Topology and its Applications 13 (1982) 137–154
North-Holland Publishing Company

6-DIMENSIONAL MANIFOLDS WITH EFFECTIVE T^4-ACTIONS

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Received 21 October 1980
Revised 5 May 1981

Suppose the four dimensional torus T^4 acts effectively on a 6-manifold M so that the orbit space M^* is a closed 2-disk, and there exist no exceptional orbits, and the isotropy groups span T^4. Then the fundamental group of M is a finite abelian group with at most two generators. In this paper, we obtain a homology classification of manifolds of this type under an additional hypothesis that one of the two generators is trivial. We then use this result to obtain a complete classification of simply connected 6-manifolds supporting effective T^4-actions.

AMS (MOS) Subj. Class.: Primary 57S15, 57S25

manifold	group action	orbit space
isotropy group	equivariantly diffeomorphic	trilinear form
Stiefel-Whitney class	Pontrjagin class	

1. Introduction

We shall be concerned throughout with closed orientable smooth manifolds M of dimension 6 supporting smooth effective T^4-actions.

In [5], we showed that the classification theorem for simply connected 6-manifolds with effective T^4-actions in [3] is not valid. The main purpose of this paper is to prove the following.

Theorem 1.1. *Suppose T^4 acts on a simply connected 6-manifold M so that the number of orbits of type T^2 is k. Then we have*

$$M \approx \#(k-4)(S^2\times S^4)\#(k-3)(S^3\times S^3), \quad \text{if } w_2(M)=0,$$
$$M \approx (S^2\tilde{\times} S^4)\#(k-5)(S^2\times S^4)\#(k-3)(S^3\times S^3), \quad \text{if } w_2(M)\neq 0,$$

where $w_2(M)$ is the second Stiefel–Whitney class and $S^2\tilde{\times}S^4$ is the non-trivial S^4-bundle over S^2.

This paper is a continuation of [5] in which we studied non-simply connected 5-manifolds with effective T^3-actions under some restrictions and obtained a complete classification of simply connected 5-manifolds with effective T^3-actions For definitions and terminology, we refer to [5]. Here we restate the notations which are frequently used in this paper: for relatively prime integers a, b, c, and d, $G(a,b,c,d)$ means the circle subgroup of T^4 defined by $G(a,b,c,d)=\{(at,bt,ct,dt)\,|\,0\le t<1\}$ (mod $\mathbb{Z}^4$). Suppose M_1 and M_2 are two G-manifolds for a compact Lie group G. By $M_1=M_2$, we mean that there exists a weakly equivariant diffeomorphism $f:M_1\to M_2$ (that is, there is $\alpha\in \mathrm{Aut}(G)$ such that $f(g\cdot x)=\alpha(g)\cdot f(x)$ for $g\in G$, $x\in M_1$). We write $M_1\approx M_2$ when they are just topologicaly equivalent.

Unless otherwise stated, we shall always use the integers $\mathbb{Z}$ as coefficients in (*co*)homology. The first Pontrjagin class of M will be denoted by $P_1(M)$. Finally, we have the trilinear form

$$\mu: H^2(M)\times H^2(M)\times H^2(M)\to \mathbb{Z}$$

defined by $\mu(\alpha,\beta,\gamma)=\alpha\cup\beta\cup\gamma\in H^6(M)\approx\mathbb{Z}$, where the isomorphism $H^6(M)\to\mathbb{Z}$ is defined by the orientation on M.

2. Preliminary lemmas

Orlik–Raymond [6] proved for a certain 4-manifold M^4 with an effective T^2-action that the orbit map $M^4\to(M^4)^*$ has a cross-section.

By applying a technique similar to that used in [6], we can prove the following.

Lemma 2.1 [5]. (Cross-sectioning Theorem). *The orbit map $M^6\to M^6/T^4$ has a cross-section, provided that the orbit space is a 2-manifold with non-empty boundary and there exist no exceptional orbits.*

Lemma 2.2 [5]. (Equivalent Classification Theorem). *Suppose the orbit maps $M_1\to M_1^*$ and $M_2\to M_2^*$ have cross sections. Then M_1 is weakly equivariantly diffeomorphic to M_2 if and only if there exists a weight-preserving diffeomorphism from M_1^* onto M_2^*.*

Example 2.3 [6]. Suppose T^2 acts on a closed orientable 3-manifold X so that the orbit space X^* is a closed interval with both boundary points corresponding to non-principal orbits whose stability groups are $G(m,n)$ and $G(m',n')$ (that is, X^* is $G(m,n)\bullet\!\!\overset{e}{—}\!\!\bullet\, G(m',n')$). Let $q:X\to X^*=[0,1]$ be the orbit map. Then $q^{-1}([0,\frac{1}{2}])$ and $q^{-1}([\frac{1}{2},1])$ are solid tori. Thus X is the space constructed by gluing two solid tori along $q^{-1}(\frac{1}{2})\approx T^2$. Hence X is a lens space $L(p,q)$ (Note: It is S^3 if $p=\pm1$ and is $S^2\times S^1$ if $p=0$). Furthermore, by checking the gluing map, we can see that p is

$$\det\begin{pmatrix} m & m' \\ n & n' \end{pmatrix},\quad \text{and}\quad mq\equiv m' \pmod p \quad\text{and}\quad nq\equiv n' \pmod p.$$

Example 2.4. Let N be a 5-manifold supporting an effective smooth T^4-action such that the orbit space N^* is as shown below

$$G(a,b,c,d) \bullet\!\!-\!\!\bullet\, G(a',b',c',d').$$

Then $N = L(p,q)\times T^2$ for some lens space $L(p,q)$. In fact, we can choose an automorphism α for T^4 such that $\alpha(G(a,b,c,d)) = G(m,n,0,0)$ and $\alpha(G(a',b',c',d')) = G(m',n',0,0)$. Hence, we may assume N^* is as shown below

$$G(m,n,0,0) \bullet\!\!-\!\!\bullet\, G(m',n',0,0).$$

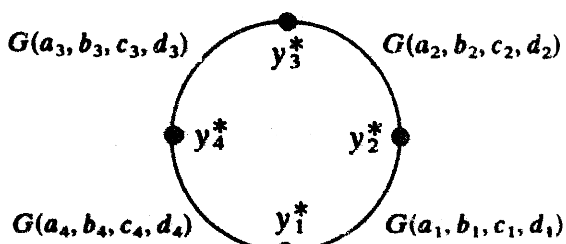
If X is a 3-manifold supporting a T^2-action such that the orbit space X^* is $G(m,n) \bullet\!\!-\!\!\bullet\, G(m',n')$, then by Example 2.3, X is a lens space. Define a T^4-action on $X\times T^2$ by the product of the T^2-action on X and the multiplication of T^2 on itself. Then the orbit space $(X\times T^2)^*$ with respect to the product action is

$$G(m,n,0,0) \bullet\!\!-\!\!\bullet\, G(m',n',0,0).$$

Since the orbit maps $N\to N^*$ and $(X\times T^2)\to(X\times T^2)^*$ have cross sections, Lemma 2.2 gives rise to $N = X\times T^2$.

If the isotropy groups span T^k for some $k<4$ (say T^3), then it follows from an argument similar to [5, Remark 1.7] that M is equivariantly diffeomorphic to $T^1\times N$ for some 5-manifold N with an effective T^3-action. So in this paper we assume that the isotropy groups span T^4 unless otherwise stated. This will then force the number of orbits of type T^2 to be at least four.

