigner coefficients of eq. (94) are very simple. Exponentials of the type $\exp\{\rho(\bar{K} \cdot \bar{K})\}$, and $\exp\{\tau(K \cdot K)\}$ are functions of the magnitudes $\bar{K}$ and K only and contribute directly to the $\bar{K}$, K-dependence of the spatial integrals of eq. (75). Exponentials of the type $\exp\{\sigma(\bar{K} \cdot K)\}$ with $\sigma = (1 - p[A/f(A-f)] \pm g[A/f(A-f)])$ are best expanded in terms of modified spherical Bessel functions to match the convergence properties of eq. (70). With this expansion the K-space angular functions can then be combined by

$$e^{\sigma(\bar{K} \cdot K)}((P(\bar{K}_1)^{(40)}_{l_c} \times Y_{l_1}(\hat{\bar{K}}))^{L_2} \times (P(K_1^*)^{(04)}_{l_c'} \times Y_{l_2}(\hat{K}))^{L_2})^0_0$$

$$= \sum_L i_L(\sigma\bar{K}K)(-1)^L \times \sum_J\sum_{ll'}\left[\frac{(2J+1)(2l_1+1)(2l_2+1)(2L+1)^2}{(2L_2+1)(2l+1)(2l'+1)}\right]^{\frac{1}{2}}$$

$$\times \langle l_1 0 L0|l0\rangle\langle l_2 0 L0|l'0\rangle \times U(l_c l_1 JL ; L_2 l)U(l_c' l_2 JL ; L_2 l')$$

$$\times ((P(\bar{K}_1)^{(40)}_{l_c} \times Y_l(\hat{\bar{K}}))^J \times (P(K_1^*)^{(04)}_{l_c'} \times Y_l(\hat{K}))^J)^0_0, \qquad (96)$$

to yield the K-space expansions in terms of the desired angular-momentum coupled channel functions. The determination of the needed coefficients is again achieved by

carrying out the sums over all *abc*, *L*-dependent terms which are contributed by the various pieces of the full $H_V(\bar{K}, K^*)$.

8. Summary and discussion

The complicated $3A$-dimensional coordinate space integrals needed for the evaluation of both RGM and GCM interaction kernels can be extracted from an appropriate expansion of the Bargmann transform of the interaction by purely algebraic techniques. The Bargmann transform of the antisymmetrizer (norm kernels) and the two-body interaction in a fully antisymmetrized basis (interaction kernels) can thus be made to play the central role in the calculation of RGM and GCM kernels. In the present investigation techniques developed for the evaluation of the Bargmann transform of the antisymmetrizer have been generalized to include the Fourier kernel, $\sum_j \exp(i\mathbf{q} \cdot \mathbf{r}_j)$, of the basic one-body operator, and a central two-body interaction of gaussian form with arbitrary exchange mixture. The present investigation is limited to binary cluster systems in which both cluster fragments in both sides of a matrix element are $4n$-self-conjugate (SU(4) scalar) nuclei with internal wave functions of space symmetry of $[44\ldots4]$ character and oscillator functions of good SU(3) symmetry. The cluster fragments are thus restricted to be in their ground states or in excited states which can be approximated as rotational band members of the ground state SU(3) representations. The calculations are simplest for such $4n$-self-conjugate cluster fragments. Yet, some of the most interesting problems which remain to be investigated by microscopic cluster model techniques fall precisely into this category. Our prime example is the $A = 24$ system with cluster decompositions $^{12}C + ^{12}C$, $^{8}Be + ^{16}O$, and $\alpha + ^{20}Ne$, all important channels for the so-called quasi-molecular resonances, and all made up of cluster fragments with internal functions which are approximated well by functions of good SU(3) symmetry.

Generalization of the present technique to systems made up of more than two $4n$-self-conjugate cluster fragments should be straightforward but will lead to more complicated functions in the several cluster-relative motion variables. The generalization to systems with more complicated cluster fragments (with $A \neq 4n$) will be more challenging since a key feature of the present technique involves the immediate reduction from A-particle orbital states of space symmetry characterized by 4-columned Young tableaux to $\frac{1}{4}A$-particle orbital states of single-column symmetry. In the more general case the summation over spin-isospin quantum numbers responsible for this reduction can lead to linear combinations of products of single-column orbital functions which may contain a large number of terms and will in general involve some two-column functions which cannot be further reduced or expressed in terms of a convolution of single-column Fourier transforms. For simple systems (e.g., an α-particle + arbitrary fragment or a nucleon + arbitrary fragment) such a generalization may be possible. For more complicated cluster fragments it may be more fruitful to use a reduction of the A-

particle problem to the $(A-2)$-particle problem, utilizing known $(A-2)$-particle norm kernels and a cluster fractional parentage expansion.

Throughout the present investigation it has been assumed that the internal functions of the cluster fragments are built from harmonic oscillator functions with equal width parameters for all fragments. The generalization to the case of unequal width parameters is straightforward as far as the calculation of the Bargmann transform is concerned, but the resultant cross terms between relative motion and center-of-mass excitations require special treatment. Dilatation techniques [1,4] or the method of double Fourier transformation [7] can in principle be used but will greatly complicate the process of evaluating the Bargmann transforms in general algebraic form.

The restriction to two-body interactions of gaussian form could also be relaxed. However, the gaussian form is very convenient, since many effective interactions can be approximated well by a superposition of a few gaussian terms with different ranges. Even the Coulomb potential can be included since the Bargmann transform of the gaussian interaction is given in general algebraic form. The $1/r$ potential can be expressed in terms of an integral over all values of the gaussian range parameter. Since the Bargmann transform is known as an algebraic function of the range parameter such integrals can be performed for the K-space functions.

Throughout this investigation the aim has been the evaluation of the needed Bargmann transforms in general algebraic form. This is important for the reasons indicated above and is vital for RGM calculations. It has also been shown how RGM interaction kernels can be extracted from appropriate expansions of the Bargmann transform of the interaction. Since the expansions, particularly in the harmonic oscillator basis, may be slowly convergent, it would be advantageous to evaluate RGM kernels in completely closed algebraic form. Since the Bargmann transform of a gaussian interaction in a particular binary cluster channel is built from a few simple polynomial and exponential functions in the relative motion K-space variable, it may be possible to gain such a closed form by performing the inverse of the Bargmann transformation. This will be the subject of a future investigation.

Appendix

TABULATION OF THE SINGLE-COLUMN BARGMANN TRANSFORMS

The full tabulation of the single-column Bargmann transforms H_E and H_U is given in ref. [15] for the following A-values and fragment decompositions: $A = 12$, $\alpha + {}^8Be$; $A = 16$, $\alpha + {}^{12}C$ and ${}^8Be + {}^8Be$; $A = 20$, $\alpha + {}^{16}O$ and ${}^8Be + {}^{12}C$; and $A = 24$, ${}^{12}C + {}^{12}C$, ${}^8Be + {}^{16}O$ and $\alpha + {}^{20}Ne$. To illustrate the nature and scope of the results the tables for the ${}^{12}C + {}^{12}C$ and ${}^8Be + {}^{16}O$ fragment decompositions of the $A = 24$ system

TABLE E1

$$H_E \begin{pmatrix} 1\,(00) & 1\,(00) \\ 2\,(10) & 2\,(10) \end{pmatrix} \text{ for } \alpha + {}^8\text{Be}$$

(00) - tensors: $\ a = (\overline{K}_1 \cdot K_1^*), \ \overline{b} = (K_1^* \cdot \overline{\mathscr{K}}), \ b^* = (\overline{K}_1 \cdot \mathscr{K}^*).$

q-dep	q-indep	$e^{\theta\,(1,1)}$	$e^{\theta\,(1,2)}$	$e^{\theta\,(2,1)}$	$e^{\theta\,(2,2)}$
	a	1	$-e^1$	$-e^1$	$2-e^1$
	$\overline{b}b^*$		$-e^1$	$-e^1$	$-e^1$
$\overline{K}_1$	$\overline{b}$			e^1	$-e^1$
K_1^*	b^*		e^1		$-e^1$
$\overline{K}_1\,K_1^*$					$1-e^1$

are included in this appendix along with the simple results for the $\alpha + {}^8\text{Be}$ system, the latter for ready comparison with the text. H_E and H_U give the BS transforms of the operators $\ \mathscr{C} = \sum_j \exp(\sqrt{2}i q \cdot r_j) \ (H_E)$, and $\ \mathscr{C} = \sum_{i<j} u(r_{ij})$, with $u(r) = \exp(-\tfrac{1}{2}\beta r^2) \ (H_U)$. The tabulations give the factors needed to construct the single-column functions

$$H_{\mathscr{C}} \begin{pmatrix} l(\lambda_l\mu_l) & l'(\lambda_l'\mu_l') \\ n(\lambda_n\mu_n) & n'(\lambda_n'\mu_n') \end{pmatrix},$$

where $l = \tfrac{1}{4}f$, $n = \tfrac{1}{4}(A-f)$, $l' = \tfrac{1}{4}f'$, $n' = \tfrac{1}{4}(A-f')$ and $(\lambda_l\mu_l)$ is the SU(3) representation of the l nucleons in fragment f, etc. The dependence on the relative motion Bargmann variables $\overline{K}$ (for bra) and K^* (for ket) are given in terms of

$$\mathscr{K} = \left[\frac{A}{f(A-f)} \right]^{\frac{1}{2}} K, \qquad \mathscr{K}^* = \left[\frac{A}{f'(A-f')} \right]^{\frac{1}{2}} K^*.$$

