THE TWO-LOOP AXIAL ANOMALY IN $N = 1$ SUPERSYMMETRIC YANG–MILLS THEORY

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We calculate the two loop correction to the divergence of the axial current in $N = 1$ supersymmetric Yang–Mills theory. We find that, if we use supersymmetric dimensional regularization, the anomaly is proportional to the β-function, a result we interpret as evidence that the supermultiplet nature of the trace, axial and supersymmetry anomalies persists at the two-loop level. Thus it appears that the Adler–Bardeen theorem is not valid in supersymmetric theories.

In 1975 Ferrara and Zumino [1] pointed out that in supersymmetric theories the axial current J^5_μ, the improved energy momentum tensor θ^ν_μ and the improved supersymmetry current S_μ can be identified with the members of a single supermultiplet. At the quantum level one is therefore led to expect that the anomalies $\partial \cdot J^5$, θ^μ_μ and $\gamma \cdot S$ are also related by supersymmetry. This has been verified at the one-loop order in a number of cases [2]. The purpose of this paper is to investigate the two-loop correction to $\partial \cdot J^5$ in the case of $N = 1$ supersymmetric Yang–Mills theory. The Adler–Bardeen (AB) theorem [3], as usually stated, asserts that $\partial \cdot J^5$ is given by $cF_{\mu\nu}\tilde{F}^{\mu\nu}$ where $c \equiv kg^2$ (where k is a pure number) and there are *no corrections* to c of order g^4, g^6 etc. On the other hand it is well known that the trace of the energy–momentum tensor θ^μ_μ is proportional to the β function which for the theory under consideration has a non-vanishing two-loop correction [4]. One is therefore forced to conclude that either the multiplet pattern of the currents somehow ceases to hold at higher orders or that the AB theorem is not valid for supersymmetric theories. In the latter case, we would expect the following equation to be true to all orders:

$$\partial_\mu J^{\mu 5} = \partial\mu(\tfrac{1}{2}\bar{\psi}\gamma^\mu\gamma^5\psi) = -\tfrac{1}{6}[\beta(g)/g]F_{\rho\sigma}\tilde{F}^{\rho\sigma}\,, \quad (1)$$

where $\tilde{F}_{\rho\sigma} = \epsilon_{\rho\sigma\delta\tau}F^{\delta\tau}$. The factor of $\frac{1}{2}$ in the definition of the axial current is there because the fermions are Majorana.

We investigate the validity of (1) by calculating the Greens functions

$$\langle 0|T(\partial \cdot J^5(x)A_\mu(y)A_\nu(z))|0\rangle$$

and

$$\langle 0|T(F\tilde{F}(x)A_\mu(y)A_\nu(z))|0\rangle\,.$$

For generality we will perform the calculation for Dirac fermions ψ transforming according to an arbitrary representation of the gauge group, and determine r, which we define by the equation

$$\partial_\mu J^{\mu 5} = \partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi) = rF\tilde{F}\,. \quad (2)$$

The special case of Majorana fermions transforming according to the adjoint representation is then easily extracted by judicious choice of group theory factors.

Thus if we write in momentum space

$$\langle 0|\partial \cdot J^5 A_\mu(P_1)A_\nu(P_2)|0\rangle$$
$$= [Ag^2/16\pi^2 + Bg^4/(16\pi^2)^2 + \ldots]4\epsilon_{\mu\nu\rho\sigma}P_1^\rho P_2^\sigma \quad (3)$$

and

$$\langle 0|F\tilde{F}A_\mu(P_1)A_\nu(P_2)|0\rangle$$
$$= 2(1 + Cg^2/16\pi^2 + \ldots)4\epsilon_{\mu\nu\rho\sigma}P_1^\rho P_2^\sigma\,, \quad (4)$$

our purpose is to determine A, B and C in eqs. (3) and (4); calculate r from eq. (2), and test the validity of eq. (1) in the supersymmetric case.