Suppose T^4 acts effectively on a 6-manifold M so that the orbit space M^* is as shown below,



and let

$$\Delta = \begin{pmatrix} a_1 & a_1 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_5 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}.$$

Then we have the following.

Theorem 2.5. (1) *If* $\det\Delta = \pm 1$, *then M is equivariantly diffeomorphic to* $S^3\times S^3$.

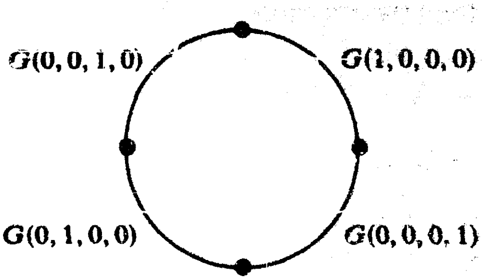
(2) *If the subgroup spanned by the circle isotropy groups* $G(a_1,b_1,c_1,d_1)$ *and* $G(a_3,b_3,c_3,d_3)$ *has a trivial intersection with the subgroup spanned by the circle*

isotropy groups $G(a_2, b_2, c_2, d_2)$ and $G(a_4, b_4, c_4, d_4)$, then M is equivariantly diffeomorphic to the product of two lens spaces, $L(p_1, q_1)\times L(p_2, q_2)$.

Proof. (1) Define a T^2-action on $S^3=\{(z_1, z_2)\,|\,|z_1|+|z_2|=1\}$ by $(t, t')\cdot(z_1, z_2)=(z_1e^{2\pi ti}, z_2e^{2\pi t'i})$. Then the orbit space $(S^3)^*$ is as shown below.

$$G(1,0)\ \bullet\!\overset{e}{—}\!\bullet\ G(0,1).$$

Define a T^4-action on $S^3\times S^3$ by the product of two copies of the T^2-action on S^3. Then the orbit space $(S^3\times S^3)^*$ is as shown below.



If $\det\Delta=\pm1$, then there is an automorphism α^{-1} of T^4 which maps $G(a_1, b_1, c_1, d_1)$, $G(a_2, b_2, c_2, d_2)$, $G(a_3, b_3, c_3, d_3)$ and $G(a_4, b_4, c_4, d_4)$ to $G(1,0,0,0)$, $G(0,0,1,0)$, $G(0,1,0,0)$ and $G(0,0,0,1)$, respectively. Define a T^4-action on M by

$$\theta_\alpha((t_1, t_2, t_3, t_4), x)=\theta(\alpha(t_1, t_2, t_3, t_4), x),$$

where θ is the original T^4-action on M. Then the orbit space M^* with respect to θ_α is (weight-preserving) diffeomorphic to $(S^3\times S^3)^*$. Hence, it follows from Lemma 2.2 that M is weakly equivariantly diffeomorphic to $S^3\times S^3$.

(2) Under the hypothesis, we can choose an automorphism β^{-1} of T^4 which maps $G(a_1, b_1, c_1, d_1)$, $G(a_2, b_2, c_2, d_2)$, $G(a_3, b_3, c_3, d_3)$ and $G(a_4, b_4, c_4, d_4)$ to $G(1,0,0,0)$, $G(0,0,1,0)$, $G(a, p_1, 0, 0)$ and $(0, 0, b, p_2)$, respectively.

Let q_1 (and q_2) be the unique solution of $ax\equiv 1 \bmod p_1$ ($by\equiv 1 \bmod p_2$). Then by Example 2.3, $X=L(p_1, q_1)$ and $Y=L(p_2, q_2)$ admit effective T^2-actions so that the orbit spaces X^* and Y^* are, respectively,

$$G(1,0)\bullet\!—\!\bullet G(a, p_1)\quad\text{and}\quad G(1,0)\bullet\!—\!\bullet G(b, p_2).$$

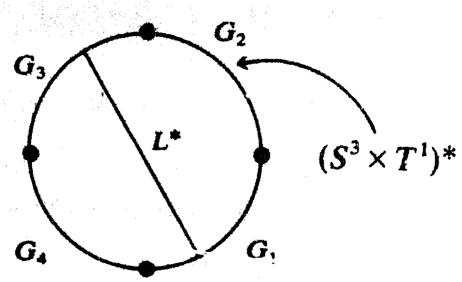
Define a T^4-action on $X\times Y$ by the product of the T^2-action on X and the T^2-action on Y. Then the orbit space $(X\times Y)^*$ with respect to the product action is (weight-preserving) diffeomorphic to the orbit space M^* with respect to a T^4-action θ_β defined by an automorphism β and the original action θ. By Lemma 2.2, M is weakly equivariantly diffeomorphic to $X\times Y$. $\square$

In the orbit space M^* specified above, the boundary of M^* is divided into four arcs by four points y_i^*, $i=1,2,3,4$, each of which corresponds to an orbit of type

T^2. By an argument similar to that used in Example 2.4, we can see that each of four arcs in ∂M^* corresponds to $L(r, s) \times T^1$.

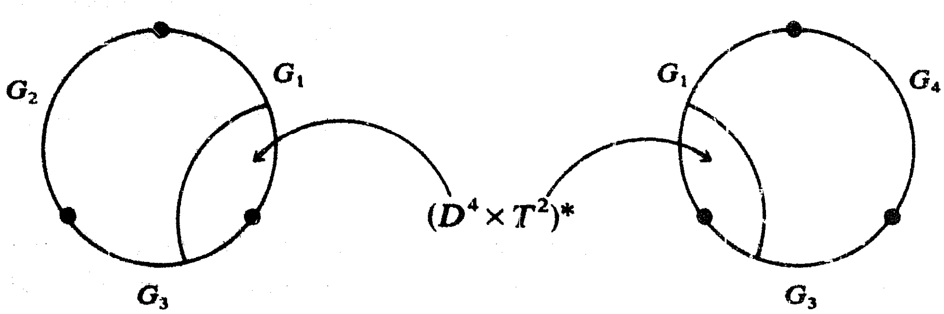
Lemma 2.6. *Suppose T^4 acts effectively and smoothly on a 6-manifold M so that the number of orbits of type T^2 is four, and the orbit space M^* is a 2-disk, and there is at least one arc in ∂M^* corresponding to $S^3 \times T^1$. Then $H_2(M, \mathbb{Z}) = 0$.*

Proof. From the hypothesis, we may assume that M^* is as shown below and G_1, G_2, and G_3 generate T^3. Hence, G_1 and G_3 have a trivial intersection.



Thus it follows from Example 2.4 that the arc L^* corresponds to $L = S^3 \times T^2$.

Cutting along L and attaching $D^4 \times T^2$ equivariantly to the boundary of each piece results in two manifolds M_1 and M_2 with the orbit spaces as shown below.



By applying Lemma 2.2 and [5, Lemma 3.1], we have $M_1 = S^5 \times T^1$ and $M_2 = L(p) \times T^1$, where $L(p)$ is a five dimensional lens space. Since by Lemma 2.7, an orbit of type T^1 is a generator of $\pi_1(L(p)) = \mathbb{Z}_p$, it follows from the Van Kampen theorem that $\pi_1(L(p) - (D^4 \times T^1)) = \pi_1(L(p))$ and $H_1(S^3 \times T^1) \to H_1(L(p) - (D^4 \times T^1))$ is surjective. From the Mayer–Vietoris sequence for $(L(p), D^4 \times T^1, L(p) - (D^4 \times T^1))$, we have $H_2(L(p) - (D^4 \times T^1)) = H_2(L(p)) = 0$.