Tables E1–E4 give the factors needed to construct the functions

$$H_E = \exp\left[\frac{1}{4} \frac{f'(A-f)}{A} \mathscr{K} \cdot \mathscr{K}^* \right] e^{-\frac{1}{2}q \cdot q} \sum_{i,j} h_{ij} e^{\theta(i,j)}.$$

The choices of f, $A-f$, f', $A-f'$ in the bras and kets of the tables are arranged so that the f' fragment is the smallest of the four fragments when $f' \neq f$. The $\theta(i,j)$ are defined by

$$\theta(1,1) = iq \cdot \left[-\left(\frac{A-f}{A}\right)\mathscr{K} - \left(\frac{A-f'}{A}\right)\mathscr{K}^* \right], \quad \theta(2,1) = iq \cdot \left[\frac{f}{A}\mathscr{K} - \left(\frac{A-f'}{A}\right)\mathscr{K}^* \right],$$

$$\theta(1,2) = iq \cdot \left[-\left(\frac{A-f}{A}\right)\mathscr{K} + \frac{f'}{A}\mathscr{K}^* \right], \qquad \theta(2,2) = iq \cdot \left[\frac{f}{A}\mathscr{K} + \frac{f}{A}\mathscr{K}^* \right],$$

TABLE E2

$$H_E \begin{Bmatrix} 3(01) & \Big| & 3(01) \\ 3(01) & \Big| & 3(01) \end{Bmatrix} \text{ for } {}^{12}C + {}^{12}C$$

(00) – tensors: $\quad a = (\bar{K}_{12} \cdot K^*_{12})$, $\ b = (\bar{K}_{34} \cdot K^*_{34})$, $\ \bar{c} = (\bar{K}_{12} \cdot K^*_{34})$,

$\qquad\qquad\qquad c^* = (K^*_{12} \cdot \bar{K}_{34})$, $\ \bar{d} = (\bar{K}_{12} \cdot \bar{\mathcal{K}})$, $\ d^* = (K^*_{12} \cdot \mathcal{K}^*)$,

$\qquad\qquad\qquad \bar{e} = (\bar{K}_{34} \cdot \bar{\mathcal{K}})$, $\ e^* = (K^*_{34} \cdot \mathcal{K}^*)$, $\ f = (\bar{\mathcal{K}} \cdot \mathcal{K}^*)$.

q-dep	q-indep	$e^{\theta}(1,1)$	$e^{\theta}(1,2)$	$e^{\theta}(2,1)$	$e^{\theta}(2,2)$
	ab	$3-4e^1+e^2$	$-2e^1+2e^2$	$-2e^1+2e^2$	$3-4e^1+e^2$
	$\bar{c}c^*$	$-2e^1+2e^2$	$-e^1+4e^2-3e^3$	$-e^1+4e^2-3e^3$	$-2e^1+2e^2$
	$a\bar{e}e^*+b\bar{d}d^*$ $-\bar{c}d^*\bar{e}-c^*\bar{d}e^*$	$-2e^1+e^2$	$-e^1+2e^2$	$-e^1+2e^2$	$-2e^1+e^2$
	$abf^2-a\bar{e}e^*f$ $-b\bar{d}d^*f$ $+\bar{d}d^*\bar{e}e^*$	$-2e^1$	$-e^1$	$-e^1$	$-2e^1$
	$\bar{c}c^*f^2-\bar{c}d^*\bar{e}f$ $-c^*\bar{d}e^*f$ $+\bar{d}d^*\bar{e}e^*$	e^2	$2e^2$	$2e^2$	e^2
$\bar{K}_{12}$	bd^*		e^1-e^2		$-e^1+e^2$
$\bar{K}_{12}$	c^*e^*	$-e^1+e^2$		e^1-e^2	
$\bar{K}_{12}$	$bd^*f-\bar{d}\bar{e}e^*$	$-e^1$		e^1	
$\bar{K}_{12}$	$c^*e^*f-\bar{d}\bar{e}e^*$		e^2		$-e^2$
K^*_{12}	$b\bar{d}$			e^1-e^2	$-e^1+e^2$
K^*_{12}	$\bar{c}\bar{e}$	$-e^1+e^2$	e^1-e^2		
K^*_{12}	$b\bar{d}f-\bar{d}\bar{e}e^*$	$-e^1$	e^1		
K^*_{12}	$\bar{c}\bar{e}f-\bar{d}\bar{e}e^*$			e^2	$-e^2$
$\bar{K}_{34}$	ae^*	e^1-e^2		$-e^1+e^2$	
$\bar{K}_{34}$	$\bar{c}d^*$		$-e^1+e^2$		e^1-e^2

TABLE E2 (cont.)

$\bar{K}_{34}$	$ae^*f-\bar{d}d^*e^*$		$-e^1$		e^1
$\bar{K}_{34}$	$\bar{c}d^*f-\bar{d}d^*e^*$	e^2		$-e^2$	
K_{34}^*	$a\bar{e}$	e^1-e^2	$-e^1+e^2$		
K_{34}^*	$c^*\bar{d}$			$-e^1+e^2$	e^1-e^2
K_{34}^*	$a\bar{e}f-\bar{d}d^*\bar{e}$			$-e^1$	e^1
K_{34}^*	$c^*\bar{d}f-\bar{d}d^*\bar{e}$	e^2	$-e^2$		
$\bar{\mathcal{K}}$	$abf-a\bar{e}e^*$	e^1	$-e^1$		
$\bar{\mathcal{K}}$	$abf-b\bar{d}d^*$			e^1	$-e^1$
$\bar{\mathcal{K}}$	$\bar{c}c^*f-\bar{c}d^*\bar{e}$	$-e^2$	e^2		
$\bar{\mathcal{K}}$	$\bar{c}c^*f-c^*\bar{d}e^*$			$-e^2$	e^2
$\mathcal{K}^*$	$abf-a\bar{e}e^*$	e^1		$-e^1$	
$\mathcal{K}^*$	$abf-b\bar{d}d^*$		e^1		$-e^1$
$\mathcal{K}^*$	$\bar{c}c^*f-c^*\bar{d}e^*$	$-e^2$		e^2	
$\mathcal{K}^*$	$\bar{c}c^*f-\bar{c}d^*\bar{e}$		$-e^2$		e^2
$\bar{K}_{12}\ K_{12}^*$	b	$-1+e^1$	e^1-e^2	e^1-e^2	$-e^1+e^2$
$\bar{K}_{12}\ K_{12}^*$	bf	e^1	$-e^2$	$-e^2$	e^1
$\bar{K}_{12}\ K_{12}^*$	$\bar{e}e^*$				$-e^1+e^2$
$\bar{K}_{12}\ K_{34}^*$	c^*	e^1-e^2	$-e^2+e^3$	$-e^1+e^2$	e^1-e^2
$\bar{K}_{12}\ K_{34}^*$	c^*f	$-e^1$	e^2	e^2	$-e^1$
$\bar{K}_{12}\ K_{34}^*$	$d^*\bar{e}$			e^1-e^2	
$K_{12}^*\ \bar{K}_{34}$	$\bar{c}$	e^1-e^2	$-e^1+e^2$	$-e^2+e^3$	e^1-e^2
$K_{12}^*\ \bar{K}_{34}$	$\bar{c}f$	$-e^1$	e^2	e^2	$-e^1$
$K_{12}^*\ \bar{K}_{34}$	$\bar{d}e^*$		e^1-e^2		

TABLE E2 (cont.)