Consider eq. (3). It is clear that to calculate A and B we can differentiate with respect to P_1^α and then set $P_1 = -P_2 = P$. This procedure does not introduce any infrared divergences (as long $P^2 \neq 0$). Thus the calculation of A and B becomes a calculation of the matrix elements of J_α^5 between gluon states at zero momentum transfer. At the one-loop level we have compared this procedure with the calculation (for both massive and massless fermions) for general P_1 and P_2 and verified that the same result is obtained [‡1]. At the two-loop level the fact that the Feynman integrals depend on only one external momentum simplifies the calculation considerably. The same operation on eq. (4) leads to a straightforward calculation of C.

We perform the calculation using two regularization methods: conventional dimensional regularization (CDR) and supersymmetric dimensional regularization (SDR) [6]. The SDR method consists of dimensionally *reducing* rather than dimensionally continuing to n dimensions. For an exposition of SDR in the component field formalism with examples see ref. [7]. The method appears to be consistent with supersymmetry at low orders; it has recently been pointed out, however, that the method will give rise to anomalous terms at sufficiently high orders [8]. We do not, however, expect these considerations to affect the reliability of our calculations.

In the earliest applications of dimensional regularization [9] it was recognized that the accomodation of γ^5 presents difficulties. For our purposes, however, we can avoid the necessity for a general definition of γ^5 as follows: we write all fermion traces starting with the $\gamma^5\gamma_\alpha$ vertex and perform the momentum integrations and Dirac algebra using no property of γ^5 other than the n-dimensional identity $\mathrm{tr}(\gamma^5\gamma_\alpha\gamma_\beta) = 0$. This enables one to reduce initially (apparently) divergent expressions to ones finite as $n \to 4$, after which limit the Dirac trace can be performed with impunity [‡2]. The advantage of this prescription is that it gives *automatic* vector current conservation (we have verified this through two loops in QED). The usual (indeed *inevitable*) ambiguity in the position of the anomaly corresponds to the arbitrariness in the choice of initial position for the γ^5.

[‡1] For further details, see ref. [5].

[‡2] The one-loop calculation has been frequently performed in this spirit: see ref. [10].

The results for A, B, C are:
In CDR

$$A = 2T(R)\,,$$
$$B = [(12 - 4\gamma)C_2(G) + 8C_2(R)]T(R)\,,$$
$$C = (6 - 2\gamma)C_2(G)\,. \tag{5}$$

In SDR

$$A = 2T(R)\,,$$
$$B = [(12 - 4\gamma)C_2(G) + 4C_2(R)]\,T(R)\,,$$
$$C = (6 - 2\gamma)C_2(G)\,, \tag{6}$$

where we use the conventional definitions

$$T(R)\delta^{ab} = \mathrm{tr}\, R^a R^b\,,$$
$$C_2(R)I = R^a R^a\,,$$
$$C2(G)\delta^{ab} = f^{acd} f^{bcd}\,,$$

for fermions transforming under a representation R of the gauge group, and $\gamma = \gamma_E + \ln P^2/\mu^2$ where γ_E is Euler's constant, and μ is the renormalization scale.

From (5), (6) we obtain

$$r_{\mathrm{CDR}} = T(R)g^2/16\pi^2 + 4C_2(R)T(R)g^4/(16\pi^2)^2\,, \tag{7}$$
$$r_{\mathrm{SDR}} = T(R)g^2/16\pi^2 + 2C_2(R)T(R)g^4/(16\pi^2)^2\,, \tag{8}$$

$N = 1$ supersymmetric Yang–Mills theory corresponds to $C_2(R) = 2T(R) = C_2(G)$. [$2T(R)$ because of the Majorana nature of the fermions.] Eq. (8) then yields

$$r_{\mathrm{SDR}} = \tfrac{1}{2}C_2(G)g^2/16\pi^2 + [C_2(G)^2/(16\pi^2)^2]g^4\,. \tag{9}$$

The β function for this theory is [4]

$$\beta(g) = -(3g^3/16\pi^2)C_2(G) - [6g^5/(16\pi^2)^2]\,[C_2(G)]^2\,. \tag{10}$$

So we see that, in accordance with eq. (1),

$$r_{\mathrm{SDR}} = -\tfrac{1}{6}\beta(g)/g\,. \tag{11}$$

Note that from eq. (7) it is clear that CDR gives a result for r which does not obey eq. (1).