Let

$$N_1 \equiv M_1 - (D^4 \times T^2) = (S^5 - (D^4 \times T^1)) \times T^1$$

and

$$N_2 \equiv N_2 - (D^4 \times T^2) = (L(p) - (D^4 \times T^1)) \times T^1,$$

then $N_1 \cap N_2 = (S^3 \times T^1) \times T^1$ and by the Künneth formula we have the following commutative diagram:

$$\begin{array}{ccc} H_1(S^3\times T^1)\otimes H_1(T^1) & \xrightarrow{\cong} & H_2(N_1\cap N_2) \\ \downarrow f_* & & \downarrow \alpha_* \\ H_1(L(p)-(D^4\times T^1))\otimes H_1(T_1) & \xrightarrow{\cong} & H_2(N_2) \end{array}$$

where f and α are appropriate inclusion maps. Since f_* is surjective, so is α_*.

From the Mayer–Vietoris sequence for (M, N_1, N_2), we have

$$\to H_2(S^3\times T^2)\xrightarrow{\alpha_*} H_2(S^3\times D^2\times T^1)\oplus H_2(M_2-(D^4\times T^2))\to H_2(M)$$

$$\to H_1(S^3\times T^2)\to H_1(S^3\times D^2\times T^1)\oplus H_1(M_2-(D^4\times T^2))\to H_1(M).$$

Since α_* is surjective, $H_2(M)$ is a subgroup of $\mathbb{Z}\oplus\mathbb{Z}$ and hence it is torsion free. If we use rational coefficients in the exact sequence above, we have

$$0\to H_2(M;\mathbb{Q})\to\mathbb{Q}\oplus\mathbb{Q}\to\mathbb{Q}\oplus\mathbb{Q}\to 0.$$

Hence we have $H_2(M;\mathbb{Z})=0$. □

In [5] we showed that the only possible isotropy groups for T^4 on M^6 are T^2, T^1, $\mathbb{Z}/k\mathbb{Z}$, and the identity. Furthermore, the boundary of the orbit space M^* corresponds to the singular orbits and the interior of M^* corresponds to the principal orbits and the exceptional orbits.

In the following we provide a modified version of [5, Theorem 2.1] so that it can be usable in subsequent arguments.

Lemma 2.7. *If T^4 acts on a 6-manifold M so that the orbit space M^* is a closed 2-disk and there exist no exceptional orbits, then the fundamental group of M is a finite abelian group with at most two generators.*

Proof. If α is an element of $\pi_1(M)$, then by the Whitney embedding theorem, there is an embedding $f: S^1\to M$ which represents α.

Let $q: M\to M^*$ be the orbit map and $P=q^{-1}(\operatorname{int} M^*)$. Then P is the union of principal orbits. By the general position theorem, f is homotopic to an embedding $g: S^1\to P$. Hence

$$\pi_1(P)=\pi_1(D^2\times T^4)\xrightarrow{i_\#}\pi_1(M)$$

is surjective, where $i_\#$ is a homomorphism induced by the inclusion.

Define $h^x:(T^4,1)\to(M,x)$ by $h^x((t_1,t_2,t_3,t_4))=(t_1,t_2,t_3,t_4)\cdot x$ and let $h^x_\#:\pi_1(T_4,1)\to\pi_1(M,x)$ be the induced map. Here x is a point in P. Since we assumed the isotropy groups span T^4, there are four circle isotropy groups $G_1=G(1,0,0,0)$, $G_2=G(0,1,0,0)$, $G_3=G(a_3,b_3,c_3,d_3)$ and $G_4=G(a_4,b_4,c_4,d_4)$ whose determinant is not zero.

Since, for a point $y \in M$ with $(T^4)_y = G_i$, $h^y(G_i) = \{y\}$, we have $h^x_*(G_i) = 1$ for $i = 1, 2, 3$, and 4. Hence

$$\pi_1(M) \approx Z^4/\text{Kernel } j_* \subset Z^2/\langle h^x_*(G_3), h^x_*(G_4)\rangle.$$

Since the circle groups span T^4, the quotient group must be finite. □

If we assume in the proof of Lemma 2.7 that G_3 and G_4 are $G(0, 0, 1, 0)$ and $G(0, 0, 0, 1)$, respectively, then $Z^2/\langle h^x_*(G_3), h^x_*(G_4)\rangle$ is trivial. Thus we have:

Corollary 2.8 [5]. *Under the hypothesis of Lemma 2.7, M is simply connected, provided that there are four circle isotropy groups whose determinant is* ± 1.

3. The proof of main theorem

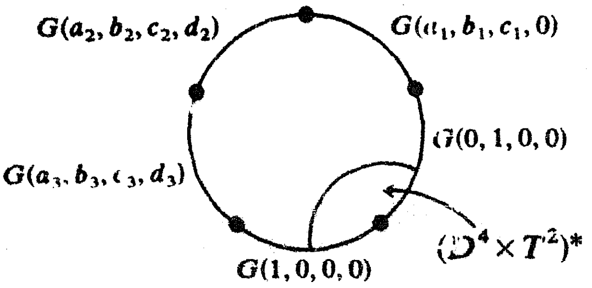
Throughout this section, we assume that the orbit space M^* for a T^4-action on M^6 is a closed 2-disk and that there exist no exceptional orbits. We recall that all actions are assumed to be smooth and effective.

By the slice theorem, an invariant tubular neighborhood of an orbit of type T^2 is D^4-bundle over T^4/T^2 with the structure group T^2. By applying Lemma 2.2, we can show that this bundle is trivial (that is, $D^4 \times T^2$).

Lemma 3.1. *Suppose* T^n, $n \geq 4$, *acts on a manifold M of dimension* $(n+2)$ *so that the number of orbits of type* T^{n-2} *is k. Then we can properly choose an orbit* $T^n(x)$ *of type* T^{n-2} *so that if* M_+ *is obtained from M by equivariantly replacing the invariant tubular neighborhood of* $T^n(x)$ *with* $S^3 \times D^2 \times T^{n-3}$, *then* M_+ *is obtained from an* $(n+2)$*-manifold* M_- *with* $(k-1)$ *orbits of type* T^{n-2} *by equivariantly replacing two copies of* $D^4 \times T^{n-2}$ *with two copies of* $S^3 \times D^2 \times T^{n-3}$.

Proof. We prove it for $k = 5$ and $n = 4$, but the general case can be proved in the same way as this case.

We may assume that M^* is as shown below and the four circle isotropy groups $G(0, 1, 0, 0)$, $G(a_1, b_1, c_1, 0)$, $G(a_2, b_2, c_2, d_2)$, $G(a_3, b_3, c_3, d_3)$ span T^4.