$\bar{K}_{34}\ K^*_{34}$	a	$-e^1+e^2$	e^1-e^2	e^1-e^2	$-1+e^1$
$\bar{K}_{34}\ K^*_{34}$	af	e^1	$-e^2$	$-e^2$	e^1
$\bar{K}_{34}\ K^*_{34}$	$\bar{d}d^*$	$-e^1+e^2$			
$\bar{K}_{12}\ \bar{\varkappa}$	bd^*			$-e^1+e^2$	
$\bar{K}_{12}\ \bar{\varkappa}$	c^*e^*				e^1-e^2
$K^*_{12}\ \varkappa^*$	$b\bar{d}$		$-e^1+e^2$		
$K^*_{12}\ \varkappa^*$	$\bar{c}\bar{e}$				e^1-e^2
$\bar{K}_{34}\ \bar{\varkappa}$	ae^*		$-e^1+e^2$		
$\bar{K}_{34}\ \bar{\varkappa}$	$\bar{c}d^*$	e^1-e^2			
$K^*_{34}\ \varkappa^*$	$a\bar{e}$			$-e^1+e^2$	
$K^*_{34}\ \varkappa^*$	$c^*\bar{d}$	e^1-e^2			
$\bar{\varkappa}\ \varkappa^*$	ab		e^1-e^2	e^1-e^2	
$\bar{\varkappa}\ \varkappa^*$	$\bar{c}c^*$	$-e^1+e^2$			$-e^1+e^2$
$(iq\cdot iq)$	ab	$1-e^1$	$-e^1+e^2$	$-e^1+e^2$	$1-e^1$
$(iq\cdot iq)$	$\bar{c}c^*$	$-e^1+e^2$	e^2-e^3	e^2-e^3	$-e^1+e^2$
$(iq\cdot iq)$	$abf-\bar{c}c^*f$	$-e^1$	e^2	e^2	$-e^1$
$[\bar{K}_{12}\times\varkappa^*]$ $[K^*_{12}\times\bar{\varkappa}]$	$bf-\bar{e}e^*$	$-e^1$			
$[\bar{K}_{12}\times\varkappa^*]$ $[K^*_{34}\times\bar{\varkappa}]$	$c^*f-d^*\bar{e}$		e^2		
$[K^*_{12}\times\bar{\varkappa}]$ $[\bar{K}_{34}\times\varkappa^*]$	$\bar{c}f-\bar{d}e^*$			e^2	
$[\bar{K}_{34}\times\varkappa^*]$ $[K^*_{34}\times\bar{\varkappa}]$	$af-\bar{d}d^*$				$-e^1$

$$H_E \begin{pmatrix} 2(10) & 2(10) \\ 4(00) & 4(00) \end{pmatrix} \text{ for } {}^8Be + {}^{16}O$$

(00) - tensors: $a = (\bar{K}_1 \cdot K_1^*)$, $\bar{b} = (K_1^* \cdot \bar{\mathcal{K}})$, $b^* = (\bar{K}_1 \cdot \mathcal{K}^*)$, $c = (\bar{\mathcal{K}} \cdot \mathcal{K}^*)$.

q-dep	q-indep	$e^0(1,1)$	$e^0(1,2)$	$e^0(2,1)$	$e^0(2,2)$
	a	$2-2e^1$	$-2e^1+2e^2$	$-2e^1+2e^2$	$4-6e^1+2e^2$
	$ac-\bar{b}b^*$	$-e^1$	$-e^1+2e^2$	$-e^1+2e^2$	$-3e^1+2e^2$
	$\bar{b}b^*c$	$-e^1$	$-e^1$	$-e^1$	$-3e^1$
$\bar{K}_1$	$\bar{b}$			$-e^1+e^2$	e^1-e^2
$\bar{K}_1$	$\bar{b}c$	e^1	$-e^1$		
K_1^*	b^*		$-e^1+e^2$		e^1-e^2
K_1^*	b^*c	e^1		$-e^1$	
$\bar{\mathcal{K}}$	a			e^1-e^2	$-e^1+e^2$
$\bar{\mathcal{K}}$	$\bar{b}b^*$			e^1	$-e^1$
$\mathcal{K}^*$	a		e^1-e^2		$-e^1+e^2$
$\mathcal{K}^*$	$\bar{b}b^*$		e^1		$-e^1$
$\bar{K}_1\ K_1^*$		$1-e^1$	$-e^1+e^2$	$-e^1+e^2$	e^1-e^2
$\bar{K}_1\ K_1^*$	c	$-e^1$	e^2	e^2	$-e^2$
$\bar{K}_1\ \mathcal{K}^*$	$\bar{b}$		e^1-e^2		$-e^1+e^2$
$K_1^*\ \bar{\mathcal{K}}$	b^*			e^1-e^2	$-e^1+e^2$
$\bar{\mathcal{K}}\ \mathcal{K}^*$	a				e^1-e^2
$\bar{\mathcal{K}}\ \mathcal{K}^*$	$\bar{b}b^*$				e^1
$(iq \cdot iq)$	a				$1-2e^1+e^2$
$(iq \cdot iq)$	$ac-\bar{b}b^*$				$-e^1+e^2$
$(iq \cdot iq)$	$\bar{b}b^*c$				$-e^1$

TABLE E4

$$H_E\begin{Bmatrix} 3(01) & 2(10) \\ 3(01) & 4(00) \end{Bmatrix} \text{ for } {}^{12}C + {}^{12}C \mid {}^{8}Be + {}^{16}O$$

(00) – tensors: $a = ([\overline{K}_{12} \times \overline{K}_{34}] \cdot K_1^*)$, $b = (\overline{K}_{12} \cdot \overline{\mathcal{K}})$, $c = (\overline{K}_{34} \cdot \overline{\mathcal{K}})$,

$d = (K_1^* \cdot \overline{\mathcal{K}})$, $e = ([\overline{K}_{12} \times \overline{K}_{34}] \cdot \mathcal{K}^*)$, $f = ([\overline{K}_{12} \times K_1^*] \cdot \mathcal{K}^*)$,

$g = ([\overline{K}_{34} \times K_1^*] \cdot \mathcal{K}^*)$, $h = (\overline{\mathcal{K}} \cdot \mathcal{K}^*)$.

q-dep	q-indep	$e^{\theta}(1,1)$	$e^{\theta}(1,2)$	$e^{0}(2,1)$	$e^{\theta}(2,2)$
	a	$2-2e^1$	$1-4e^1+3e^2$	$-2e^1+2e^2$	$3-4e^1+e^2$
	bg	$-e^1$	$-2e^1+3e^2$	$-e^1+2e^2$	$-2e^1+e^2$
	cf	$2-e^1$	$1-2e^1$	$-e^1$	$3-2e^1$
	deh	$-e^1$	$-2e^1$	$-e^1$	$-2e^1$
$\overline{K}_{12}$	g		e^1-e^2		$-e^1+e^2$
$\overline{K}_{34}$	f		$-1+e^1$		$1-e^1$
K_1^*	eh	e^1		$-e^1$	
$[\overline{K}_{12} \times \overline{K}_{34}]$	dh	e^1	$-e^1$	e^1	$-e^1$
$[\overline{K}_{12} \times K_1^*]$	c	$-1+e^1$	$1-e^1$		
$[\overline{K}_{34} \times K_1^*]$	b			e^1-e^2	$-e^1+e^2$
$\mathcal{K}^*$	de		e^1		$-e^1$
$[\overline{K}_{12} \times \mathcal{K}^*]$	cd	$-e^1$	e^1		

TABLE E4 (cont.)

$[\overline{K}_{34} \times \mathcal{K}^*]$	bd			e^1	$-e^1$
$\overline{K}_{12}$ $[\overline{K}_{34} \times K_1^*]$					$1-2e^1+e^2$
$\overline{K}_{34}$ $[\overline{K}_{12} \times K_1^*]$				$-1+2e^1-e^2$	
$[\overline{K}_{12} \times \overline{K}_{34}]$ K_1^*		$1-e^1$	$-e^1+e^2$	$-e^1+e^2$	$1-e^1$
$[\overline{K}_{12} \times \overline{K}_{34}]$ K_1^*	h		e^1		$-e^1$
$\overline{K}_{12}$ $\mathcal{K}$	g		e^1-e^2		
$\overline{K}_{34}$ $\mathcal{K}$	f				$-1+e^1$
K_1^* $\mathcal{K}$	e	$-e^1$		e^1	
$\overline{K}_{12}$ $[\overline{K}_{34} \times \mathcal{K}^*]$	dh				$-e^1$
$\overline{K}_{34}$ $[\overline{K}_{12} \times \mathcal{K}^*]$	dh		e^1		
$[\overline{K}_{12} \times \mathcal{K}^*]$ K_1^*	c	-1		e^1	
$[\overline{K}_{34} \times \mathcal{K}^*]$ K_1^*	b	e^1		$-e^2$	
$[\overline{K}_{12} \times \mathcal{K}^*]$ $\mathcal{K}^*$	cd		$-e^1$		
$[\overline{K}_{34} \times \mathcal{K}^*]$ $\mathcal{K}^*$	bd				e^1
$(iq \cdot iq)$	bg		$-e^1+e^2$		
$(iq \cdot iq)$	cf				$1-e^1$

and where the functions h_{ij},

$$h_{ij} = \prod_{k=1}^{n} (i\boldsymbol{q} \cdot \boldsymbol{a}_k)(i\boldsymbol{q} \cdot i\boldsymbol{q})^m (q\text{-independent factor}),$$

can be constructed from the tables. The first column of the tables (labeled q-dep) lists the factors $\boldsymbol{a}_k$ and $(i\boldsymbol{q} \cdot i\boldsymbol{q})^m$ (if $m \neq 0$). A blank in colomn 1 means that there are no q-dependent factors. Column 2 (labeled q-indep) gives the q-independent factors of h_{ij} made up of scalar products of the type $(\bar{\boldsymbol{K}}_1 \cdot \boldsymbol{K}_1^*)$, $(\bar{\boldsymbol{K}}_1 \cdot \mathscr{K}^*)$, ... where the $\bar{\boldsymbol{K}}_i$, $\boldsymbol{K}_j^*$ with numerical indices refer to cluster-internal variables. For ^{12}C fragments $\boldsymbol{K}_{ij}$ (e.g. $\boldsymbol{K}_{12}$) refer to the vector products $\boldsymbol{K}_{12} = \sqrt{\tfrac{1}{2}}[\bar{\boldsymbol{K}}_1 \times \bar{\boldsymbol{K}}_2]$. Barred K-variables refer to bra and starred K-variables refer to ket cluster variables. For H_E these scalar products are all SU(3) scalars [(00) tensors]. Shorthand symbols for these scalar products are given at the head of each table. A blank in column 2 means that there are no q-independent terms of this type. Additional q-independent factors of the h_{ij}, built from linear combinations of the functions e^p, are listed in columns 3–6 under the headings $e^{\theta(i, j)}$. Here

$$e^p = \exp\left[-p\mathscr{K} \cdot \mathscr{K}^*\right], \qquad p \leq p_{\max} = \min\left(\tfrac{1}{4}f, \tfrac{1}{4}f', \tfrac{1}{4}(A-f), \tfrac{1}{4}(A-f')\right).$$