Eq. (11) is our main result. We interpret it as evidence that the supermultiplet nature of the axial, supersymmetry and trace anomalies is preserved at the two-loop level. (The situation in higher orders is unclear in view of the apparent inherent inconsistency of SDR [8].) Of course to fully substantiate this conclusion we would have to perform the analogous calculations for the supersymmetry and trace anomalies. With regard to the supersymmetry anomaly, the following remarks are in order:

The supersymmetry current in the model under consideration is

$$S^\mu = \sigma^{\alpha\beta}\gamma^\mu \psi F_{\alpha\beta} .$$

Interpreting μ as an n-dimensional index (which is the natural procedure in view of gauge invariance) Nicolai and Townsend [10] showed (at the one-loop level) that using SDR,

$$\partial_\mu S^\mu = 0 , \quad \gamma \cdot S = (3/4\pi^2)\sigma \cdot F\psi .$$

It is this result that we would wish to extend to the next order. (Of course it is possible to regulate the theory in such a way [11] that $\gamma \cdot S = 0$ and the anomaly resides in $\partial_\mu S^\mu$, just as one can regulate the axial anomaly in such a way that it is the vector current which is not conserved. It is more natural, however, to choose (or impose) a prescription such that the supersymmetry current is conserved, if this can be done consistently.)

With regard to the Adler–Bardeen theorem, note that from eqs. (7) and (8) the theorem apparently fails even for the case of QED (which we can recover by setting $C_2(G) = 0$, $C_2(R) = T(R) = 1$), for both CDR and SDR. In the case of CDR, we can, however, recover the AB theorem by a redefinition of the subtraction constant at the $\bar{\psi}\gamma_\mu\gamma^5\psi$ vertex, while preserving vector current conservation. Previously Bardeen [12] considered a modification of CDR with the same purpose. Thus in non-supersymmetric theories one can choose a gauge invariant regularization procedure such that the AB theorem is valid. (For a recent discussion of the status of the AB theorem in non-abelian gauge theories, see ref. [12].) It is presumably the case, however, that the subset of regularization procedures which respect the AB theorem violate supersymmetry. What we require, ideally, is a superfield formulation of the anomalies and an unambiguous supersymmetric regulator. Piguet and Sibold [14] have recently made some progress in this direction.

A necessary condition for the renormalizability of a gauge theory is the absence of anomalies. If the Adler–Bardeen theorem is valid, it is sufficient to impose the cancellation of anomalies at one loop. If the AB theorem fails in supersymmetric theories, as suggested above, a problem is raised: must one impose an infinite set of anomaly cancellation conditions? If that is the case the class of physically consistent supersymmetric theories may be reduced considerably; perhaps only to left–right symmetric ones. For further discussions of the implications of our result and the details of our calculation, the reader is referred to a forthcoming publication [5].

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Note added in proof. The redefinition of the subtraction constant at the $\Gamma^5_\mu = (\bar{\psi}\gamma_\mu\gamma^5\psi)$ vertex (to give the AB theorem) referred to above involves subtracting an additional finite piece as well as the pole term. In fact this additional finite subtraction is precisely that which is obtained if one uses the non-anticommuting γ^5 of 't Hooft and Veltman [9] and imposes the identity $\Gamma^5_\mu = \Gamma_\mu\gamma^5$ for the renormalized axial and vector vertice: (true at this order). Thus, in CDR, we have verified that with this prescription for γ^5 the AB theorem holds at this order in both abelian adn non-abelian theories. In SDR, with this procedure, one also obtaines (at least formally) the AB theorem. (The significance of this result is unclear, however, since SDR involves continuing to $n < 4$ while the non anti-commuting γ^5 requires $n > 4$, so it is not obvious that this procedure is consistent with supersymmetry.) It seems however that the result (1) that $\partial_\mu J^5_\mu$ is proportional to $\beta(g)$ can be obtained only at the expense of a breakdown of the chiral identity $\Gamma^5_\mu = \Gamma_\mu\gamma^5$.

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