Since $G(a_1, b_1, c_1, 0) \cap G(0, 1, 0, 0) = 1$, and $G(a_3, b_3, c_3, d_3) \cap G(1, 0, 0, 0) = 1$, it follows from [5, Corollary 1.3] that $\gcd(a_1, c_1) = 1$ and $\gcd(b_3, c_3, d_3) = 1$. Hence we can choose relatively prime integers p and q such that $a_1q - c_1p = 1$. Now we have

$$\det\begin{pmatrix} 1 & 0 & p & 0 \\ 0 & 1 & X & 0 \\ 0 & 0 & q & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = -1 \quad \text{for any integer } X.$$

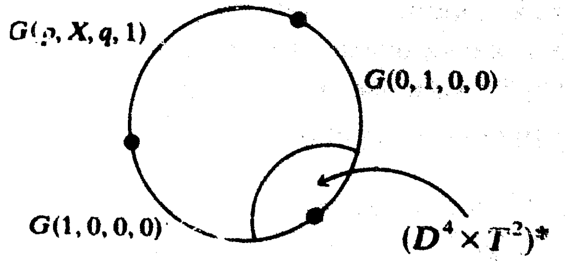
Define a T^4-action on $S^5 \times T^1 = \{(z_1, z_2, z_3, w) \mid |z_1| + |z_2| + |z_3| = 1, |w| = 1\}$ by

$$(\alpha, \beta, \gamma, \delta) \times (z_1, z_2, z_3, w) \to (z_1 e^{2\pi\bar{\alpha}i}, z_2 e^{2\pi\bar{\beta}i}, z_3 e^{2\pi\bar{\gamma}i}, w e^{2\pi\bar{\delta}i}),$$

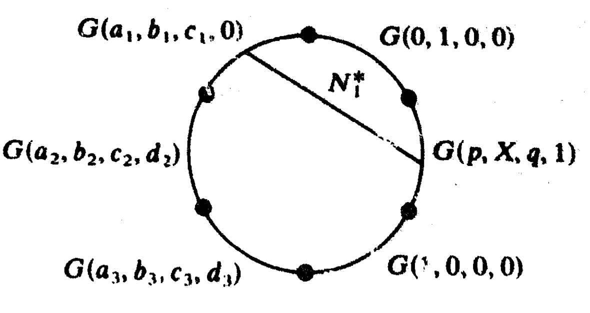
where

$$\begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \\ \bar{\gamma} \\ \bar{\delta} \end{pmatrix} = \begin{pmatrix} p & 1 & 0 & 0 \\ X & 0 & 1 & 0 \\ q & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}.$$

Then the orbit space $(S^5 \times T^1)/T^4$ is as shown below.



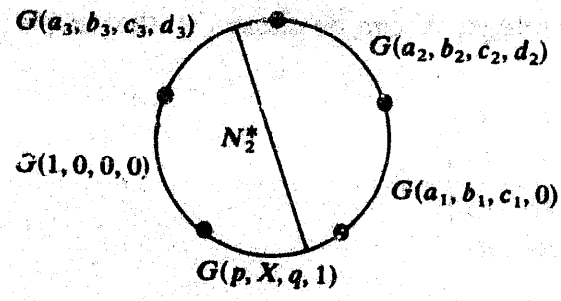
Gluing $(S^5 \times T^1) - (D^4 \times T^2)$ and $M - (D^4 \times T^2)$ together along their boundary results in M_+ whose orbit space M_+^* is as shown below.



From the choice of p and q, we have

$$\det\begin{pmatrix} 0 & a_1 & p & 0 \\ 1 & b_1 & X & 0 \\ 0 & c_1 & q & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \pm 1.$$

Hence $N_1 \approx S^3 \times D^2 \times T^1$. By equivariantly replacing N_1 with $D^4 \times T^2$, we obtain M_0 whose orbit space is as shown below.



Next we want to find integers X, Y, Z, U, V, such that

$$\det\begin{pmatrix} 1 & a_3 & p & V \\ 0 & b_3 & X & Y \\ 0 & c_3 & q & Z \\ 0 & d_3 & 1 & U \end{pmatrix} = \pm 1.$$

This is the same as finding integers X, Y, Z, U, such that

$$\det\begin{pmatrix} b_3 & X & Y \\ c_3 & q & Z \\ d_3 & 1 & U \end{pmatrix} = 1,$$

$$b_3(qU - Z) - c_3(XU - Y) + d_3(XZ - qY) = 1,$$

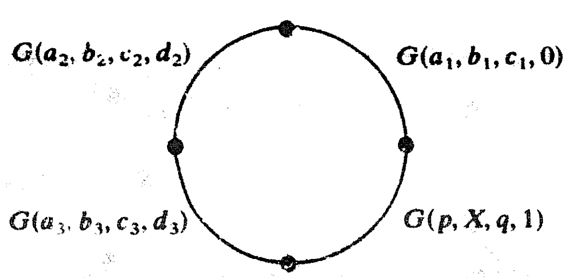
$$(b_3 q - c_3 X)U + (d_3 X - b_3)Z + (c_3 - d_3 q)Y = 1.$$

If $d_3 = 0$, then $\gcd(b_3, c_3, d_3) = 1$ makes it possible to select Y and Z such that $-b_3 Z + c_3 Y = 1$. Thus these choices for Y and Z and $U = 0$ yield the desired determinant. Hence we assume $d_3 \neq 0$.

Let $\gcd(b_3, d_3) = d$, then $-b_3 + d_3 X = d(-b_3' + d_3' X)$ where $\gcd(b_3', d_3') = 1$. Since $\gcd(b_3, c_3, d_3) = 1$, $c_3 - d_3 q$ and d have no common factors. By applying the Chinese Remainder Theorem as we did in [5, Section 5], we can choose an integer X so that X is greater than any given integer and no factor of $c_3 - d_3 q$ is a divisor of $-b_3' + d_3' X$.

Hence $c_3 - d_3 q$ and $-b_3 + d_3 X$ are relatively prime for some integer X. So we choose integers X_0, Y_0, Z_0, U_0, and $V_0 = 1$ so that the determinant is 1.

Equivariantly replacing N_2, which is homeomorphic to $S^3 \times D^2 \times T^1$ by the choice of integer X, with $D^4 \times T^2$, we have a 6-manifold M_- with the orbit space M_-^* as shown below.



□

Remark 3.2. (1) The integer X can be chosen so that $\pi_1(M_-)$ is finite. In fact, we can assume

$$\det\begin{pmatrix} 0 & a_1 & a_1 & a_3 \\ 1 & b_1 & b_2 & b_3 \\ 0 & c_1 & c_2 & c_3 \\ 0 & 0 & d_2 & d_3 \end{pmatrix} \neq 0,$$

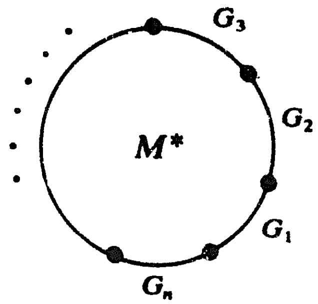
in the proof of Lemma 3.1. Hence

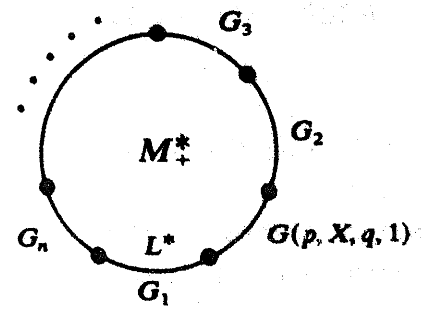
$$\det\begin{pmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ 0 & d_2 & d_3 \end{pmatrix} = K_0 \neq 0.$$

The determinant of $G(p, X, q, 1)$, $G(a_1, b_1, c_1, 0)$, $G(a_2, b_2, c_2, d_2)$ and $G(a_3, b_3, c_3, d_3)$ is $-XK_0 + L_0$ for some integer L_0. Hence $-XK_0 + L_0$ is not zero for a sufficiently large number X. This implies $\pi_1(M_-)$ is finite.