The integer p counts the number of exchanged nucleons. A blank in one of the columns 3–6 means that there is no $e^{\theta(i, j)}$ term corresponding to the i, j of this column. A comparison of table E1 and eq. (50) will further illustrate the construction of the H_E from the tables.

Tables U1–U4 give the factors needed to construct the functions

$$H_U = (1-2g)^{\frac{3}{2}}\exp\left[\frac{1}{4}\frac{f'(A-f)}{A}\mathscr{K} \cdot \mathscr{K}^*\right]\sum_{ij} H_{ij}e^{\phi(i, j)},$$

with

$$\phi(i, j) = -\tfrac{1}{2}g(i\mathscr{K} + j\mathscr{K}^*)^2, \quad (i,j) = (0,0), (1,0), (0,1), (1,1), (1,-1),$$

where g gives the range of the gaussian interaction, $\exp(-\tfrac{1}{2}\beta r^2)$, through $g = \beta/2(1+\beta)$, where β (like r) is dimensionless. The H_{ij} are built from scalar product factors of the type $(\bar{\boldsymbol{K}}_1 \cdot \boldsymbol{K}_1^*)$, $(\boldsymbol{K}_1^* \cdot \mathscr{K})$, ... which are given in the first column in terms of shorthand symbols defined at the head of each table. These scalar products now include not only SU(3) scalars [(00) tensors], but also SU(3) (20) tensors and SU(3) (02) tensors. Small Roman letters $a, b, \ldots$ are used for the (00) tensors. Capital Roman letters are used for the (20) and (02) tensors; for the case of identical bra and ket cluster decompositions the symbols $\bar{A}, \bar{B}, \ldots$ designate the (20) tensors, while the symbols A^*, $B^*, \ldots$ designate the corresponding (02) tensors. With different cluster decompositions in bra and ket the letters $A, B, \ldots$ designate the (20) tensors, the letters $P, Q, \ldots$ the (02)

TABLE U1

$$H_U \begin{pmatrix} 1(00) & 1(00) \\ 2(10) & 2(10) \end{pmatrix} \text{ for } \alpha + {}^8Be$$

(00) – tensors: $a = (\bar{K}_1 \cdot K_1^*)$, $\bar{b} = (K_1^* \cdot \mathscr{K})$, $b^* = (\bar{K}_1 \cdot \mathscr{K}^*)$.

(20) – tensor : $\bar{A} = (\bar{K}_1 \cdot \mathscr{K})$.

(02) – tensor : $A^* = (K_1^* \cdot \mathscr{K}^*)$.

	1	$e^{\phi(1,0)}$	$e^{\phi(0,1)}$	$e^{\phi(1,1)}$	$e^{\phi(1,-1)}$
a	$(1-2g)$	$(-1+2g)e^1$	$(-1+2g)e^1$	$(2-g)$	$(-1-g)e^1$
$\bar{b}b^*$		$(-1+2g)e^1$	$(-1+2g)e^1$	g^2	$-(1-g)^2e^1$
$\bar{A}A^*$				g^2	$-g^2e^1$
$\bar{b}\bar{A}$		$(g-2g^2)e^1$		g^2	$(-g+g^2)e^1$
b^*A^*			$(g-2g^2)e^1$	g^2	$(-g+g^2)e^1$

tensors. The additional dependence of the functions H_{ij} on g and e^p dependent factors are given in columns 2–6 for each $e^{\phi(i,j)}$ term (note that $e^{\phi(0,0)} \equiv 1$). For some columns in some tables it is convenient to factor out the function $(1 - 2g)$ and include it with the factor $e^{\phi(i,j)}$ in the table headings.

A comparison of table U1 and eq. (51) will serve to illustrate the construction of the H_U from the tables.

A number of symmetry properties should be noted. With the same cluster decomposition in bra and ket $(f' = f)$, the H_E and H_U are invariant under the interchange $\mathscr{K} \leftrightarrow \mathscr{K}^*$, $K_i \leftrightarrow K_i^*$ for all i (bra-ket interchange). With two identical fragments in either bra or ket there is an additional symmetry. With two identical ${}^{12}C$ fragments in the bra, e.g., the transformation $K_{12} \leftrightarrow \bar{K}_{34}, \mathscr{K} \rightarrow -\mathscr{K}$ induces an overall change of sign in the single-column transform since it corresponds to an interchange of two identical fragments of odd particle number, $l = n = 3$. Note, also, that $\phi(1, 1) \leftrightarrow \phi(1, -1)$ under this transformation, while $\phi(1,0)$ and $\phi(0, 1)$ remain invariant in H_U. In H_E this transformation leads to $\theta(1, 1) \leftrightarrow \theta(2, 1)$, and $\theta(1, 2) \leftrightarrow \theta(2, 2)$. The factor $\exp([l'n/(l+n)]\mathscr{K} \cdot \mathscr{K}^*)e^p$ is also affected by this transformation and leads to the interchanges $e^0 \leftrightarrow e^3$, $e^1 \leftrightarrow e^2$ in tables E2 and U2 with ${}^{12}C + {}^{12}C$ fragment decompositions in both bra and ket while it leads to the interchanges $e^0 \leftrightarrow e^2, e^1 \leftrightarrow e^1$ in tables E4 and U4 with ${}^{12}C + {}^{12}C$ fragments in bra and ${}^8Be + {}^{16}O$ fragments in ket.

TABLE U2

$$H_U\begin{pmatrix} 3(01) & 3(01) \\ 3(01) & 3(01) \end{pmatrix} \text{ for } {}^{12}C + {}^{12}C$$

(00) - tensors: $a = (\bar{K}_{12}\cdot K_{12}^*)$, $b = (\bar{K}_{34}\cdot K_{34}^*)$, $c^* = (\bar{K}_{12}\cdot K_{34}^*)$, $\bar{c} = (\bar{K}_{12}\cdot \bar{K}_{34})$, $c^* = (K_{12}^*\cdot \bar{K}_{34})$, $\bar{d} = (\bar{K}_{12}\cdot \bar{\mathcal{K}})$,
$d^* = (K_{12}^*\cdot \mathcal{K}^*)$, $\bar{e} = (\bar{K}_{34}\cdot \bar{\mathcal{K}})$, $e^* = (K_{34}^*\cdot \mathcal{K}^*)$, $f = (\bar{\mathcal{K}}\cdot \mathcal{K}^*)$.

(20) - tensors: $\bar{A} = (K_{12}^*\cdot \bar{K}_{34})$, $\bar{B} = (K_{12}^*\cdot \bar{\mathcal{K}})$, $\bar{C} = (K_{34}^*\cdot \bar{\mathcal{K}})$, $\bar{D} = (\bar{K}_{12}\cdot[\bar{\mathcal{K}}\times \mathcal{K}^*])$, $\bar{E} = (K_{34}\cdot[\bar{\mathcal{K}}\times \mathcal{K}^*])$,
$\bar{F} = (\bar{\mathcal{K}}\cdot \mathcal{K}^*)$.

(02) - tensors: $A^* = (\bar{K}_{12}\cdot \bar{K}_{34})$, $B^* = (\bar{K}_{12}\cdot \mathcal{K}^*)$, $C^* = (\bar{K}_{34}\cdot \mathcal{K}^*)$, $D^* = (K_{12}^*\cdot[\bar{\mathcal{K}}\times \mathcal{K}^*])$,
$E^* = (K_{34}^*\cdot[\bar{\mathcal{K}}\cdot \mathcal{K}^*])$, $F^* = (\mathcal{K}^*\cdot \mathcal{K}^*)$.