(2) Lemma 3.1 implies that an equivariant replacement of $D^4 \times T^2$ with $S^3 \times D^2 \times T^1$ can be chosen properly so that the replacement produces three arcs in the orbit space which correspond to $S^3 \times T^1$.

(3) The following are schematic interpretations of what we shall apply Lemma 3.1 to in the subsequent arguments.





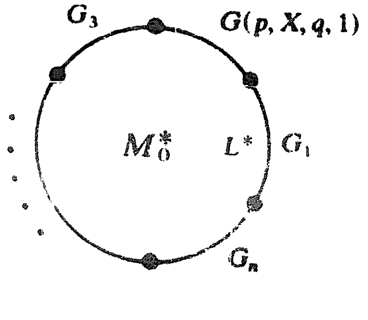
M^*: $G_1 = G(1, 0, 0, 0)$, $G_2 = G(0, 1, 0, 0)$, $G_3 = G(a_1, b_1, c_1, d_1) \ldots,$ $G_n = G(a_{n-2}, b_{n-2}, c_{n-2}, d_{n-2})$. $\det(G_2, G_3, G_i, G_j) \neq 0$, and the number of orbits of type T^2 is n, $n \geqslant 3$.

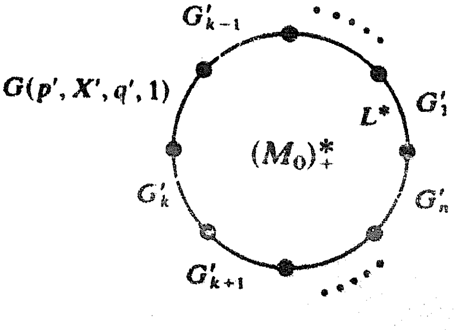
M_+^*: The number of orbits of type T^2 is $n+1$. By Lemma 3.1, we can choose an integer X so that

(i) $\det(G, G_3, G_i, G_j) \neq 0$, $G = G(p, X, q, 1)$,

(ii) $G_n \times G_1 \times G \approx T^3$,

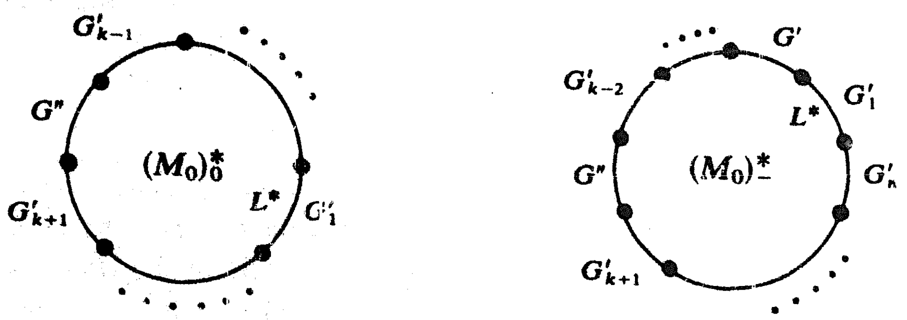
(iii) $L \approx S^3 \times T^1$.





M_0^*: The number of orbits of type T^2 is n. There is a circle isotropy group G_k such that $\det(G_k, G_n, G_1, G(p, X, q, 1)) \neq 0$ (Proof: We can choose an automorphism α which maps G_n, G_1, and $G(p, X, q, 1)$ to $G(1,0,0,0)$, $G(0,1,0,0)$ and $G(0,0,1,0)$, respectively. Suppose for each G_i, $\det(G_n, G_1, G, G_i) = 0$. Then the fourth component of $\alpha(G_i)$ is 0 for each i. This is a contradiction, since the isotropy groups span T^4.)

$(M_0)_\pm^*$: Here G_i' is the image of G_i under an automorphism β of T^4 that maps G_{k-1} and G_k to $G_{k-1}' = G(1,0,0,0)$ and $G_k' = G(0,1,0,0)$, respectively. Since $\det(G_k', G_n', G_1', G') \neq 0$, we can choose integers p', q', X', so that $\det(G(p', X', q', 1), G_n', G_1', G') = 0$ (by using an argument similar to that used in (1) of the remark).



$(M_0)_0^*$: The number of orbits of type T^2 is n. $\det(G'', G_n', G_1', G') \neq 0$ and hence $\pi_1((M_0)_0)$ is finite cyclic. Here $G'' = G(p', X', q', 1)$.

$(M_0)_-^*$: The number of orbits of type T^2 is $n-1$. Since $\det(G'', G_n', G_1', G') \neq 0$ and $L \approx S^3 \times T^1$, $\pi_1((M_0)_-)$ is finite cyclic.

Note that all of these manifolds can be constructed so that the isotropy groups span T^4 and except for M, the fundamental groups are generated by at most one generator of finite order, which means that one of the two generators is trivial.

Lemma 3.3. *Let M be a 6-manifold with a T^4-action and k be the number of orbits of type T^2 Suppose $\pi_1(M)$ is a finite cyclic group (that is, it has at most one generator) and M_1 is obtained from M by equivariantly replacing $D^4 \times T^2$, an invariant tubular neighborhood of an orbit of type T^2, with $S^3 \times D^2 \times T^1$. Then $H_2(M_1) \approx \mathbb{Z}^{k+1}$ if and only if $H_2(M) \approx \mathbb{Z}^k$.*

Proof. Let $V_1 = D^4 \times T^2$ be an invariant tubular neighborhood of an orbit through $y \in M$, $V_2 = S^3 \times D^2 \times T^1$, and $U = M - (D^4 \times T^2)$. Then $V_i \cap U = S^3 \times T^2$, $U \cup V_1 = M$ and $U \cup V_2 = M_1$.

Let $j_1 : S^3 \times T^2 \to V_1$ and $j_2 : S^3 \times T^2 \to U$ be inclusions. The induced maps $H_1(j_1)$ and $H_2(j_1)$ are injective.

By the Mayer–Vietoris sequence of U and V_1, we have

$$H_2(U \cap V_1) \xrightarrow{\alpha} H_2(U) \oplus H_2(V_1) \to H_2(M) \to$$

$$\to H_1(V_1 \cap U) \xrightarrow{\beta} H_1(U) \oplus H_1(V_1) \to \mathbb{Z}_p.$$

Here $\alpha = (j_{2*}, -j_{1*})_2$ and $\beta = (j_{2*}, -j_{1*})_1$.

Since $(j_{1*})_1 \equiv H_1(j_1)$ and $(j_{1*})_2 \equiv H_2(j_1)$ are injective, α and β are injective. Hence we have a short exact sequence

$$0 \to H_2(S^3 \times T^2) \to H_2(U) \oplus \mathbb{Z} \to H_2(M) \to 0.$$

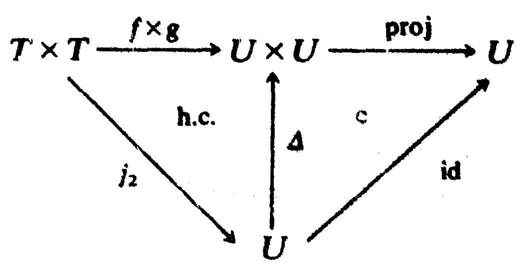
Let $T^2 \to U$ be the inclusion restricted to the second factor of $S^3 \times T^2 = U \cap V_i$ and $T^2 = \{(x, y) \mid 0 \leq x, y \leq 1\}$.

Let $f = j_2 | T^1 \times 1$, $g = j_2 | 1 \times T^1$ be the restrictions of each generator of T^2. Then, by the Künneth formula, we have a commutative diagram below.

$$\begin{array}{ccc} H_1(T^1) \otimes H_1(T^1) & \xrightarrow[\approx]{\times} & H_2(T^2) \\ \downarrow {\scriptstyle f_* \otimes g_*} & & \downarrow {\scriptstyle (f \times g)_*} \\ H_1(U) \otimes H_1(U) & \xrightarrow{\times} & H_2(U \times U) \end{array}$$

We may assume that the isotropy group of T^4 at y is $G(1, 0, 0, 0) \times G(0, 1, 0, 0)$. Let x be a point in the union of principal orbits. Then by Lemma 2.7, $h^x_*(G(0, 0, 1, 0))$ and $h^x_*(G(0, 0, 0, 1))$ are the generators of $\pi_1(M)$. Since $T^1 \times 1$ and $1 \times T^1$ are homotopic to $h^x(G(0, 0, 1, 0))$ and $h^x(G(0, 0, 0, 1))$, respectively, it follows from the hypothesis that $T^1 \times 1$ or $1 \times T^1$ is homotopically trivial. By the Van Kampen Theorem, a homomorphism $\pi_1(U) \to \pi_1(M)$, induced by the inclusion, is an isomorphism. Thus we have $(f \times g)_*(\xi) = 0$, where ξ is a generator of $H_2(T^2)$. The left half of the following diagram is homotopy commutative by a homotopy defined by

$$H(x, y, t) = j_2(x, t + (1 - t)y) \times j_2(t + (1 - t)x, y).$$



where Δ is the diagonal map. Since $\Delta_* j_{2*}(\xi) = 0$, and Δ_* is injective, $j_{2*}(\xi) = 0$. Hence

$$H_2(T^2 \times S^3) \xrightarrow{j_{2*}} H_2(U)$$

is trivial. Thus we have $H_2(U) \approx H_2(M)$ from the short sequence above.

From the Mayer–Vietoris sequence for (M_1, U, V_2), we have

$$\to H_2(U \cap V_2) \xrightarrow{\alpha'} H_2(U) \oplus H_2(V_2) \to H_2(M_1) \to$$

$$\to H_1(U \cap V_2) \xrightarrow{\beta'} H_1(U) \oplus H_1(V_2) \to \mathbb{Z}_p.$$

We have another short exact sequence,

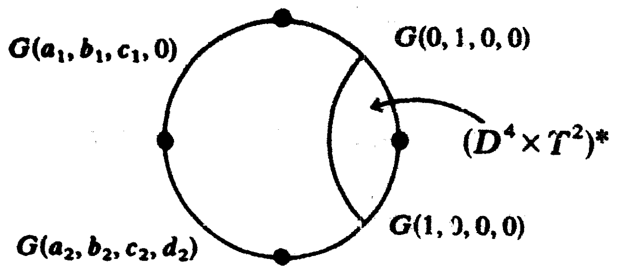
$$0 \to H_2(U) \to H_2(M_1) \to \text{kernel}\ \beta' \to 0.$$

The kernel β' is a subgroup of $\mathbb{Z}\oplus\mathbb{Z}$ and hence is torsion free. We can see kernel $\beta' \approx \mathbb{Z}$.

Hence $H_2(M) \approx \mathbb{Z}\oplus H_2(U) \approx \mathbb{Z}\oplus H_2(M)$. So $H_2(M)\approx \mathbb{Z}^k$ if and only if $H_2(M_1)\approx \mathbb{Z}^{k+1}$. □

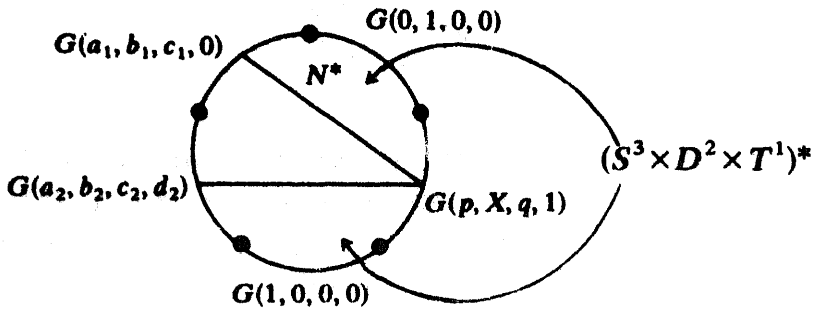
Lemma 3.4. *Suppose T^4 acts on a 6-manifold M so that the number of orbits of type T^2 is k, and $\pi_1(M)$ is generated by at most one generator of finite order. Then $H_2(M)\approx \mathbb{Z}^{k-4}$.*

Proof. If $k=4$, then M^* is as shown below.

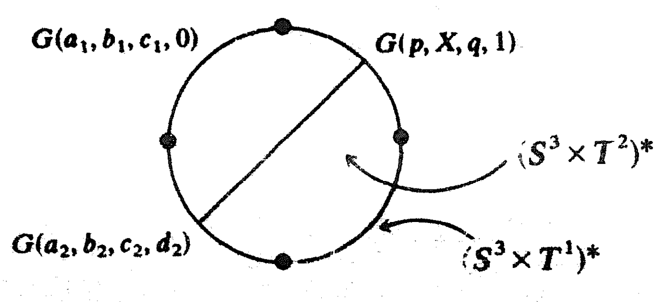


The determinant of isotropy groups was assumed to be non-zero.

By applying Lemma 3.1, we can select integers p, X, q, so that an equivariant replacement of $D^4\times T^2$ with $S^3\times D^2\times T^1$ produces M_1 with the orbit space M_1^* as shown below.



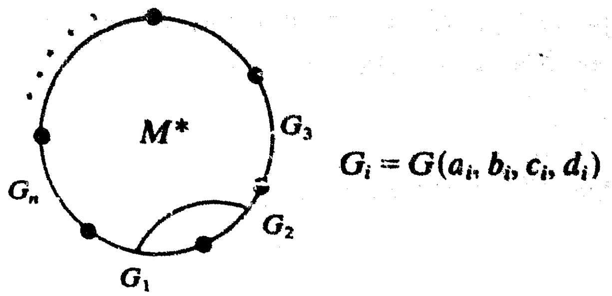
Replacing N with $D^4\times T^2$ equivariantly results in M_2 with M_2^* as shown below.



Since the determinant of $G(1,0,0,0)$, $G(0,1,0,0)$, $G(a_1,b_1,c_1,0)$, and $G(a_2,b_2,c_2,d_2)$ is not zero, the determinant of $G(p,X,q,1)$, $G(1,0,0,0)$, $G(a_1,b_1,c_1,0)$ and $G(a_2,b_2,c_2,d_2)$ is also nonzero for a sufficiently large X. Hence $\pi_1(M_2)$ is finite.

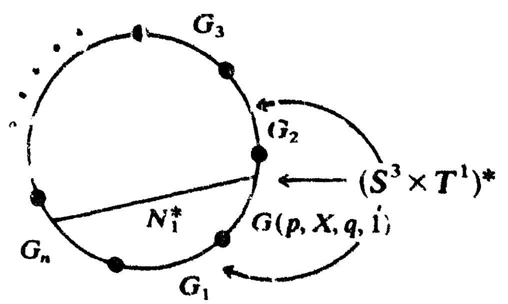
By Lemma 2.6, we have $H_2(M_2)=0$. It follows from Lemma 3.3 that $H_2(M_1)=\mathbb{Z}$ and $H_2(M)=0$.