	$(1-2g)$	$(1-2g)e^{\phi(1,0)}$	$(1-2g)e^{\phi(0,1)}$	$e^{\phi(1,1)}$	$e^{\phi(1,-1)}$
ab	$6-4e^1+2e^2$	$-8e^1+4e^2$	$-8e^1+4e^2$	$(9-12g+6g^2)$ $-(8-10g+7g^2)e^1$ $+(1-g)^2e^2$	$(-2-2g-6g^2)e^1$ $+(4-2g+7g^2)e^2$ $-g^2e^3$
$\bar{c}c^*$	$-2e^1+4e^2-6e^3$	$-4e^1+8e^2$	$-4e^1+8e^2$	$g^2-(4-2g+7g^2)e^1$ $+(2+2g+6g^2)e^2$	$-(1-g)^2e^1$ $+(8-10g+7g^2)e^2$ $-(9-12g+6g^2)e^3$
$\bar{A}A^*$				$g^2(1-2e^1+e^2)$	$g^2(-e^1+2e^2-e^3)$

$\bar{a}\bar{e}e^*+b\bar{d}\bar{d}^*$	$-2e^1+2e^2$	$-4e^1+4e^2$	$-4e^1+4e^2$	$(-3g^2+6g^3)$ $-(4-4g+g^2+5g^3)e^1$ $+(1-g)^3e^2$	$(-1+6g-11g^2$ $+6g^3)e^1+(4-7g$ $+12g^2-5g^3)e^2$ $-g^3e^3$
$\bar{c}\bar{d}\bar{e}+c^*\bar{d}e^*$	$2e^1-2e^2$	$4e^1-4e^2$	$4e^1-4e^2$	$-g^3+(4-7g$ $+12g^2-5g^3)e^1$ $-(1-6g+11g^2$ $-6g^3)e^2$	$(1-g)^3e^1$ $-(4-4g+g^2+5g^3)e^2$ $+(-3g^2+6g^3)e^3$
abf				$(12g^2-16g^3)$ $+(6g-18g^2$ $+16g^3)e^1$	$(-6g+16g^2$ $-16g^3)e^1$ $+(-10g^2+16g^3)e^2$
$\bar{c}c^*f$				$(-10g^2+16g^3)e^1$ $-(6g-16g^2$ $+16g^3)e^2$	$(6g-18g^2+16g^3)e^2$ $+(12g^2-16g^3)e^3$
$a\bar{C}C^*+b\bar{B}B^*$				$(-3g^2+6g^3)$ $+(g^2-5g^3)e^1$ $+(g^2-g^3)e^2$	$(-3g^2+6g^3)e^1$ $+(4g^2-5g^3)e^2$ $-g^3e^3$
$\bar{c}\bar{B}C+c^*\bar{C}B^*$				$-g^3$ $+(4g^2-5g^3)e^1$ $+(-3g^2+6g^3)e^2$	$(g^2-g^3)e^1$ $+(g^2-5g^3)e^2$ $+(-3g^2+6g^3)e^3$

TABLE U2 (cont.)

$\bar{d}\bar{A}C^* + e\bar{A}B^*$ $+d^*\bar{C}A + e^*\bar{B}A$			$-g^3-(g^2-2g^3)e^1$ $+(g^2-g^3)e^2$	$(g^2-g^3)e^1$ $-(g^2-2g^3)e^2-g^3e^3$
$a\bar{e}\bar{C}+bd\bar{B}$		$g(e^1-e^2)$	$(-3g^2+6g^3)$ $+(g+g^2-5g^3)e^1$ $-g(1-g)^2e^2$	$(-2g+7g^2-6g^3)e^1$ $+(2g-7g^2+5g^3)e^2$ $+g^3e^3$
$\bar{c}\bar{e}\bar{B}+c^*\bar{d}\bar{C}$		$g(e^1-e^2)$	$-g^3$	$g(1-g)^2e^1$ $-(g+g^2-5g^3)e^2$ $+(3g^2-6g^3)e^3$
$ab\bar{F}$		$-2ge^1$	$(6g^2-8g^3)$ $+(g-8g^2+8g^3)e^1$	$(g-8g^2+8g^3)e^1$ $+(6g^2-8g^3)e^2$
$\bar{c}c^*\bar{F}$		$2ge^2$	$(-6g^2+8g^3)$ $-(g-8g^2+8g^3)e^2$	$(-g+8g^2-8g^3)e^2$ $+(-6g^2+8g^3)e^3$
$\bar{d}e\bar{A}$		$2g(-e^1+e^2)$	$-g^3$ $+(g-2g^2+2g^3)e^1$ $-g(1-g)^2e^2$	$g(1-g)^2e^1$ $-(g-2g^2+2g^3)e^2$ $+g^3e^3$
$\bar{B}\bar{C}A^* + \bar{A}B^*C^*$	$g(e^1-e^2)$		$g^3(-1+2e^1-e^2)$	$g^3(e^1-2e^2+e^3)$
$ae^*C^* + bd^*B^*$			$(-3g^2+6g^3)$ $+(g+g^2-5g^3)e^1$ $-g(1-g)^2e^2$	$(-2g+7g^2-6g^3)e^1$ $+(2g-7g^2+5g^3)e^2$ $+g^3e^3$

$\bar{c}d^* C^* + c^* e^* B^*$			$g(e^1-e^2)$	$-g^3$ $-(2g-7g^2+5g^3)e^1$ $+(2g-7g^2+6g^3)e^2$	$g(1-g)^2 e^1$ $-(g+g^2-5g^3)e^2$ $+(3g^2-6g^3)e^3$
abF^*			$-2ge^1$	$(6g^2-8g^3)$ $+(g-8g^2+8g^3)e^1$	$(g-8g^2+8g^3)e^1$ $+(6g^2-8g^3)e^2$
$\bar{c}c^* F^*$			$2ge^2$	$(-6g^2+8g^3)e^1$ $-(g-8g^2+8g^3)e^2$	$(-g+8g^2-8g^3)e^2$ $+(-6g^2+8g^3)e^3$
$d^* e^* A^*$			$2g(-e^1+e^2)$	$-g^3$ $+(g-2g^2+2g^3)e^1$ $-g(1-g)^2 e^2$	$g(1-g)^2 e^1$ $-(g-2g^2+2g^3)e^2$ $+g^3 e^3$
$\bar{d}\bar{e}d^* e^*$	$-2e^1+2e^2$	$-4e^1+4e^2$	$-4e^1+4e^2$	$g^4-(4-12g+17g^2$ $-4g^3+2g^4)e^1$ $+(1-g)^4 e^2$	$-(1-g)^4 e^1$ $+(4-12g+17g^2$ $-4g^3+2g^2)e^2$ $-g^4 e^3$
$(a\bar{e}e^* + \bar{b}\bar{d}d^*)f$	$2e^1$	$4e^1$	$4e^1$	$-2g^4+(4-12g$ $+14g^2-6g^3+g^4)e^1$ $+g^2(1-g)^2 e^2$	$(1-4g+8g^2-8g^3$ $+3g^4)e^1$ $-(3g^2+3g^4)e^2$
$(\bar{c}d^* e + c^* \bar{d}e^*)f$	$-2e^2$	$-4e^2$	$-4e^2$	$(3g^2+3g^4)e^1$ $-(1-4g+8g^2-8g^3$ $+3g^4)e^2$	$-g^2(1-g)^2 e^1$ $-(4-12g+14g^2$ $-6g^3+g^4)e^2+2g^4 e^3$

TABLE U2 (cont.)

abf^2	$-2e^1$	$-4e^1$	$-4e^1$	$4g^4-(4-12g+18g^2$ $-15g^3+3g^4)e^1$ $-g^2(1-g)^2e^2$	$(-1+4g-9g^2$ $+10g^3-5g^4)e^1$ $-(g^2-7g^3-5g^4)e^2$
$\bar{c}c^*f^2$	$2e^2$	$4e^2$	$4e^2$	$(g^2-7g^3-5g^4)e^1$ $+(1-4g+9g^2$ $-10g^3+5g^4)e^2$	$g^2(1-g)^2e^1$ $+(4-12g+18g^2$ $-15g^3+3g^4)e^2$ $-4g^4e^3$
$(a\bar{C}C^*+b\bar{B}B^*)f$				$-2g^4+(3g^2-6g^3$ $+g^4)e^1$ $+g^2(1-g)^2e^2$	$(2g^2-4g^3+3g^4)e^1$ $+(2g^2-4g^3-3g^4)e^2$
$(\bar{c}BC^*+c\,\bar{C}B^*)f$				$-(2g^2-4g^3-3g^4)e^1$ $-(2g^2-4g^3+3g^4)e^2$	$-g^2(1-g)^2e^1$ $-(3g^2-6g^3+g^4)e^2$ $+2g^4e^3$
$\bar{d}d^*\bar{C}C^*+\bar{e}e^*\bar{B}B^*$				$g^4(1-e^1)$	$-2g^2(1-g)^2e^1$ $+(2g^2-4g^3+3g^4)e^2$ $-g^4e^3$
$\bar{d}e^*\bar{B}C^*+d^*\bar{e}\bar{C}B^*$				$g^4-(2g^2-4g^3$ $+3g^4)e^1$ $+2g^2(1-g)^2e^2$	$g^4(e^2-e^3)$
$(ae\bar{C}+bd\bar{B})F^*$ $+(ae^*C+bd^*B)\bar{F}$				$-g^4$ $+(g^2-2g^3+2g^4)e^1$ $-g^2(1-g)^2e^2$	