Suppose the Lemma is true for $k<n$, and M has n orbits of type T^2.

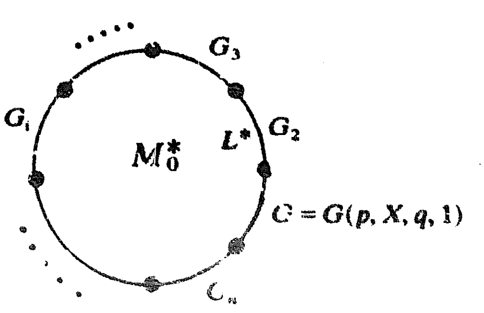


We may assume $G_1=G(1,0,0,0)$, $G_2=G(0,1,0,0)$, $G_3=G(a_1,b_1,c_1,0)$ and the determinant of G_2, G_i, G_j, G_k is not zero for some i, j, and k.

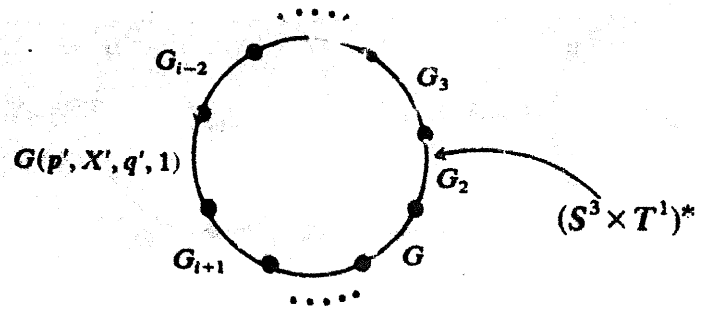
By Lemma 3.1, we have a 6-manifold M'_+ with M^*_+ as shown below.



M_0 obtained from M_+ by equivariantly replacing N_1 with $D^4\times T^2$ has a finite fundamental group generated by at most one generator. In fact, M_0^* has one arc L^* corresponding to $S^3\times T^1$ and hence $G(p,X,q,1)$, G_2, and G_3 represent three generators out of four of $\pi_1(T^4)=\mathbb{Z}^4$. On the other hand, all of these three circle groups are mapped to 1 by the homomorphism h_*^x defined in Lemma 2.7. It follows from Lemma 2.7 that $\pi_1(M_0)=\mathbb{Z}^4/(h_*^x)^{-1}(1)$ has only one generator.



We can assume $\det(G, G_2, G_3, G_i) \neq 0$, and $G_{i-1} = G(1,0,0,0)$, $G_i = G(0,1,0,0)$ and so on. By applying Lemma 3.1, we have T^4-manifolds of dimension 6, $(M_0)_+$, $(M_0)_-$ such that the orbit space $(M_0)_-^*$ is shown below.



By the choice of X', $\pi_1((M_0)_-)$ is generated by at most one generator of finite order.

By the induction hypothesis, $H_2((M_0)_-) = \mathbb{Z}^{n-5}$. By Lemma 3.3, $H_2((M_0)_+) = \mathbb{Z}^{n-3}$ and hence $H_2(M_0) = \mathbb{Z}^{n-4}$. By applying Lemma 3.3 again to M_0, M_+, and M, we have $H_2(M_+) = \mathbb{Z}^{n-3}$ and $H_2(M) = \mathbb{Z}^{n-4}$. □

Theorem 3.5. *Suppose T^4 acts smoothly and effectively on a simply connected 6-manifold M so that the number of orbits of type T^2 is k. Then we have*

$$H_0(M) = H_6(M) = \mathbb{Z}, \qquad H_1(M) = H_5(M) = 0,$$
$$H_2(M) = H_4(M) = \mathbb{Z}^{k-4}, \qquad H_3(M) = \mathbb{Z}^{2(k-3)}.$$

Proof. By the Poincaré duality, $H_1(M) = H^5(M) = 0$. By the universal coefficient theorem, $H^5(M) = \mathrm{Hom}(H_5(M), \mathbb{Z}) \oplus \mathrm{Ext}(H_4(M), \mathbb{Z})$. Hence $H_4(M)$ is torsion-free. The torsion of $H^3(M)$ is $\mathrm{Ext}(H_2(M), \mathbb{Z})$ and $H_2(M)$ is free by Lemma 3.4. Hence $H^3(M) = H_3(M)$ is also torsion-free.

Suppose $T = G(a, b, c, d)$ is a circle subgroup of T^4 which is different from any circle isotropy groups. Then the action restricted to T does not have any fixed point. Hence $\chi(M) = \chi(F(T, M)) = 0$.

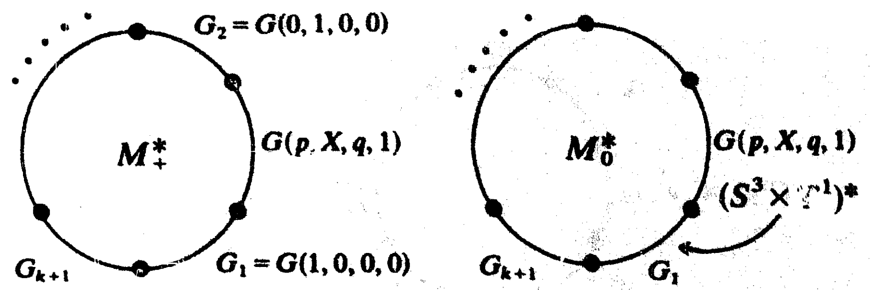
Thus we have $\chi(M) = -\mathrm{rank}\, H_3(M) + \mathrm{rank}\, H_2(M) + \mathrm{rank}\, H_4(M) + 2 = 0$. Hence $\mathrm{rank}\, H_3(M) = 2(k-4) + 2 = 2(k-3)$. □

Lemma 3.6. *Suppose M is a 6-manifold with an effective T^4-action and $\pi_1(M)$ is generated by at most one generator of finite order. Then the first Pontrjagin class, $p_1(M)$, is zero.*

Proof. Let k be the number of orbits of type T^2 in M. Then $k \geq 4$ and if $k = 4$, then $H_2(M) = 0$ by Lemma 3.4. Hence $p_1(M) = 0$.

Suppose the lemma is true for some $k > 4$, and T^4 acts on M smoothly and effectively so that the number of orbits of type T^2 is $k+1$.

Applying Lemma 3.1, we obtain T^4-manifolds M_+ and M_0 with the orbit spaces as shown below.



By Remark 3.2, we can choose an integer X so that $\pi_1(M_+)$ and $\pi_1(M_0)$ are generated by at most one generator of finite order, respectively.

Applying Lemma 3.1, as we did in the proof of Lemma 3.4, we have T^4-manifolds of dimension 6, $(M_0)_+$ and $(M_0)_-$ such that $\pi_1((M_0)_-)$ is generated by at most one generator of finite order, and $(M_0)_+$ is obtained from $(M_0)_-$ by equivariantly replacing two copies of $(D^4 \times T^2)$ with two copies of $(S^3 \times D^2 \times T^1)$.

By the induction hypothesis, $p_1((M_0)_-) = 0$. By Lemma 3.4, $H^2((M_0)_-)$, $H_2((M_0)_+)$, $H_2(M_0)$, $H_2(M_+)$, and $H_2(M)$ are all torsion free. So the third homology groups of these manifolds are torsion-free.