Operator			
$(\bar{c}\bar{e}B+c^*\bar{d}\bar{C})F^* + (\bar{c}d^*C^*+c^*e^*B^*)\bar{F}$			$g^2(1-g)^2e^1 - (g^2-2g^3+2g^4)e^2 + g^4e^3$
$\bar{d}e\bar{A}F^* + d^*e^*A^*\bar{F} - f^2\bar{A}A^*$			$-g^2e^2$
$ab\bar{F}F^*$		g^2e^1	$-g^2(1-g)^2e^1 + (2g^2-3g^3+g^4)e^2$
$\bar{c}c^*\bar{F}F^*$		$2g^4 - (g^2-g^3+3g^4)e^1 + g^2(1-g)^2e^2$	$-g^2(1-g)^2e^1 + (g^2-g^3+3g^4)e^2 - 2g^4e^3$
$\bar{B}CB^*C^*$		$g^4(1-2e^1+e^2)$ $-(2g^2-3g^3+g^4)e^1 + g^2(1-g)^2e^2$	$-g^4(e^1-2e^2+e^3)$
$(a\bar{e}\bar{C}+b\bar{d}\bar{B})f$	ge^1	$-2g^4 - (2g-8g^2 + 8g^3-2g^4)e^1$	$(g-3g^2+4g^3 -2g^4)e^1 - (3g^2-4g^3-2g^4)e^2$
$(\bar{c}\bar{e}B+c^*\bar{d}\bar{C})f$	ge^2	$(-3g^2+4g^3+2g^4)e^1 + (g-3g^2+4g^3 -2g^4)e^2$	$(-2g+8g^2-8g^3 +2g^4)e^2 - 2g^4e^3$
$\bar{d}e(d^*\bar{C}+e^*\bar{B})$	$g(-e^1-e^2)$	$g^4 + (2g-5g^2 +3g^3-2g^4)e^1 - g(1-g)^3e^2$	$-g(1-g)^3e^1 + (2g-5g^2+3g^3 -2g^4)e^2 + g^4e^3$
$(a\bar{e}e+b\bar{d}d^*)\bar{F}$	ge^1	$-g^4 - (2g-2g^2 +g^3-g^4)e^1$	$g(1-g)^3e^1 + (3g^2-2g^3+g^4)e^2$

TABLE U2 (cont.)

$(\bar{c}d^*\bar{e}+c^*\bar{d}e^*)\bar{F}$	ge^2		$(3g^2-2g^3+g^4)e^1$ $+g(1-g)^3e^2$	$(-2g+2g^2-g^3$ $+g^4)e^2-g^4e^3$
$abf\bar{F}$	$-2ge^1$		$4g^4+(4g-10g^2$ $+10g^3-4g^4)e^1$	$(-2g+6g^2$ $-8g^3+4g^4)e^1$ $-(2g^3+4g^4)e^2$
$\bar{c}c^*f\bar{F}$	$-2ge^2$		$(-2g^3-4g^4)e^1$ $-(2g-6g^2+8g^3$ $-4g^4)e^2$	$(4g-10g^2+10g^3$ $-4g^4)e^2+4g^4e^3$
$(ae^*C^*+bd^*B^*)f$		ge^1	$-2g^4-(2g-8g^2$ $+8g^3-2g^4)e^1$	$(g-3g^2+4g^3$ $-2g^4)e^1-(3g^2$ $-4g^3-2g^4)e^2$
$(\bar{c}d^*C^*+c^*e^*B^*)f$		ge^2	$(-3g^2+4g^3+2g^4)e^1$ $+(g-3g^2$ $+4g^3-2g^4)e^2$	$(-2g+8g^2-8g^3$ $+2g^4)e^2-2g^4e^3$
$d^*e^*(\bar{d}C^*+\bar{e}B^*)$		$g(-e^1-e^2)$	$g^4+(2g-5g^2+3g^3$ $-2g^4)e^1$ $-g(1-g)^3e^2$	$-g(1-g)^3e^1$ $+(2g-5g^2+3g^3$ $-2g^4)e^2+g^4e^3$
$(a\bar{e}e^*+b\bar{d}d^*)F$		ge^1	$-g^4-(2g-2g^2$ $+g^3-g^4)e^1$	$g(1-g)^3e^1$ $+(3g^2-2g^3+g^4)e^2$
$(\bar{c}d^*e+c^*\bar{d}e^*)F$		ge^2	$(3g^2-2g^3+g^4)e^1$ $+g(1-g)^3e^2$	$(-2g+2g^2-g^3$ $+g^4)e^2-g^4e^3$

$ab\bar{f}F^*$		$-2ge^1$	$4g^4+(4g-10g^2+10g^3-4g^4)e^1$	$(-2g+6g^2-8g^3+4g^4)e^1 -(2g^3+4g^4)e^2$
$\bar{c}c^*fF^*$		$-2ge^2$	$(-2g^3-4g^4)e^1 -(2g-6g^2+8g^3-4g^4)e^2$	$(4g-10g^2+10g^3-4g^4)e^2+4g^4e^3$
$(a\bar{C}C^*+b\bar{B}B^*)\times(\bar{F}+F^*)$			$-g^4-(g^3-g^4)e^1$	$(g^3-g^4)e^1+g^4e^2$
$(\bar{c}BC^*+c^*\bar{C}B)\times(\bar{F}+F^*)$			$g^4e^1+(g^3-g^4)e^2$	$(-g^3+g^4)e^2-g^4e^3$
$(\bar{d}C^*+eB^*)\bar{B}\bar{C} +(d^*\bar{C}+e^*\bar{B})B^*C^*$			$g^4+(g^3-2g^4)e^1+(-g^3+g^4)e^2$	$(-g^3+g^4)e^1+(g^3-2g^4)e^2+g^4e^3$
$\bar{d}e\bar{B}\bar{C}$	$2g^2(e^1-e^2)$		$g^4-(g^2-2g^3+2g^4)e^1+g^2(1-g)^2e^2$	$-g^2(1-g)^2e^1+(g^2-2g^3+2g^4)e^2-g^4e^3$
$(ab\bar{F}-ae\bar{C}-b\bar{d}B)\bar{F}$	$2g^2e^1$		$g^4-g^2(1-g)^2e^1$	$-g^2(1-g)^2e^1+g^4e^2$
$(\bar{c}c^*\bar{F}-\bar{c}e\bar{B}-c^*\bar{d}C)\bar{F}$	$-2g^2e^2$		$-g^4e^1+g^2(1-g)^2e^2$	$g^2(1-g)^2e^2-g^4e^3$
$d^*e^*B^*C^*$	$2g^2(e^1-e^2)$		$g^4-(g^2-2g^3+2g^4)e^1+g^2(1-g)^2e^2$	$-g^2(1-g)^2e^1+(g^2-2g^3+2g^4)e^2-g^4e^3$

TABLE U2 (cont.)

$(ab\bar{F}^* - ae^*\bar{C} - \overline{bd}^*\bar{B})\bar{F}$	$2g^2 e^1$			$g^4 - g^2(1-g)^2 e^1$	$-g^2(1-g)^2 e^1 + g^4 e^2$
$(\bar{c}c^*\bar{F}^* - \overline{cd}^*\bar{C} - c^*e\bar{B})\bar{F}$	$-2g^2 e^2$			$-g^4 e^1 + g^2(1-g)^2 e^2$	$g^2(1-g)^2 e^2 - g^4 e^3$
$(a\bar{E}\bar{E}^* + b\bar{D}\bar{D}^*)f$				$(2g^2 - 3g^3 + 2g^4)e^1$	$(-g^3 - 2g^4)e^2$
$(\bar{c}\bar{E}D + c^*\bar{D}\bar{E}^*)f$				$(-g^3 - 2g^4)e^1$	$(2g^2 - 3g^3 + 2g^4)e^2$
$\bar{d}d^*\bar{E}\bar{E} + \bar{e}e^*\bar{D}\bar{D}$				$(-2g^2 + 3g^3 - g^4)e^1$	$(g^3 + g^4)e^2$
$\bar{d}e^*\bar{E}D + d^*\bar{e}\bar{D}E$				$(g^3 + g^4)e^1$	$(-2g^2 + 3g^3 - g^4)e^2$
$\bar{B}\bar{D}C^*E + \bar{C}\bar{E}B^*D - \bar{B}\bar{E}B^*E - \bar{C}\bar{D}C^*D$				$g^4 e^1$	$-g^4 e^2$
$(a\bar{F}+aF^* -\bar{d}\bar{B} -d^*B)\bar{E}\bar{E} + (b\bar{F}+bF^* -e\bar{C} -e^*C)\bar{D}\bar{D}$				$(-g^3 + g^4)e^1$	$g^4 e^2$
$(\bar{c}\bar{F}+\bar{c}F^* -\bar{d}\bar{C} -e^*B)\bar{E}D + (c\bar{F}+c^*F^* -d^*C -e\bar{B})\bar{D}E$				$-g^4 e^1$	$(g^3 - g^4)e^2$
$(\bar{A}\bar{F}-\bar{B}\bar{C})\bar{D}E + (A^*F^* -B^*C^*)D^*E$				$g^3 e^1$	$-g^3 e^2$
$\bar{D}\bar{E}D^*E$				$-g^4 e^1$	$g^4 e^2$

TABLE U3

$$H_U\begin{pmatrix}2(10) & 2(10)\\4(00) & 4(00)\end{pmatrix} \text{ for } {}^8\text{Be} + {}^{16}\text{O}$$

(00) - tensors: $\quad a = (K_1 \cdot K_1^*)$, $\quad \bar{b} = (K_1^* \cdot \bar{\mathcal{K}})$, $\quad b^* = (\bar{K}_1 \cdot \mathcal{K}^*)$, $\quad c = (\bar{\mathcal{K}} \cdot \mathcal{K}^*)$.