Hence the first Pontrjagin class p_1 can be regarded as a homomorphism $p_1(M): H_4(M) \to \mathbb{Z}$.

Since $p_1((M_0)_-)$ and $p_1(S^5 \times S^1)$ are zero, $p_1 \circ i_{1*}$ and $p_1 \circ i_{2*}$ are zero in the following diagram:

$$\begin{array}{ccccc} H_4(S^3 \times D^2 \times T^1) & \xrightarrow{i_{1*}} & H_4((M_0)_0) & \xleftarrow{i_{2*}} & H_4((M_0)_- - (D^4 \times T^2)) \\ & \searrow^{p_1(S^5 \times T^1)} & \downarrow p_1 & \swarrow_{p_1((M_0)_-)} & \\ & & \mathbb{Z} & & \end{array}$$

From the Mayer–Vietoris sequence of $(S^5 \times T^1) - (D^4 \times T^2)$ and $(M_0)_- - (D^4 \times T^2)$,

$$H_3(\partial(S^3 \times D^2 \times T^1)) \to H_3(S^3 \times D^2 \times T^1) \oplus H_3((M_0)_- - (D^4 \times T^2))$$

is injective. Hence

$$H_4(S^3 \times D^2 \times T^1) \oplus H_4((M_0)_- - (D^4 \times T^2)) \xrightarrow{\alpha} H_4((M_0)_0) \to 0.$$

Since $\alpha = i_{1*} + i_{2*}$, for all $y \in H_4((M_0)_0)$,

$$\begin{aligned} p_1((M_0)_0)(y) &= p_1 \circ (i_{1*} + i_{2*})(\bar{y}_1 + \bar{y}_2) \\ &= p_1(S^5 \times T^1)(\bar{y}_1) + p_1((M_0)_-)(\bar{y}_2) = 0. \end{aligned}$$

By applying this trick and by using the naturality of Pontrjagin class, we have $p_1(M)=0$. □

Lemma 3.7. *If M is a 6-manifold supporting an effective T^4-action such that $\pi_1(M)$ is generated by at most one generator of finite order, then the trilinear form*

$$\mu : H^2(M)\times H^2(M)\times H^2(M) \to \mathbb{Z}$$

is zero.

Proof. Since $\mu(x_1, x_2, x_3)=0$ if one of x_i is a torsion element, it is sufficient to prove for rational coefficients.

If the number of orbits of type T^2 is $k=4$, then $H^2(M;\mathbb{Z})=\mathbb{Z}_p$ and hence $H^2(M;\mathbb{Q})=0$. So $\mu\equiv 0$.

Suppose the lemma is true for some $k\geq 4$ and M has $(k+1)$ orbits of type T^2. By applying Lemma 3.1, we have T^4-manifolds of dimension 6, M_+, M_0, $(M_0)_+$, $(M_0)_0$, $(M_0)_-$ such that the fundamental groups of all these manifolds are generated by at most one generator of finite order.

Let

$$V_1=D^4\times T^2, \qquad V_2=S^3\times D^2\times T^1,$$
$$U=(M_0)_- -(D^4\times T^2), \qquad (M_0)_0=U\cup V_2.$$

In the proof of Lemma 3.3, the inclusion $i:U\cap V_2=S^3\times T^2\to U$ induces a trivial homomorphism $i_*:H_2(U\cap V_2;\mathbb{Q})\to H_2(U;\mathbb{Q})$ and hence $H^2(U;\mathbb{Q})\to H^2(U\cap V_2;\mathbb{Q})$ is trivial.

Thus the composition of inclusions $j:U\cap V_2\to U\to U\cup V_2=(M_0)_0$ induces a trivial homomorphism $j^*:H^2(U\cup V_2;\mathbb{Q})\to H^2(U\cap V_2;\mathbb{Q})$.

By the Mayer–Vietoris sequence for U and V_2, we have

$$\begin{array}{ccccccc} 0 \longrightarrow & H^1((M_0)_0;\mathbb{Q}) & \longrightarrow & H^1(U;\mathbb{Q}) & \oplus\, H^1(V_2;\mathbb{Q}) & \longrightarrow & H^1(U\cap V_2;\mathbb{Q}) \\ & \wr\wr & & \wr\wr & \wr\wr & & \wr\wr \\ & 0 & & 0 & \mathbb{Q} & & \mathbb{Q}\oplus\mathbb{Q} \end{array}$$

$$\begin{array}{ccccc} \xrightarrow{\delta^*} H^2((M_0)_0;\mathbb{Q}) & \xrightarrow{k^*} & H^2(U;\mathbb{Q})\oplus & H^2(V_2;\mathbb{Q}) & \longrightarrow 0. \\ & & & \wr\wr & \\ & & & 0 & \end{array}$$

Hence we get a short exact sequence

$$\begin{array}{l} 0\to \text{image } \delta^* = \text{kernel } k^* \to H^2(U\cup V_2)\to H^2(U)\to 0. \\ \qquad\qquad \wr\wr \\ \qquad\qquad \mathbb{Q} \end{array}$$

So $H^2(U\cup V_2)=H^2(U)+\text{kernel } k^*=\langle\alpha_1\rangle+\cdots+\langle\alpha_{k-4}\rangle+\langle\beta\rangle$ where α_i and β are basis of the vector space $H^2(U\cup V_2;\mathbb{Q})$.

Since $\langle\beta\rangle = \text{kernel } k^*$, there exist $\bar{\beta} \in H^1(U \cap V_2)$ such that $\delta^*(\bar{\beta}) = \beta$. Now we have

$$\begin{aligned} \alpha_i \cup \beta &= \delta^*(\bar{\beta}) \cup \alpha_i \\ &= \delta^*(\beta \cup j^*(\alpha_i)) \qquad \text{(see [7, p. 252])} \\ &= \delta^*(\beta \cup 0) = 0. \end{aligned}$$

Hence $\mu_0(\alpha_i, \alpha_j, \beta) = 0$ for any $\alpha_i, \alpha_j \in H^2((M_0)_0; \mathbf{Q})$.

By the induction hypothesis, $\mu_{0-} : H^2(U) \times H^2(U) \times H^2(U) \to \mathbf{Q}$ is zero. The following diagram is commutative:

$$\begin{array}{ccc} H^2((M_0)_0) \times H^2((M_0)_0) \times H^2((M_0)_0) & \xrightarrow{\mu_{00}} & \mathbf{Q} \\ \Big\downarrow j_0^* \times j_0^* \times j_0^* & \nearrow \mu_{0-} & \\ H^2(U) \times H^2(U) \times H^2(U) & & \end{array}$$

Hence $\mu_{00}(\alpha_i, \alpha_j, \alpha_k) = 0$. So $\mu_{00} \equiv 0$.

It follows from the same argument that the trilinear form μ_{0+} on $(M_0)_+$ is zero. By applying the above diagram and the short exact sequence for M_{0+} and M_0, we have $\mu_0 \equiv 0$. For a similar reason this fact implies $\mu_+ \equiv 0$ and hence $\mu \equiv 0$. □

Proof of Theorem 1.1. With the results of Theorem 3.5, Lemma 3.6 and Lemma 3.7, Wall [8] and Jupp's [2] classification theorems can be applied to complete this proof.

Acknowledgement

The author wishes to acknowledge his gratitude to Professor Frank Raymond for his generous advice and encouragement.

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