(20) - tensors: $\quad \bar{A} = (\bar{K}_1 \cdot \bar{\mathcal{K}})$, $\quad \bar{B} = (\bar{\mathcal{K}} \cdot \bar{\mathcal{K}})$.

(02) - tensors: $\quad A^* = (K_1^* \cdot \mathcal{K}^*)$, $\quad B^* = (\mathcal{K}^* \cdot \mathcal{K}^*)$.

	$(1-2g)$	$(1-2g)e\phi(1,0)$	$(1-2g)e\phi(0,1)$	$e\phi(1,1)$	$\cdot e\phi(1,-1)$
a	$7-6e^1+3e^2$	$-8e^1+4e^2$	$-8e^1+4e^2$	$(8-10g+5g^2)$ $-(6-6g+5g^2)e^1$	$(-2-2g-5g^2)e^1$ $+(4-2g+5g^2)e^2$
$\bar{b}b^*$	$3e^1-3e^2$	$4e^1-4e^2$	$4e^1-4e^2$	$(4g^2-7g^3)$ $+(3-g-3g^2+7g^3)e^1$	$(1-7g+13g^2$ $-7g^3)e^1$ $-(4-8g+14g^2$ $-7g^3)e^2$
ac	$-3e^1+3e^2$	$-4e^1+4e^2$	$-4e^1+4e^2$	$(4g^2-2g^3)$ $-(3-7g+12g^2$ $-2g^3)e^1$	$(-1+g-2g^3)e^1$ $+(4-8g+8g^2$ $+2g^3)e^2$
$\bar{A}A^*$				$(4g^2-7g^3)$ $-(3g^2-7g^3)e^1$	$(3g^2-7g^3)e^1$ $-(4g^2-7g^3)e^2$

TABLE U3 (cont.)

$\bar{b}\bar{A}$		$2g(-e^1+e^2)$		$(4g^2-7g^3)(1-e^1)$	$(2g-8g^2+7g^3)$ $\times(e^1-e^2)$
$a\bar{B}$		$-2ge^2$		$(2g^2-g^3)$ $+(g-4g^2+g^3)e^1$	$(-g+g^3)e^1$ $+(2g-2g^2-g^3)e^2$
b^*A^*			$2g(-e^1+e^2)$	$(4g^2-7g^3)(1-e^1)$	$(2g-8g^2+7g^3)$ $\times(e^1-e^2)$
aB^*			$-2ge^2$	$(2g^2-g^3)$ $+(g-4g^2+g^3)e^1$	$(-g+g^3)e^1$ $+(2g-2g^2-g^3)e^2$
$\bar{b}b^*c$	$-3e^1$	$-4e^1$	$-4e^1$	$2g^4$ $-(3-10g+13g^2$ $-8g^3+2g^4)e^1$	$(-1+4g-7g^2$ $+6g^3-2g^4)e^1$ $+(2g^3+2g^4)e^2$
$ac^2-a\bar{B}B^*$				$(-g^2+g^3)e^1$	g^3e^2
$\bar{b}\bar{A}B^*+b^*\bar{B}A^*$				$g^4-(g^2+g^4)e^1$	$-g^2(1-g)^2e^1$ $+(2g^2-2g^3+g^4)e^2$
$c\bar{A}A^*$				$2g^4$ $-(3g^2-8g^3+2g^4)e^1$	$(-g^2+2g^3-2g^4)e^1$ $-(4g^2-6g^3-2g^4)e^2$
$\bar{b}b^*(\bar{B}+B^*)$				g^4 $+(g-g^2+g^3-g^4)e^1$	$-g(1-g)^3e^1$ $-(2g-2g^3+g^4)e^2$

$\bar{A}A^*(\bar{B}+B^*)$			$g^4+(g^3-g^4)e^1$	$(-g^3+g^4)e^1-g^4e^2$
$\bar{b}c\bar{A}$	$-2ge^1$		$2g^4$ $+(3g-9g^2$ $+8g^3-2g^4)e^1$	$(-g+3g^2$ $-4g^3+2g^4)e^1$ $+(2g^2-4g^3-2g^4)e^2$
b^*cA^*		$-2ge^1$	$2g^4$ $+(3g-9g^2$ $+8g^3-2g^4)e^1$	$(-g+3g^2$ $-4g^3+2g^4)e^1$ $+(2g^2-4g^3-2g^4)e^2$
$\bar{b}\bar{A}B$	$2g^2e^1$		$g^4-g^2(1-g)^2e^1$	$-g^2(1-g)^2e^1+g^4e^2$
b^*A^*B		$2g^2e^1$	$g^4-g^2(1-g)^2e^1$	$-g^2(1-g)^2e^1+g^4e^2$
$\bar{b}b^*(c^2-\bar{B}B^*)$			$-g^2(1-g)^2e^1$	g^4e^2
$(c^2-\bar{B}B^*)\bar{A}A^*$			$-g^4e^1$	g^4e^2
$(c^2-\bar{B}B^*)$ $\times(\bar{b}\bar{A}+b^*A^*)$			$(g^3-g^4)e^1$	$-g^4e^2$

TABLE U4

$$H_U\left[\begin{matrix}3\,(01)\\3\,(01)\end{matrix}\middle|2\,(10)\middle|4\,(00)\right]\ \text{for}\ ^{12}C + {}^{12}C\ \middle|\ ^{8}Be + {}^{16}O$$

(00) – tensors:
$a = ([\bar{K}_{12}\times\bar{K}_{34}]\cdot K_1^*)$, $b = (\bar{K}_{12}\cdot\bar{\mathcal{K}})$, $c = (\bar{K}_{34}\cdot\bar{\mathcal{K}})$, $d = (K_1^*\cdot\bar{\mathcal{K}})$, $e = ([\bar{K}_{12}\times\bar{K}_{34}]\cdot\mathcal{K}^*)$,
$f = ([\bar{K}_{12}\times K_1^*]\cdot\mathcal{K}^*)$, $h = ([\bar{K}_{34}\times K_1^*]\cdot\mathcal{K}^*)$, $i = (\bar{\mathcal{K}}\cdot\mathcal{K}^*)$.

(20) – tensors:
$A = ([\bar{K}_{12}\times\bar{K}_{34}]\cdot\bar{\mathcal{K}})$, $B = ([\bar{K}_{12}\times K_1^*]\cdot\bar{\mathcal{K}})$, $C = ([\bar{K}_{34}\times K_1^*]\cdot\bar{\mathcal{K}})$, $D = (\bar{K}_{12}\cdot[\bar{\mathcal{K}}\times\mathcal{K}^*])$,
$E = (\bar{K}_{34}\cdot[\bar{\mathcal{K}}\times\mathcal{K}^*])$, $F = (K_1^*\cdot[\bar{\mathcal{K}}\times\mathcal{K}^*])$, $G = (\bar{\mathcal{K}}\cdot\bar{\mathcal{K}})$.

(02) – tensors:
$P = (\bar{K}_{12}\cdot\bar{K}_{34})$, $Q = (\bar{K}_{12}\cdot K_1^*)$, $R = (\bar{K}_{34}\cdot K_1^*)$, $S = (\bar{K}_{12}\cdot\mathcal{K}^*)$, $T = (\bar{K}_{34}\cdot\mathcal{K}^*)$,
$U = (K_1^*\cdot\mathcal{K}^*)$, $V = (\mathcal{K}^*\cdot\mathcal{K}^*)$.

	$(1-2g)$	$(1-2g)e\phi(1,0)$	$(1-2g)e\phi(0,1)$	$e\phi(1,1)$	$e\phi(1,-1)$
a	$4(1-e^1+e^2)$	$3-10e^1+3e^2$	$2-8e^1+2e^2$	$(6-6g+5g^2)$ $-(4-2g+5g^2)e^1$	$(-4+2g-5g^2)e^1$ $+(6-6g+5g^2)e^2$
bh	$-2e^1+4e^2$	$-5e^1+3e^2$	$-4e^1+2e^2$	$(-3g^2+7g^3)$ $-(2+g-2g^2+7g^3)e^1$	$(-2+9g-14g^2$ $+7g^3)e^1+(6-12g$ $+15g^2-7g^3)e^2$
cf	$4-2e^1$	$3-5e^1$	$2-4e^1$	$(6-12g+15g^2-7g^3)$ $-(2-9g+14g^2$ $-7g^3)e^1$	$(-2-g+2g^2-7g^3)e^1$ $+(-3g^2+7g^3)e^2$

	$-2e^1$	$-5e^1$	$-4e^1$		
dei	$-2e^1$	$-5e^1$	$-4e^1$	$(2g^3+6g^4)$ $-(2-7g+14g^2$ $-14g^3+6g^4)e^1$	$(-2+7g-14g^2$ $+14g^3-6g^4)e^1$ $+(2g^3+6g^4)e^2$
bEU				$(-g^3-g^4)$ $+(2g^3+g^4)e^1$	$(2g^2-3g^3+g^4)e^1$ $-(3g^2-4g^3+g^4)e^2$
bFT				$(g^2-g^3)(2-e^1)$	$g^3(-2e^1+e^2)$
cDU				$(3g^2-4g^3+g^4)$ $-(2g^2-3g^3+g^4)e^1$	$(-2g^3-g^4)e^1$ $+(g^3+g^4)e^2$
cFS				$g^3(-1+2e^1)$	$(g^2-g^3)(e^1-2e^2)$
dDT				$(-2g^2+3g^3-g^4)$ $+g^2(1-g)^2e^1$	$(g^3+g^4)e^1-g^4e^2$
dES				$g^4-(g^3+g^4)e^1$	$-g^2(1-g)^2e^1$ $+(2g^2-3g^3+g^4)e^2$
iDR				$(2g^2-3g^3-3g^4)$ $+(3g^2-5g^3+3g^4)e^1$	$(3g^2-7g^3+3g^4)e^1$ $+(-g^3-3g^4)e^2$
iEQ				(g^3+3g^4) $-(3g^2-7g^3+3g^4)e^1$	$(-3g^2+5g^3-3g^4)e^1$ $-(2g^2-3g^3-3g^4)e^2$

TABLE U4 (cont.)

ai			$(11g^2-16g^3) + (6g-17g^2+16g^3)e^1$	$(-6g+17g^2-16g^3)e^1 + (-11g^2+16g^3)e^2$
BT			$(5g^2-7g^3) - (4g^2-7g^3)e^1$	$(2g^2-7g^3)e^1 - (3g^2-7g^3)e^2$
CS			$(-3g^2+7g^3) + (2g^2-7g^3)e^1$	$(-4g^2+7g^3)e^1 + (5g^2-7g^3)e^2$
bC	$g(e^1-3e^2)$		$(3g^2-g^3) + (g-5g^2+g^3)e^1$	$(-2g+g^2+g^3)e^1 + (3g-3g^2-g^3)e^2$
cB	$g(3-e^1)$		$(-3g+3g^2+g^3) + (2g-g^2-g^3)e^1$	$(-g+5g^2-g^3)e^1 + (-3g^2+g^3)e^2$
dA	$-2ge^1$		$(6g^2-8g^3) + (g-8g^2+8g^3)e^1$	$(g-8g^2+8g^3)e^1 + (6g^2-8g^3)e^2$
eU		$-2ge^1$	$(6g^2-8g^3) + (g-8g^2+8g^3)e^1$	$(g-8g^2+8g^3)e^1 + (6g^2-8g^3)e^2$
fT		$2g$	$(-2g+2g^2+g^3) + (g-g^3)e^1$	$(-g+4g^2-g^3)e^1 + (-2g^2+g^3)e^2$
hS		$-2ge^2$	$(2g^2-g^3) + (g-4g^2+g^3)e^1$	$(-g+g^3)e^1 + (2g-2g^2-g^3)e^2$

iFP					$(-2g^2+3g^3)+(g^2-3g^3)e^1$	$(-g^2+3g^3)e^1+(2g^2-3g^3)e^2$
bdE			$-ge^1$		$(-g^3-2g^4)+(-g-g^3+2g^4)e^1$	$(2g-5g^2+5g^3-2g^4)e^1+(3g^2-3g^3+2g^4)e^2$
cdD			$-ge^1$		$(3g^2-3g^3+2g^4)+(2g-5g^2+5g^3-2g^4)e^1$	$(-g-g^3+2g^4)e^1+(-g^3-2g^4)e^2$
diA					$(g^3+4g^4)+(3g-8g^2+9g^3-4g^4)e^1$	$(-3g+8g^2-9g^3+4g^4)e^1+(-g^3-4g^4)e^2$
(bc-GP)F					$g^3(-1+e^1)$	$g^3(e^1-e^2)$
(DR-EQ)G					$-g^4+(-g^3+g^4)e^1$	$(g^3-g^4)e^1+g^4e^2$
deV					$(2g^2-2g^3+g^4)+g(1-g)^3e^1$	$-g(1-g)^3e^1-(2g^2-2g^3+g^4)e^2$
eiU					$(-2g+3g^2+2g^4)+(2g-5g^2+5g^3-2g^4)e^1$	$(-2g+5g^2-5g^3+2g^4)e^1+(2g^2-3g^3-2g^4)e^2$

TABLE U4 (cont.)

			$(2g^2-2g^3)(1-e^1)$	$(2g^2-2g^3)(-e^1+e^2)$
h(bv-iS)				
f(cV-iT)				$(2g^2-2g^3)(-e^1+e^2)$
AUV			$g^4+(g^3-g^4)e^1$	$(-g^3+g^4)e^1-g^4e^2$
dAG	$2g^2e^1$		$g^4-g^2(1-g)^2e^1$	$-g^2(1-g)^2e^1+g^4e^2$
eUV		$2g^2e^1$	$g^4-g^2(1-g)^2e^1$	$-g^2(1-g)^2e^1+g^4e^2$
dD(cV-iT)			g^4	$-g^2(1-g)^2e^1$
dE(bV-iS)			$-g^2(1-g)^2e^1$	g^4e^2
(bi-GS)EU			$-g^4e^1$	g^4e^2
(ci-GT)DU			g^4	$-g^4e^1$
(ci-GT)dD +(cV-iT)DU			g^4	$(-g^3+g^4)e^1$
(bi-GS)dE +(bV-iS)EU			$(g^3-g^4)e^1$	$-g^4e^2$

References

1) K. Wildermuth and Y. C. Tang, A unified theory of the nucleus (Vieweg, Braunschweig, 1977);
   P. Kramer, G John and D. Schenzle, Group theory and the interaction of composite nucleon systems (Vieweg, Braunschweig, 1981);
   K. Ikeda *et al.*, Progr. Theor. Phys. Suppl. **68** (1980) 1
2) K. Wildermuth and W. McClure, Cluster representations of nuclei (Springer Verlag, Berlin, 1966)
3) H. H. Hackenbroich, Symp. on present status and novel developments in the nuclear many-body problem, Rome (editrice Bologna, 1973)
4) W. Zahn, Nucl. Phys. **A269** (1976) 138;
   H. Stöwe and W. Zahn, Nucl. Phys. **A286** (1977) 89; **A289** (1977) 317; Z. Phys. **A286** (1978) 173; J. of Phys. **G4** (1978) 1423
5) Y. C. Tang, M. LeMere and D. R. Thompson, Phys. Reports **47** (1978) 168;
   Y. C. Tang, Microscopic description of the nuclear cluster theory, Lecture notes in physics (Springer, Berlin, 1981)
6) H. Friedrich, K. Langanke and A. Weiguny, Phys. Lett. **63B** (1976) 125;
   H. Friedrich and K. Langanke, Nucl. Phys. **A252** (1975) 47
7) A. Tohsaki-Suzuki, Progr. Theor. Phys. Suppl. **62** (1977) 191
8) H. Horiuchi, Progr. Theor. Phys. Suppl. **62** (1977) 90
9) D. M. Brink, Int. School of Physics Enrico Fermi, Course **36** (1965) 247
10) Y. Fujiwara and H. Horiuchi, Progr. Theor. Phys. **63** (1980) 895; **65** (1981) 1632, 1901
11) K. T. Hecht, E. J. Reske, T. H. Seligman and W. Zahn, Nucl. Phys. **A356** (1981) 146
12) K. T. Hecht and W. Zahn, Nucl. Phys. **A318** (1979) 1
13) Y. Suzuki, Progr. Theor. Phys. **55** (1976) 1751; **56** (1976) 111
14) J. P. Draayer and Y. Akiyama, J. Math. Phys. **14** (1973) 1904
15) Tables of Interaction Kernels for $A = 4n$ binary cluster systems, Y. Suzuki, E. J. Reske and K. T. Hecht, University of Michigan Report Phy 37 (1981)